Short Proofs for the Determinant Identities

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ABSTRACT

We study arithmetic proof systems $\mathbb{P}_c(\mathbb{F})$ and $\mathbb{P}_f(\mathbb{F})$ operating with arithmetic circuits and arithmetic formulas, respectively, that prove polynomial identities over a field $\mathbb{F}$. We establish a series of structural theorems about these proof systems, the main one stating that $\mathbb{P}_c(\mathbb{F})$ proofs can be balanced: if a polynomial identity of syntactic degree d and depth k has a $\mathbb{P}_c(\mathbb{F})$ proof of size s, then it also has a $\mathbb{P}_c(\mathbb{F})$ proof of size $\text{poly}(s,d)$ and depth $O(k+\log^2 d+\log d\cdot\log s)$. As a corollary, we obtain a quasipolynomial simulation of $\mathbb{P}_c(\mathbb{F})$ by $\mathbb{P}_f(\mathbb{F})$, for identities of a polynomial syntactic degree.

Using these results we obtain the following: consider the identities

$$\det(XY)=\det(X)\cdot\det(Y) \quad \text{and} \quad \det(Z)=z_{11}\cdots z_{nn},$$

where X,Y and Z are $n\times n$ square matrices and Z is a triangular matrix with $z_{11},\ldots,z_{nn}$ on the diagonal (and det is the determinant polynomial). Then we can construct a polynomial-size arithmetic circuit det such that the above identities have $\mathbb{P}_c(\mathbb{F})$ proofs of polynomial-size and $O(\log^2 n)$ depth. Moreover, there exists an arithmetic formula det of size $n^{O(\log n)}$ such that the above identities have $\mathbb{P}_f(\mathbb{F})$ proofs of size $n^{O(\log n)}$.

This yields a solution to a basic open problem in propositional proof complexity, namely, whether there are polynomial-size $\mathbf{NC}^2$-Frege proofs for the determinant identities and the *hard matrix identities*, as considered, e.g. in Soltys and Cook [14] (cf., Beame and Pitassi [1]). We show that matrix identities like $AB=I\rightarrow BA=I$ (for matrices over the two element field) as well as basic properties of the determinant have polynomial-size $\mathbf{NC}^2$-Frege proofs, and quasipolynomial-size Frege proofs.

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1. INTRODUCTION

The field of proof complexity is dominated by the question of how hard is it to prove propositional tautologies. For weak proof systems, such as resolution, many hardness results are known (cf., [10] for a recent technical survey), but for strong propositional proof systems like Frege or extended Frege the question remains completely open. In this paper we continue to investigate a different but related problem: how hard is it to prove polynomial identities? For this purpose, various systems for proving polynomial identities were introduced in [4]. The main feature of these systems is that they manipulate arithmetic equations of the form $F=G$, where F,G are arithmetic formulas over a given field. Such equations are manipulated by means of simple syntactic rules, in such a way that $F=G$ has a proof if and only if F and G compute the same polynomial. The central question in this framework is the following:

> What is the *length* of such proofs, namely, does every true polynomial equation have a short proof, or are there hard equations that require extremely long proofs?

In this paper, we focus on two arithmetic equational proof systems (arithmetic proofs systems, for short) for proving polynomial identities: $\mathbb{P}_f$ and $\mathbb{P}_c$. The former system was introduced in [4] and the latter is an extension of it. The difference between the two systems is that $\mathbb{P}_f$ operates with *arithmetic formulas*, whereas $\mathbb{P}_c$ operates with *arithmetic circuits*—this is analogous to the distinction between Frege and extended Frege proof systems (Frege and extended Frege proofs are propositional proof systems establishing propositional tautologies, essentially operating with boolean formulas and circuits, respectively).

The study of proofs of polynomial identities is motivated by at least two reasons. First, as a study of the Polynomial Identity Testing (PIT) problem. As a decision problem, polynomial identity testing can be solved by an efficient randomized algorithm [9, 19], but no efficient deterministic algorithm is known. In fact, it is not even known whether there is a polynomial time non-deterministic algorithm or, equivalently, whether PIT is in **NP**. A proof system such as $\mathbb{P}_c$ can be interpreted as a specific non-deterministic algorithm for PIT: in order to verify that an arithmetic formula F computes the zero polynomial, it is sufficient to guess a proof of $F = 0$ in $\mathbb{P}_c$. Hence, if every true equality has a polynomial-size proof then PIT is in **NP**. Conversely, $\mathbb{P}_f$ and $\mathbb{P}_c$ systems capture the common syntactic procedures used to establish equality of algebraic expressions. Thus, showing the existence of identities that require superpolynomial arithmetic proofs would imply that those syntactic procedures are not enough to solve PIT efficiently.

The second motivation comes from propositional proof complexity. The systems $\mathbb{P}_f$ and $\mathbb{P}_c$ are in fact *restricted* versions of their propositional counterparts, Frege and extended Frege, respectively (when operating over $GF(2)$). One may hope that the study of the former would help to understand the latter. Arithmetic proof systems have the advantage that they work with arithmetic circuits. The structure of arithmetic circuits is perhaps better understood than the structure of their Boolean counterparts, or is at least different, suggesting different techniques and fresh perspectives.

In order to understand the strength of the systems $\mathbb{P}_f$ and $\mathbb{P}_c$, as well as their relative strength, we investigate quite a specific question, namely, how hard is it to prove basic properties of the determinant? In other words, we investigate lengths of proofs of identities such as $\det(AB) = \det(A)\cdot\det(B)$, or the cofactor expansion of the determinant. We show that such identities have polynomial-size $\mathbb{P}_c$ proofs of depth $O(\log^2 n)$ and quasipolynomial size $\mathbb{P}_f$ proofs (both results hold over any field).[1]

The determinant polynomial has a central role in both linear algebra and arithmetic circuit complexity. Therefore, an immediate motivation for our inquiry is to understand whether arithmetic proof systems are strong enough to reason efficiently about the determinant. More importantly, we take the determinant question as a pretext to present several structural properties of $\mathbb{P}_c$ and $\mathbb{P}_f$. A large part of this work is not concerned with the determinant at all, but is rather a series of general theorems showing how classical results in arithmetic circuit complexity can be translated to the setting of arithmetic proofs. We thus show how to capture efficiently the following results: (i) homogenization of arithmetic circuits (implicit in [16]); (ii) Strassen's technique for eliminating division gates over large enough fields (also in [16]); (iii) eliminating division gates over small fields—this is done by simulating large fields in small ones; and (iv) balancing arithmetic circuits (Valiant et al. [18]; see also [5]). Most notably, the latter result gives a collapse of polynomial-size $\mathbb{P}_c$ proofs to polynomial-size $O(\log^2 n)$-depth $\mathbb{P}_c$ proofs (for proving identities of polynomial syntactic degrees) and a quasipolynomial simulation of $\mathbb{P}_c$ by $\mathbb{P}_f$. This is one important point where the arithmetic systems differ from Frege and extended Frege, for which no non-trivial simulation is known.

Furthermore, the proof complexity of linear algebra attracted a lot of attention in the past. This was motivated, in part, by the goal of separating the propositional proof systems Frege and extended Frege. A classical example, originally proposed by Cook and Rackoff (cf., [1, 14, 15, 12, 13]), is the so called *inversion principle* asserting that $AB = I \rightarrow BA = I$. When A, B are $n \times n$ matrices over $GF(2)$, the inversion principle is a collection of propositional tautologies. Soltys and Cook [14, 12] showed that the principle has polynomial size extended Frege proofs. On the other hand, no feasible Frege proof is known, and hence the inversion principle is a candidate for separating the two proof systems. Other candidates, including several based on linear algebra, were presented by Buss et al. [3]. The inversion principle is one of the "hard matrix identities" explored in [14]. Inside Frege, the hard matrix identities have feasible proofs from one another, and they have short proofs from the aforementioned determinant identities. This connection between the hard matrix identities and the determinant identities serves as an evidence for the conjecture that hard matrix identities require superpolynomial Frege proofs: it seems that every Frege proof must in some sense construct the determinant, which is believed to require a superpolynomial-size formula.

A related question is whether the hard matrix identities and the determinant identities have polynomial-size $\mathbf{NC}^2$-Frege proofs[2]. This was conjectured in, e.g., [3], based on the intuition that the determinant is $\mathbf{NC}^2$ computable, and so by the analogy between circuit classes and proofs, it is natural to assume that the determinant properties are efficiently provable in $\mathbf{NC}^2$-Frege. Again, a polynomial-size extended Frege proofs of the determinant identities have been constructed in [14]. Whether these identities have polynomial-size $\mathbf{NC}^2$-Frege proofs (and hence, quasipolynomial-size Frege proofs) remained open. In this paper, we positively answer this question: we show that over $GF(2)$, the hard hard matrix identities and the determinant identities have polynomial-size $\mathbf{NC}^2$-Frege proofs. This is a simple corollary of the results on arithmetic proof systems. Over the two element field, an $O(\log^2 n)$-depth $\mathbb{P}_c$ proof is formally also $\mathbf{NC}^2$-Frege proof[3]. Thus, if determinant identities like $\det(AB) = \det(A) \cdot \det(B)$ have polynomial-size $\mathbb{P}_c(GF(2))$ proofs with depth $O(\log^2 n)$, then the corresponding propositional tautologies have polynomial-size $\mathbf{NC}^2$-Frege proofs.

Let us remark that one can also consider propositional translations of the determinant identities (and the hard matrix identities) over different finite fields or even the rationals. We do not explicitly study these translations, but there is no apparent obstacle to extending the result to these cases.

To understand our construction of short arithmetic proofs for the determinant identities, let us consider the following example. In [2], Berkowitz constructed a quasipolynomial size arithmetic formula for the determinant. He used a clever combinatorial argument designed *specifically* for the determinant function. However, one can build such a formula in a completely oblivious way: first compute the determinant by, say, Gaussian elimination algorithm. This gives an

[1]The parameter n is the dimension of the matrices A, B, and quasipolynomial size means size $n^{O(\log n)}$.

[2]That is, polynomial size proofs using circuits of $O(\log^2 n)$-depth

[3]When $+$ and $\cdot$ modulo 2 are interpreted as Boolean connectives and $=$ is interpreted as logical equivalence.

arithmetic circuit with division gates. Second, show that any circuit with division gates computing a polynomial can be efficiently simulated by a division-free circuit [16], and finally, show that any arithmetic circuit of a polynomial degree can be transformed to an $O(\log^2 n)$-depth circuit computing the same polynomial, with only a polynomial increase in size [18] (or to a formula with at most a quasipolynomial increase in size [5]). This paper follows a similar strategy, but in the proof-theoretic framework.

It should be stressed that in full generality, the structural theorems about $\mathbb{P}_c$ and $\mathbb{P}_f$ *cannot* be reproduced for propositional Frege and extended Frege systems. As already mentioned, no non-trivial simulation between Frege and extended Frege is known, and the other theorems are difficult to even formulate in the Boolean context. This also illustrates one final point: in order to construct a Frege proof of a tautology T, it may be useful to interpret T as a polynomial identity and prove it in some of the—weaker but better structured—arithmetic proof systems.

1.1 Arithmetic proofs with circuits and formulas

Before presenting and explaining the main results of this paper (in Section 2), we need to introduce our basic arithmetic proof systems.

Arithmetic circuits and formulas.

Let $\mathbb{F}$ be a field. *An arithmetic circuit* F is a finite directed acyclic graph as follows. Nodes (or gates) of in-degree zero are labeled by either a variable or a field element in $\mathbb{F}$. All the other nodes have in-degree two and they are labeled by either $+$ or $\times$. Unless stated otherwise, we assume that F has exactly one node of out-degree zero, called the *output node*, and that moreover the two edges going into a gate v labeled by $\times$ or $+$ are labeled by *left* and *right*. This is to determine the order of addition and multiplication[4]. An arithmetic circuit is called a *formula*, if the out-degree of each node in it is one (and so the underlying graph is a directed tree). The *size* of a circuit is the number of nodes in it, and the *depth* of a circuit is the length of the longest directed path in it. Arithmetic circuits and formulas will be referred to simply as *circuits* and *formulas*.

For a circuit F and a node u in F, F_u denotes the subcircuit of F with output node u. If F, G are circuits then

$$F \oplus G \text{ and } F \otimes G$$

abbreviate any circuit H whose output node is $u+v$ and $u \cdot v$, respectively, where $H_u = F$ and $H_v = G$. Furthermore,

$$F + G \text{ and } F \cdot G$$

denote the unique circuit of the form $F' \oplus G'$ and $F' \otimes G'$, respectively, where F', G' are disjoint copies of F and G. In particular, if F and G are formulas then so are $F + G$ and $F \cdot G$.

A circuit F computes a polynomial $\widehat{F}$ with coefficients from $\mathbb{F}$ in the obvious manner. That is, if F consists of a single node labeled with z, a variable or an element of $\mathbb{F}$, we have $\widehat{F} := z$. Otherwise, F is either of the form $G \oplus H$ or $G \otimes H$, and we let $\widehat{F} := \widehat{G} + \widehat{H}$ or $\widehat{F} := \widehat{G} \cdot \widehat{H}$, respectively.

[4]Although ultimately, addition and multiplication are commutative.

Substitution is understood in the following sense. Let $F = F(z)$ be a circuit and z a variable. For a circuit G, the circuit $F(G)$ is defined as follows: let $z_1, \ldots, z_k$ be the nodes in F labeled by z. Introduce k disjoint copies $G_1, \ldots, G_k$ of G, and let $F(G)$ be the union of $F, G_1, \ldots, G_k$ where we replace the node z_i by the output node of G_i. Specifically, if F and G are formulas then so is $F(G)$. $F(G)$ will also be written as $F(z/G)$.

The system $\mathbb{P}_f(\mathbb{F})$

We now define two proof systems for deriving polynomial identities. The systems manipulate *arithmetic equations*, that is, expressions of the form $F = G$. In the case of $\mathbb{P}_f(\mathbb{F})$, F, G are formulas, and in the case of $\mathbb{P}_c(\mathbb{F})$, F, G are circuits (see [4] for similar proof systems).

Let $\mathbb{F}$ be a field. The system $\mathbb{P}_f(\mathbb{F})$ proves equations of the form $F = G$, where F, G are *formulas* over $\mathbb{F}$. The inference rules are:

$$\text{R1} \quad \frac{F = G}{G = F}$$

$$\text{R2} \quad \frac{F = G \qquad G = H}{F = H}$$

$$\text{R3} \quad \frac{F_1 = G_1 \qquad F_2 = G_2}{F_1 + F_2 = G_1 + G_2}$$

$$\text{R4} \quad \frac{F_1 = G_1 \qquad F_2 = G_2}{F_1 \cdot F_2 = G_1 \cdot G_2}.$$

The axioms are equations of the following form, with F, G, H formulas:

A1 $F = F$

A2 $F + G = G + F$

A3 $F + (G + H) = (F + G) + H$

A4 $F \cdot G = G \cdot F$

A5 $F \cdot (G \cdot H) = (F \cdot G) \cdot H$

A6 $F \cdot (G + H) = F \cdot G + F \cdot H$

A7 $F + 0 = F$

A8 $F \cdot 0 = 0$

A9 $a = b + c\,, a' = b' \cdot c'\,,$ if $a, b, c, a', b', c' \in \mathbb{F},$ are such that the equations hold in $\mathbb{F}$.

The rules and axioms can be divided into two groups. The rules R1-R4 and axiom A1 determine the logical properties of equality "=", and axioms A2-A9 assert that polynomials form a commutative ring over $\mathbb{F}$.

A proof S in $\mathbb{P}_f(\mathbb{F})$ is a sequence of equations $F_1 = G_1$, $F_2 = G_2, \ldots, F_k = G_k$, with F_i, G_i formulas, such that every equation is either an axiom A1-A9, or was obtained from previous equations by one of the rules R1-R4. An equation $F_i = G_i$ appearing in a proof is also called a *proof line*. We consider two measures of complexity for S: the *size of S* is the sum of the sizes of F_i and G_i, $i \in [k]$, and the *number of proof lines in S* is k. (Throughout the paper, $[k]$ stands for $\{1, \ldots, k\}$.)

The system $\mathbb{P}_c(\mathbb{F})$

The system $\mathbb{P}_c(\mathbb{F})$ differs from $\mathbb{P}_f(\mathbb{F})$ in that it manipulates equations with *circuits*. $\mathbb{P}_c(\mathbb{F})$ has the same rules R1-R4 and axioms A1-A9 as $\mathbb{P}_f(\mathbb{F})$, but with $F, G, H, F_1, F_2, G_1, G_2$

ranging over circuits, augmented with the following two axioms:

C1 $F_1 \oplus F_2 = F_1 + F_2$ C2 $F_1 \otimes F_2 = F_1 \cdot F_2$.

A *proof in* $\mathbb{P}_c(\mathbb{F})$ is a sequence of equations $F_1 = G_1, \ldots, F_k = G_k$, where F_i, G_i are circuits, and every equation is either an axiom or was derived by one of the rules. As for $\mathbb{P}_f(\mathbb{F})$, the *size* of a proof is the sum of the sizes of all the circuits F_i and G_i, $i \in [k]$, and the *number of proof lines* of the proof is k. The *depth* of a $\mathbb{P}_c(\mathbb{F})$ proof is the maximal depth of a circuit appearing in the proof.

The main property of the two proof systems $\mathbb{P}_c(\mathbb{F})$ and $\mathbb{P}_f(\mathbb{F})$ is that they are sound and complete with respect to polynomial identities. The systems prove an equation $F = G$ if and only if F, G compute the same polynomial:

PROPOSITION 1. *Let $\mathbb{F}$ be a field.*

(i) For any pair F, G of arithmetic formulas, $\mathbb{P}_f(\mathbb{F})$ proves $F = G$ iff $\widehat{F} = \widehat{G}$.

(ii) For any pair F, G of arithmetic circuits, $\mathbb{P}_c(\mathbb{F})$ proves $F = G$ iff $\widehat{F} = \widehat{G}$.

Part i was shown in [4], part ii is almost identical. Soundness can be easily proved by induction on the number of lines and completeness by rewriting F and G as a sum of monomials.

It should be noted that $\mathbb{P}_f$ and $\mathbb{P}_c$ proofs are closed under substitution. If $F_1 = G_1, \ldots, F_k = G_k$ is a $\mathbb{P}_c$ proof, z a variable and H a circuit then $F_1(z/H) = G_1(z/H), \ldots, F_k(z/H) = G_k(z/H)$ is also a $\mathbb{P}_c$ proof (similarly for $\mathbb{P}_f$ and a formula H). This means that from a general proof, one can obtain the proof of its instance.

For simplicity, we often suppress the explicit dependence on the field $\mathbb{F}$ in $\mathbb{P}_c$ and $\mathbb{P}_f$, if the relevant statement holds over any field.

Comments on the proof systems.

The system $\mathbb{P}_c$ is an algebraic analogue of the propositional proof system *circuit Frege* (CF). Circuit Frege is polynomially equivalent to the more well-known *extended Frege* system (EF) (see [7, 6]). Following this analogy, one can define an *extended* $\mathbb{P}_f$ proof system, E$\mathbb{P}_f$, as follows: an E$\mathbb{P}_f$ proof is a $\mathbb{P}_f$ proof in which we are allowed to introduce new "extension" variables $z_1, z_2, \ldots$ via the axiom $z_i = F_i$, where we require that (i) the variable z_i appears in neither F_i nor in any previous proof-line; and (ii) the last equation in the proof contains none of the extension variables $z_1, z_2, \ldots$.

The following is completely analogous to the propositional case (see [7, 6]):

PROPOSITION 2.

(i) The systems $\mathbb{P}_c$ and E$\mathbb{P}_f$ polynomially simulate each other. More exactly, there is a polynomial p such that for every pair of formulas F, G, if $F = G$ has a $\mathbb{P}_c$ proof of size s then it has an E$\mathbb{P}_f$ proof of size $p(s)$, and if $F = G$ has an E$\mathbb{P}_f$ proof of size s then it has a $\mathbb{P}_c$ proof of size $p(s)$.

(ii) If F and G are circuits of size s and $F = G$ has a $\mathbb{P}_c$ proof with k proof lines then $F = G$ has a $\mathbb{P}_c$ proof of size poly(s, k).

The second part of this statement is especially useful, because it is often easier to estimate the number of lines in a proof rather than its size.

REMARK 3. *An alternative, polynomially equivalent, definition for $\mathbb{P}_c$ can be given as follows. For a circuit F, define $F^\bullet$ as the unfolding of F into a* formula. *That is, $F^\bullet := F$, if F is a leaf, and $(G \oplus H)^\bullet := G^\bullet + H^\bullet$, $(G \otimes H)^\bullet := G^\bullet \cdot H^\bullet$. We say that F and G are* similar circuits, *if $F^\bullet$ is the same formula as $G^\bullet$. Then A1, C1, C2 could be replaced by the following single axiom:*

A1' $F = G$, *whenever F and G are similar.*

The axiom A1' can be proved from A1, C1, C2 by a polynomial-size proof, and vice versa.

Notation for matrices inside proofs.

In this paper, matrices are understood as matrices whose entries are circuits and operations on matrices are operations on circuits. We illustrate this for square matrices. Let $F = \{F_{ij}\}_{i,j \in [n]}$ be an $n \times n$ matrix whose entries are circuits F_{ij}; and similarly $G = \{G_{ij}\}_{i,j \in [n]}$. Addition and multiplication is defined in the obvious way, namely

$$F + G = \{F_{ij} + G_{ij}\}_{i,j \in [n]}\,, \quad F \cdot G = \{\sum_{p=1}^{n} F_{ip} \cdot G_{pj}\}_{i,j \in [n]}\,,$$

where $+$ and $\cdot$ on the right-hand side is addition and multiplication on circuits. If a is a single circuit, $a \cdot F$ is the matrix $\{a \cdot F_{ij}\}_{i,j \in [n]}$. An equation $F = G$ denotes the set of equations $F_{ij} = G_{ij}$, $i, j \in [n]$.

2. OVERVIEW OF RESULTS AND TECHNIQUES

2.1 Main theorem

It is well known that the determinant can be uniquely characterized as the function that satisfies the following two identities for any pair of $n \times n$ matrices X, Y and any (upper or lower) triangular matrix Z with $z_{11}, \ldots, z_{nn}$ on the diagonal:

$$\det(X \cdot Y) = \det(X) \cdot \det(Y), \quad (1)$$

$$\det(Z) = z_{11} \cdots z_{nn}. \quad (2)$$

Moreover, other properties of the determinant, such as the cofactor expansion, easily follow from (1) and (2).

The main goal of this paper is to prove the following theorem:

THEOREM 4 (MAIN THEOREM). *For any field $\mathbb{F}$:*

(i) There exists a circuit det *such that* (1) *and* (2) *have polynomial-size $\mathbb{P}_c(\mathbb{F})$ proofs. Moreover, every*[5] *circuit in the proof has depth at most $O(\log^2(n))$.*

(ii) There exists a formula det *such that* (1) *and* (2) *have $\mathbb{P}_f(\mathbb{F})$ proofs of size $n^{O(\log n)}$.*

As mentioned before, a large part of the construction is not related directly to the determinant. It is rather a series of structural theorems about the systems $\mathbb{P}_f$ and $\mathbb{P}_c$. These are obtained by reproducing classical results in arithmetic circuit complexity in the setting of arithmetic proofs (for a recent survey on arithmetic circuit complexity see [11]). The

[5]We assume that the product $z_{11} \cdots z_{nn}$ in (2) is written as a formula of depth $O(\log n)$.

most important of those results is showing that $\mathbb{P}_c$ proofs can be balanced, in the sense that $\mathbb{P}_c$ proofs of size s (of polynomially bounded syntactic degree equations) can be polynomially simulated by $\mathbb{P}_c$ proofs in which each circuit has depth $O(\log^2 s)$.

We do not know whether it is possible to prove Theorem 4 directly, perhaps by formalizing the elegant algorithm of Berkowitz [2]. One advantage of the algorithm is that, being division-free, it would dispense of Theorem 9 and allow to generalize Theorem 4 to an arbitrary commutative ring (as opposed to a field). We also admit that working with circuits and proofs with divisions turned out to be quite tedious. However, our construction is intended to emphasize general properties of arithmetic proof systems, and the structural theorems are in fact our main contribution.

2.2 Balancing $\mathbb{P}_c$ proofs and simulating $\mathbb{P}_c$ by $\mathbb{P}_f$

In the seminal paper [18], Valiant et al. showed that if a polynomial f of degree d can be computed by an arithmetic circuit of size s, then f can be computed by a circuit of size $\mathrm{poly}(s,d)$ and depth $O(\log s \log d + \log^2 d)$. This is a strengthening of an earlier result by Hyafil [5], showing that f can be computed by a formula of size $(s(d+1))^{O(\log d)}$. We will show that those results can be efficiently simulated within the framework of arithmetic proofs.

Instead of the degree of a polynomial, we focus on the syntactic degree of a circuit. Let F be an arithmetic circuit. *The syntactic degree* of F, $\deg F$, is defined as follows:

(i) If F is a field element or a variable, then $\deg F = 0$ and $\deg F = 1$, respectively;

(ii) $\deg(F \oplus G) = \max(\deg F, \deg G)$, and $\deg(F \otimes G) = \deg F + \deg G$.

The *syntactic degree of an equation* $F = G$ is $\max(\deg F, \deg G)$, and the *syntactic degree of a proof* S is the maximum of the syntactic degrees of equations in S. If F is a circuit and u is a node in F we also write $\deg(v)$ to denote $\deg F_v$.

In accordance with [18], we will construct a map $[\cdot]$ that maps any given circuit F of size s and syntactic degree d to a circuit $[F]$ computing the same polynomial, such that $[F]$ has size $\mathrm{poly}(s,d)$ and depth $O(\log s \log d + \log^2 d)$. We will show the following:

THEOREM 5. *Let F, G be circuits of syntactic degree at most d such that $F = G$ has a $\mathbb{P}_c$ proof of size s. Then:*

(i) The equation $[F] = [G]$ has a $\mathbb{P}_c$ proof of size $\mathrm{poly}(s,d)$ and depth $O(\log s \cdot \log d + \log^2 d)$.

(ii) If F, G have depth at most k then $F = G$ has a $\mathbb{P}_c$ proof of size $\mathrm{poly}(s,d)$ and depth $O(k + \log s \cdot \log d + \log^2 d)$.

We also obtain the following simulation of $\mathbb{P}_c$ by $\mathbb{P}_f$:

THEOREM 6. *Assume that F, G are formulas of syntactic degree $\leq d$ such that $F = G$ has a $\mathbb{P}_c$ proof of size s. Then $F = G$ has a $\mathbb{P}_f$ proof of size $(s(d+1))^{O(\log d)} \leq s^{O(\log s)}$.*

This simulation is polynomial if F and G have a constant syntactic degree. Let us emphasize that the syntactic degree of a formula of size s is at most s, and hence the simulation is at most *quasipolynomial*.

Homogenization and degree bound in arithmetic proofs.

One ingredient of Theorems 5 and 6 is to show that using circuits of high syntactic degree cannot significantly shorten $\mathbb{P}_c$ proofs. That is, if we want to prove an equation of syntactic degree d, we can without loss of generality use only circuits of syntactic degree at most d. This result is the proof-theoretic analog of a result by Strassen, who showed how to separate arithmetic circuits into their homogeneous parts (implicit in [16]).

We say that a circuit F is *syntactically homogeneous*, if for every sum-gate $u_1 + u_2$ in F we have $\deg(u_1) = \deg(u_2)$. For a circuit F and a number k, we introduce a circuit $F^{(k)}$ which computes the syntactically k-homogeneous part of F (see Section 3 for the definition). The *syntactic degree of a $\mathbb{P}_c$ proof* is the maximal syntactic degree of a circuit appearing in it. We show the following:

PROPOSITION 7. *Assume that $F = G$ has a $\mathbb{P}_c$ proof of size s. Then*

(i) $F^{(k)} = G^{(k)}$ has a $\mathbb{P}_c$ proof of size $s \cdot \mathrm{poly}(k)$ and a syntactic degree at most k, for any k.

(ii) If $\deg F, \deg G \leq d$ then $F = G$ has a $\mathbb{P}_c$ proof of syntactic degree at most d and size $s \cdot \mathrm{poly}(d)$.

2.3 Circuits and proofs with division

We denote by $\mathbb{F}(X)$ the field of formal rational functions in the variables X over the field $\mathbb{F}$. It is convenient to extend the notion of a circuit so that it computes rational functions in $\mathbb{F}(X)$. This is done in the following way: *a circuit with division* F is a circuit which may contain an additional type of gate with fan-in 1, called an *inverse* or a *division* gate, denoted $(\cdot)^{-1}$. If a node v computes the rational function f, then v^{-1} computes the rational function $1/f$. Moreover, we require that for every division node v^{-1} in F, v does not compute the zero rational function. If no division gate computes the zero rational function we say that F *is defined*, and otherwise, we say that F is *undefined*. One should note, for instance, that the circuit $(x^2 + x)^{-1}$ over $GF(2)$ is defined, since $x^2 + x$ is *not* the zero rational function (although it vanishes as a function over $GF(2)$).

We define the system $\mathbb{P}_c^{-1}(\mathbb{F})$, operating with equations $F = G$ for F and G circuits with division computing rational functions in $\mathbb{F}(X)$. First, we extend the axioms of $\mathbb{P}_c(\mathbb{F})$ to apply to circuits with division. Second, we add the following axiom:

$$\mathrm{D} \qquad F \cdot F^{-1} = 1\,,$$

provided that F^{-1} is defined.

REMARK 8. *The system $\mathbb{P}_c^{-1}(\mathbb{F})$ polynomially simulates the rule*

$$\frac{F = G}{F^{-1} = G^{-1}}\,.$$

Moreover, the identities $(F^{-1})^{-1} = F$ and $(F \cdot G)^{-1} = G^{-1} \cdot F^{-1}$ have linear size proofs in $\mathbb{P}_c^{-1}(\mathbb{F})$.

As before, we sometimes suppress the explicit dependence on the field in $\mathbb{P}_c^{-1}(\mathbb{F})$ if the relevant statement is field independent.

Strassen [16] showed that division gates can be eliminated from arithmetic circuits computing polynomials over large

enough fields, with only a polynomial increase in size. We will show the proof-theoretic analog of Strassen's result over *arbitrary* fields, namely that $\mathbb{P}_c(\mathbb{F})$ polynomially simulates $\mathbb{P}_c^{-1}(\mathbb{F})$ for any field $\mathbb{F}$:

THEOREM 9. *Let $\mathbb{F}$ be any field and assume that F and G are circuits without division gates such that $\deg F, \deg G \leq d$. Suppose that $F = G$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size s. Then $F = G$ has a $\mathbb{P}_c(\mathbb{F})$ proof of size $s \cdot poly(d)$.*

A corollary of Theorem 9 is that $\mathbb{P}_c(\mathbb{F})$ polynomially simulates the rule

$$\frac{F \cdot G = 0}{F = 0}, \qquad \text{if } \widehat{G} \neq 0$$

(provided the syntactic degree of G is polynomially bounded).

To prove Theorem 9, we first assume that the underlying field $\mathbb{F}$ has an exponential size. Under this assumption, we cannot eliminate division gates in $GF(2)$ which is, from the Boolean proof complexity viewpoint, the most interesting field. To deal with small fields and specifically $GF(2)$ we have to show how to simulate large fields in small ones, as we explain in what follows.

Simulating large fields in small fields.

The idea behind simulating large fields in small ones is to treat the elements of $GF(p^n)$ as $n \times n$ matrices over $GF(p)$. This enables one to simulate computations and proofs over $GF(p^n)$ by those over $GF(p)$. We prove the following:

THEOREM 10. *Let p be a prime power and n a natural number and let F, G be circuits over $GF(p)$. Assume that $F = G$ has a $\mathbb{P}_c(GF(p^n))$ proof of size s. Then $F = G$ has a $\mathbb{P}_c(GF(p))$ proof of size $s \cdot poly(n)$.*

2.4 The determinant as a rational function and as a polynomial

To prove the main theorem (Theorem 4) one needs to construct a circuit (and a formula) computing the determinant polynomial which can be used efficiently inside arithmetic proofs. We first compute the determinant as a *rational function*, using a circuit with divisions denoted $\mathrm{DET}(X)$, and show that $\mathbb{P}_c^{-1}$ admits short proofs of the properties of $\mathrm{DET}(X)$. This is achieved by defining $\mathrm{DET}(X)$ in terms of the matrix inverse X^{-1} and inferring properties of DET from the identities $X \cdot X^{-1} = X^{-1}X = I$, which are shown to have polynomial-size $\mathbb{P}_c^{-1}$ proofs. The argument is basically a Gaussian elimination.

However, we cannot yet conclude Theorem 4 which speaks about (division-free) $\mathbb{P}_c$ proofs (it is worth mentioning that we also cannot yet conclude the short $\mathbf{NC}^2$-Frege proofs for the determinant identities, because $\mathbb{P}_c^{-1}$ proofs do not immediately correspond to propositional Frege proofs). Theorem 9 cannot be directly applied because it allows to eliminate division gates in $\mathbb{P}_c^{-1}$ proofs only if the equations proved are themselves division-free. We therefore proceed to construct a division-free circuit $\det(X)$, computing the determinant as a *polynomial*. Assuming we can prove efficiently in $\mathbb{P}_c^{-1}$ that $\det(X) = \mathrm{DET}(X)$, we are done, since we can now eliminate division gates from $\mathbb{P}_c^{-1}$ proofs of division-free equations, using Theorem 9. To this end, we define the $\det(X)$ polynomial as the nth term of the Taylor expansion of $\mathrm{DET}(I + zX)$ at $z = 0$. This enables us to demonstrate short proofs of $\det(X) = \mathrm{DET}(X)$ and conclude the argument.

2.5 Applications

Equipped with feasible proofs of the determinant identities, short proofs of several related identities follow. Cofactor expansion of the determinant will be given in Section 9. Another example is the formula completeness of the determinant. In [17], Valiant showed that every formula of size s can be written as a projection of a determinant of a matrix of a linear dimension. We can conclude that this holds feasibly already in $\mathbb{P}_c$:

PROPOSITION 11. *Let F be a formula of size s. Then there exists a matrix M of dimension $2s \times 2s$ whose entries are variables or elements of $\mathbb{F}$ such that the identity*

$$F = \det(M)$$

has a polynomial-size $O(\log^2 s)$-depth $\mathbb{P}_c(\mathbb{F})$ proof (and hence also a quasipolynomial-size $\mathbb{P}_f(\mathbb{F})$ proof), where det *is the circuit (resp. the formula) from Theorem 4.*

In this paper we are mainly interested in proofs with *no* assumptions other than the axioms A1-A9. Nevertheless, we can introduce the notion of a *proof from assumptions* as follows: let S be a set of equations. Then *a $\mathbb{P}_c$ proof from the assumptions S* is a proof that can use equations in S as additional axioms (and similarly for $\mathbb{P}_f$ proofs from assumptions). Proofs from assumptions are far less well-behaved than standard arithmetic proofs. For instance, neither Theorem 6 nor Theorem 9 hold for proofs from a general nonempty set S of assumptions. We now give an important example of a proof from assumptions.

Given a pair of $n \times n$ matrices X, Y, recall that the expressions $XY = I$ and $YX = I$, are abbreviations for the list of n^2 equalities between the appropriate entries.

PROPOSITION 12. *Let $\mathbb{F}$ be any field. The equations $YX = I_n$ have polynomial-size and $O(\log^2 n)$-depth $\mathbb{P}_c(\mathbb{F})$ proofs from the equations $XY = I_n$. In the case of $\mathbb{P}_f(\mathbb{F})$, the proof has a quasipolynomial-size.*

Determinant identities in $\mathbf{NC}^2$-Frege and Frege systems.

When considering the field $\mathbb{F}$ to be $GF(2)$, there is a close connection between our proof systems and the standard propositional proof systems. Consider the propositional proof systems Frege (F), extended Frege (EF) and circuit Frege (CF). For the definitions of Frege and extended Frege see [7] and for the definition of circuit Frege see [6], where it is also shown that CF and EF are polynomially equivalent.

For simplicity, we shall assume that F, EF and CF are all in the Boolean basis $+, \cdot, 0, 1$ (addition and multiplication modulo 2, logical equivalence, and the two Boolean constants)[6]. Then every arithmetic circuit *is automatically also a Boolean circuit*, and an equality like $G = H$ can be interpreted as the logical equivalence $G \equiv H$, written as $(G + H) + 1$. Hence $\mathbb{P}_f(GF(2))$ and $\mathbb{P}_c(GF(2))$ can be considered as *fragments* of F and CF, respectively: the finite

[6]Note that by Reckhow's result, as stated in [7], the particular choice of basis is immaterial. We could also have $\equiv$ as a primitive.

set of (schematic) axioms and rules of $\mathbb{P}_f(GF(2))$ now serve as Frege axioms and rules, and similarly for $\mathbb{P}_c(GF(2))$. Note that $x^2 = x$ is a propositional tautology but not a polynomial identity, and hence F and CF are (expressively) stronger than their arithmetic counterparts. In fact, one can polynomially simulate the full F or CF systems by adding the following new axiom

$$G^2 = G$$

to $\mathbb{P}_f(GF(2))$ or $\mathbb{P}_c(GF(2))$, where G is any formula or a circuit, respectively. To see this, it is sufficient to show that the augmented systems are complete with respect to propositional tautologies: they prove $F = 1$ whenever F evaluates to 1 on every $0,1$-input.

This means that upper bounds in $\mathbb{P}_f(GF(2))$ and $\mathbb{P}_c(GF(2))$ are in fact upper bounds in F and CF (and hence also in EF), respectively.

In what follows $XY = I_n$, and similarly $YX = I_n$, denote the conjunction of n^2 formulas of the form $(x_{i,1} \cdot y_{1,j} + \cdots + x_{i,n} \cdot y_{n,j}) \equiv \delta_{ij}$, where $+, \cdot$ are addition and multiplication modulo 2, respectively, $\equiv$ is the logical equivalence, and $\delta_{ij} \in \{0,1\}$ is given by $\delta_{ij} = 1$ iff $i = j$. We have the following:

THEOREM 13.

(i). The properties of the determinant as in Theorem 4 (interpreted as Boolean tautologies over $GF(2)$) have polynomial-size circuit Frege proofs, with every circuit of depth at most $O(\log^2 n)$. In the case of Frege, the proofs have quasipolynomial-size.

(ii). The implication $(XY = I_n) \to (YX = I_n)$ has a polynomial-size circuit Frege proof, with every circuit of depth at most $O(\log^2 n)$, and a quasipolynomial-size Frege proof.

PROOF. Part (i) is a direct consequence of Theorem 4 and (ii) of Proposition 12, both using the fact that proofs in $\mathbb{P}_c(GF(2))$ and $\mathbb{P}_f(GF(2))$ can be interpreted as proofs in circuit Frege and Frege, respectively. QED

A family of polynomial-size CF proofs in which every proof-line G is of depth $O(\log^2 |G|)$, is also called an $\mathbf{NC}^2$-Frege proof. Hence, Theorem 13 states that $\mathbf{NC}^2$-Frege has polynomial-size proofs of the propositional tautologies $(XY = I) \to (YX = I)$.

Theorem 13 thus settles an important open problem in proof complexity and feasible mathematics, namely, whether basic properties of the determinant like $\det(A) \cdot \det(B) = \det(AB)$ and the cofactor expansion (see Proposition 44), as well as the hard matrix identities, have polynomial-size proofs in a proof system which corresponds to the circuit class $\mathbf{NC}^2$.

REMARK 14. *We believe that Theorem 13 can be extended to any finite field or the field of rationals (after encoding field elements as Boolean strings). For finite fields, this is rather straightforward. In the rational case, one would have to show that the $\mathbb{P}_c(\mathbb{Q})$ proofs constructed in Theorem 4 involve only constants whose Boolean representation is polynomial.*

3. HOMOGENIZATION AND BOUNDING THE DEGREE IN $\mathbb{P}_c(\mathbb{F})$ PROOFS

In this section we wish to construct the circuits $F^{(k)}$ computing the k-homogeneous part of F and prove Proposition 7. First, let us say that a circuit F is *non-redundant*, if either F is the constant 0, or F does not contain the constant 0 at all. Any circuit F can be transformed to a non-redundant circuit $F^\sharp$ as follows: successively replace all nodes of the form $u + 0$, $0 + u$ by u and $u \cdot 0$, $0 \cdot u$ by 0, until no such replacement is possible.

Let k be a natural number. We define $F^{(k)}$ as follows. For every node u in F, introduce $k+1$ new nodes $u^{(0)}, \ldots, u^{(k)}$.

(i). Assume u is a leaf. Then, $u^{(0)} := u$, in case u is a field element, and $u^{(1)} := u$ in case u is a variable, and $u^{(i)} := 0$ otherwise.

(ii). If $u = u_1 + u_2$, let $u^{(i)} := u_1^{(i)} + u_2^{(i)}$, for every $i = 0, \ldots, k$.

(iii). If $u = u_1 \cdot u_2$, let $u^{(i)} := \sum_{j=0}^{i} u_1^{(j)} \cdot u_2^{(i-j)}$.

Finally, we define $F^{(k)}$ as the circuit $G^\sharp$, where G is the circuit with the output node $w^{(k)}$ and w is the output node of F.

Note the following:

(1) $F^{(k)}$ has size $O(s(k+1)^2))$, where s is the size of F.

(2) $F^{(k)}$ is a syntactically homogeneous non-redundant circuit. Its syntactic degree is either k, or F is the constant 0.

Notation: We allow circuits and formulas to use only sum gates with fan-in two. An expression $\sum_{i=1}^{k} x_i$ is an abbreviation for a formula of size $O(k)$ and depth $O(\log k)$ with binary sum gates. For example, define $\sum_{i=1}^{k} x_i := \sum_{i=1}^{\lfloor k/2 \rfloor} x_i + \sum_{i=\lceil k/2 \rceil}^{k} x_i$. One can see that basic identities such as

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{m} x_i + \sum_{i=m+1}^{k} x_i\,, \quad \text{or} \quad y \cdot \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y x_i$$

have $\mathbb{P}_f$ proofs of size $O(k^2)$ and depth $O(\log k)$.

LEMMA 15. *Let F_1, F_2 be circuits of size $\leq s$ and k a natural number. The following have proofs of size $s \cdot poly(k)$ and syntactic degree $\leq k$.*

(i). $(F_1 \oplus F_2)^{(k)} = F_1^{(k)} + F_2^{(k)}$,

(ii). $(F_1 \otimes F_2)^{(k)} = \sum_{i=0}^{k} F_1^{(i)} \cdot F_2^{(k-i)}$.

PROOF. It is easy to see that for any circuit H of size s, $H = H^\sharp$ has a proof of size $O(s)$. This, and the definition of $F^{(k)}$, gives $(F_1 \oplus F_2)^{(k)} = F_1^{(k)} \oplus F_2^{(k)}$. Hence $(F_1 \oplus F_2)^{(k)} = F_1^{(k)} + F_2^{(k)}$ by axiom C1. Since $F_1^{(k)}, F_2^{(k)}, (F_1 \oplus F_2)^{(k)}$ have size $O(s(k+1))^2$, we obtain (i). Part (ii) is similar. QED

LEMMA 16. *If F is a circuit with syntactic degree $\leq d$ and size s then*

$$F = \sum_{k=0}^{d} F^{(k)}$$

has a $\mathbb{P}_c(\mathbb{F})$ proof of syntactic degree $\leq d$ and size $s \cdot poly(d)$.

PROOF. For every node u in F, construct a proof of $F_u = \sum_{k=0}^{\deg(u)} F_u^{(k)}$. This is done by induction on depth of u. If u is a leaf, this stems from the definition of $F_u^{(k)}$, and if $u = u_1 + u_2$ or $u = u_1 \cdot u_2$, it is an application of the previous lemma. QED

Proof of Proposition 7. Part (ii) follows from (i) by Lemma 16, hence it is sufficient to prove part (i). Let us first show that if $F = G$ is an axiom of $\mathbb{P}_c(\mathbb{F})$ of size s then $F^{(k)} = G^{(k)}$ has a proof of size $s \cdot \mathrm{poly}(k)$ and syntactic degree $\leq k$. This is an application of Lemma 15. Let c be the constant such that equations (i) and (ii) in Lemma 15 have proofs of size $O(s \cdot (k+1)^c)$.

The lemma gives a proof $(F_1 \oplus F_2)^{(k)} = (F_1 + F_2)^{(k)}$ and $(F_1 \otimes F_2)^{(k)} = (F_1 \cdot F_2)^{(k)}$, as required for the axioms C1 and C2.

A1 and A9 are immediate. For the other axioms, consider for example the axiom $F_1 \cdot (F_2 \cdot F_3) = (F_1 \cdot F_2) \cdot F_3$, where the circuits have size $\leq s$. We have to construct a proof of

$$(F_1 \cdot (F_2 \cdot F_3))^{(k)} = ((F_1 \cdot F_2) \cdot F_3)^{(k)} . \tag{3}$$

By part (ii) of Lemma 15 the equations

$$(F_1 \cdot (F_2 \cdot F_3))^{(k)} = \sum_{i=0}^{k} F_1^{(i)} \left(\sum_{j=0}^{k-i} F_2^j F_3^{k-i-j} \right) \tag{4}$$

$$((F_1 \cdot F_2) \cdot F_3)^{(k)} = \sum_{i=0}^{k} \left(\sum_{j=0}^{i} F_1^j F_2^{i-j} \right) \cdot F_3^{(k-i)} , \tag{5}$$

can be proved by proofs with size roughly $s \cdot (k+1)^c \cdot (k+1)$. In $\mathbb{P}_c(\mathbb{F})$, the right hand sides of both (4) and (5) can be written as $\sum_{i+j+l=k} F_1^{(i)} F_2^{(j)} F_3^{(l)}$, by a proof of size roughly $s(k+1)^4$. This gives the proof of (3) of size $s \cdot \mathrm{poly}(k)$.

Next, assume that $F = G$ is derived from the equations $F_1 = G_1, F_2 = G_2$ by means of the rules R1-R4, and we need to construct the proof of $F^{(k)} = G^{(k)}$ from the set of equations $F_1^{(i)} = G_1^{(i)}, F_2^{(i)} = G_2^{(i)}, i = 0, \ldots k$. The hardest case is the rule

$$\frac{F_1 = G_1 \qquad F_2 = G_2}{F_1 \cdot F_2 = G_1 \cdot G_2} .$$

We have to prove $(F_1 \cdot F_2)^{(k)} = (G_1 \cdot G_2)^{(k)}$. By Lemma 15, we have proofs of $(F_1 \cdot F_2)^{(k)} = \sum_{i=0,\ldots k} F_1^{(i)} F_2^{(k-i)}$ and $(G_1 \cdot G_2)^{(k)} = \sum_{i=0,\ldots k} G_1^{(i)} G_2^{(k-i)}$. Hence $(F_1 \cdot F_2)^{(k)} = (G_1 \cdot G_2)^{(k)}$ can be proved from the assumptions $F_1^{(i)} = G_1^{(i)}, F_2^{(i)} = G_2^{(i)}, i = 0, \ldots k$. The proof has size roughly $s \cdot (k+1)^c (k+1)$. QED

4. BALANCING $\mathbb{P}_c$ PROOFS

In this section we prove Theorem 5 which is a proof-theoretic analog of the following result:

THEOREM 17 (VALIANT ET AL. [18]). *Let F be an arithmetic circuit of size s computing a polynomial f of degree d. Then there exists an arithmetic circuit $[F]$ computing f with depth $O(\log^2 d + \log s \cdot \log d)$ and size poly(d, s).*

We first give an outline of the construction of $[F]$, which closely follows that in [18] (we also refer the reader to [8] for an especially clear exposition). We emphasize that in our case, the relevant parameter is the *syntactic* degree of F: $[F]$ will have size $\mathrm{poly}(s, d)$ and depth $O(\log^2 d + \log s \cdot \log d)$, where d is the syntactic degree of F.

We write $u \in F$ to mean that u is a node in the circuit F. The following definition is important for the construction of balanced circuits: let w, v be two nodes in F. We define the *polynomial* $\partial w F_v$ as follows:

$$\partial w F_v := \begin{cases} 0, & \text{if } w \notin F_v, \\ 1, & \text{if } w = v \text{ , and otherwise:} \\ \partial w F_{v_1} + \partial w F_{v_2}, & v = v_1 + v_2; \\ (\partial w F_{v_1}) \cdot F_{v_2}, & \text{if either } v = v_1 \cdot v_2 \text{ and } \deg(v_1) \geq \deg(v_2), \\ & \text{or } v = v_2 \cdot v_1 \text{ and } \deg(v_1) > \deg(v_2). \end{cases}$$

The idea behind this definition is the following: let w, v be two nodes in F such that $2\deg(w) > \deg(v)$. Then for any product node $v_1 \cdot v_2 \in F_v$, w can be a node in at most one of F_{v_1}, F_{v_2}, namely the one of a higher syntactic degree. If we replace the node w in F_v by a new variable z, F_v then computes a polynomial $g(z, x_1, \ldots, x_n)$ which is linear in z, and $\partial w F_v$ is the usual partial derivative $\partial z g$.

It is not hard to show the following:

CLAIM 18. *Let w, v be two nodes in a circuit F. Then the polynomial $\partial w F_v$ has degree at most* $\deg(v) - \deg(w)$.

In order to construct $[F]$, we can assume without loss of generality that F itself is a syntactic homogenous circuit of size $s' = O(d^2 \cdot s)$. This is because a circuit of size s and syntactic degree d can be written as a sum of $d+1$ syntactically homogeneous circuits of size at most s' and syntactic degree at most d. Now the construction proceeds by induction on $i = 0, \ldots, \lceil \log d \rceil$. In each step $i = 0, \ldots, \lceil \log d \rceil$ we construct:

(i). Circuits computing $\widehat{F}_v$, for all nodes v in F with $2^{i-1} < \deg(v) \leq 2^i$;

(ii). Circuits computing $\partial w F_v$, for all nodes w, v in F with $2^{i-1} < \deg(v) - \deg(w) \leq 2^i$ and $\deg(v) < 2\deg(w)$.

Each step adds depth $O(\log s')$, which at the end amounts to a depth $O(\log^2 d + \log d \cdot \log s)$ circuit. Furthermore, each node v in F adds $O(s')$ nodes in the new circuit and each pair of nodes v, w in F adds also $O(s')$ nodes in the new circuit. This finally amounts to a circuit of size $O(s'^3) = O(d^6 \cdot s^3)$.

Let us now give the formal definition of $[F]$. First, for a circuit G and a natural number m, let

$$\mathcal{B}_m(G) := \big\{ t \in G \,:\, t = t_1 \cdot t_2, \deg(t) > m \text{ and } \deg(t_1), \deg(t_2) \leq m \big\}.$$

Definition of $[F]$.

Let F be an arithmetic circuit of syntactic degree d.

If F is not syntactic homogenous, let

$$[F] := [F^{(0)}] + \ldots + [F^{(d)}] .$$

Otherwise, assume that F is a syntactically homogenous circuit of degree d. For any *node* $v \in F$ we introduce the corresponding *node* $[F_v]$ in $[F]$ (intended to compute the polynomial $\widehat{F}_v$); and for any pair of nodes $v, w \in F$ such that $2\deg(w) > \deg(v)$, we introduce the node $[\partial w F_v]$ in $[F]$ (intended to compute the polynomial $\partial w F_v$).

The construction is defined by induction on $i = 0, \ldots, \lceil \log d \rceil$, as follows:

Part (I).

Let $v \in F$:

Case 1: Assume that $\deg(v) \leq 1$, then F_v computes a linear polynomial $a_1x_1 + \ldots + a_nx_n + b$ (where, by homogeneity of F, $b \neq 0$ implies that all a_i's equal 0). Define

$$[F_v] := a_1x_1 + \ldots + a_nx_n + b.$$

Case 2: Assume that for some $0 \leq i \leq \lceil \log(d) \rceil$:

$$2^i < \deg(v) \leq 2^{i+1}.$$

Put $m = 2^i$, and define

$$[F_v] := \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_v] \cdot [F_{t_1}] \cdot [F_{t_2}],$$

where t_1, t_2 are nodes such that $t = t_1 \cdot t_2$. (Note that here $[\partial w F_v], [F_{t_1}]$ and $[F_{t_2}]$ are *nodes*.)

Part (II).

Let w, v be a pair of nodes in F with $2\deg(w) > \deg(v)$:

Case 1: Assume w is not a node in F_v. Define

$$[\partial w F_v] := 0.$$

Case 2: Assume that w is in F_v and $0 \leq \deg(v) - \deg(w) \leq 1$. Thus, by Claim 18, the polynomial $\partial w f_v$ is a linear polynomial $a_1x_1 + \ldots + a_nx_n + b$. Define

$$[\partial w F_v] := a_1x_1 + \ldots + a_nx_n + b.$$

Case 3: Assume that w is in F_v and that for some $0 \leq i \leq \lceil \log(d) \rceil$:

$$2^i < \deg(v) - \deg(w) \leq 2^{i+1}.$$

Put $m = 2^i + \deg(w)$. Define:

$$[\partial w F_v] := \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_v] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}]\,,$$

where t_1, t_2 are nodes such that $t = t_1 \cdot t_2$ and $\deg(t_1) \geq \deg(t_2)$, or $t = t_2 \cdot t_1$ and $\deg(t_2) > \deg(t_1)$. Finally, define $[F]$ as the circuit with the output node $[F_u]$, where u is the output node of F.

One should make sure that the definition of $[F]$ is well defined, and that it has the correct depth and size:

LEMMA 19. *Let F be a circuit of size s and syntactic degree d. Then $[F]$ is a circuit computing $\widehat{F}$, $[F]$ is of size poly(s,d) and depth $O(\log^2 d + \log s \log d)$. Moreover, every node $[\partial w F_v]$ in $[F]$ computes the polynomial $\partial w F_v$.*

PROOF. The proof is as in [18] (see also [8]). We shall give a partial sketch of the proof here, for the benefit of the reader.

First, assume that F is syntactic homogeneous of degree d. We need to verify that $[F]$ is well-defined. That is, at stage $i = 0, \ldots, \lceil \log d \rceil$, we compute all $[F_v]$ and $[\partial_w F_u]$ for all nodes $v, u, w \in F$ such that $2^i < \deg(v) \leq 2^{i+1}$ and $2^i < \deg(v) - \deg(u) \leq 2^{i+1}$, and we want to show that the computation uses only nodes computed in previous stages.

Take, for example, Case 2 in Part (I). For any $t \in \mathcal{B}_m(F_v)$, $m < \deg(t) \leq \deg(v) \leq 2m$. This implies that $\deg(v) - \deg(t) \leq m = 2^i$ and $\deg(t) < 2\deg(v)$. Hence, we have already computed $[\partial t F_v]$. We have also already constructed $[F_{t_1}], [F_{t_2}]$, since $\deg(t_1), \deg(t_2) < m = 2^i$.

Inspecting the construction, $[F]$ has size poly(s) and depth $O(\log s \cdot \log d)$, given that F is syntactically homogeneous of size s and degree d. If F is not syntactically homogeneous, the definition $[F] = \left[F^{(0)}\right] + \ldots \left[F^{(d)}\right]$ gives a circuit of size poly(s,d) and depth $O(\log^2 d + \log s \cdot \log d)$, since every $F^{(k)}$ has size $O(s \cdot k^2)$. QED

We need to show that properties of $[F]$ can be proved inside the system $\mathbb{P}_c$. The key ingredient is given by the following Lemma.

LEMMA 20 (MAIN SIMULATION LEMMA). *Let F_1, F_2 be circuits of syntactic degree at most d and size at most s. Then there exist $\mathbb{P}_c$ proofs of:*

$$[F_1 \oplus F_2] = [F_1] + [F_2]\,, \tag{6}$$

$$[F_1 \otimes F_2] = [F_1] \cdot [F_2]\,, \tag{7}$$

such that the proofs have size poly(s,d) and depth $O(\log^2 d + \log d \cdot \log s)$.

The proof of Lemma 20 is deferred to the end of this section. We now use Lemma 20 to prove Theorems 5 and 6.

THEOREM 21 (THEOREM 5 RESTATED). *Let F, G be circuits of syntactic degrees at most d such that $F = G$ has a $\mathbb{P}_c$ proof of size s. Then*

(i). $[F] = [G]$ has a $\mathbb{P}_c$ proof of size poly(s,d) and depth $O(\log s \cdot \log d + \log^2 d)$.

(ii). If F, G have depth at most t then $F = G$ has a $\mathbb{P}_c$ proof of size poly(s,d) and depth at most $O(t + \log s \cdot \log d + \log^2 d)$.

PROOF. Part (i). Assume that $F = G$ has syntactic degree d and a $\mathbb{P}_c$ proof of size s. By Proposition 7, $F = G$ has a $\mathbb{P}_c$ proof of syntactic degree d and size $s' = s \cdot \text{poly}(d)$. So let us consider such a proof S. By induction on the number of lines in S, construct a $\mathbb{P}_c$ proof of $[F_1] = [F_2]$, where $F_1 = F_2$ is a line in S.

Let m_0 and k_0 be such that (6) and (7) have $\mathbb{P}_c$ proofs of size at most m_0 and depth k_0, whenever $F_1 \oplus F_2$, respectively, $F_1 \otimes F_2$ have size at most s' and syntactic degree at most d. By Lemma 20, we can choose $m_0 = \text{poly}(s', d)$ and $k_0 = O(\log s' \cdot \log d + \log^2 d)$.

First, show that if a line $F = H$ in S is a $\mathbb{P}_c$ *axiom* then $[F] = [H]$ has a $\mathbb{P}_c$ proof of size $c_1 m_0$ and depth $c_2 k_0$, where c_1, c_2 are some constants independent of s', d. The axiom A1 is immediate and the axiom A9 follows from the fact that $[F] = \widehat{F}$, if $\deg(F) = 0$. The rest of the axiom are an application of Lemma 20, as follows. Axioms C1 and C2 are already the statement of Lemma 20. For the other axioms, take, for example,

$$F_1 \cdot (G_1 + G_2) = F_1 \cdot G_1 + F_1 \cdot G_2\,.$$

We are supposed to give a proof of

$$[F_1 \cdot (G_1 + G_2)] = [F_1 \cdot G_1 + F \cdot G_2]\,,$$

with a small size and depth. By Lemma 20 we have a $\mathbb{P}_c$ proof

$$\begin{aligned}[F_1 \cdot (G_1 + G_2)] &= [F_1] \cdot [G_1 + G_2] \\ &= [F_1] \cdot [G_1] + [F_1] \cdot [G_2] = [F_1] \cdot ([G_1] + [G_2])\,.\end{aligned}$$

Lemma 20 gives again

$$[F_1] \cdot ([G_1] + [G_2]) = [F_1] \cdot [G_1 + G_2] = [F_1 \cdot (G_1 + G_2)] \ .$$

Here we applied Lemma 20 to circuits of size at most s', and the proof of $[F_1 \cdot (G_1 + G_2)] = [F_1 \cdot G_1 + F \cdot G_2]$ has size at most, say, $100m_0$ and depth at most $10k_0$.

An application of rules R1, R2 translates to an application of R1, R2. For the rules R3 and R4, it is sufficient to show the following: if S uses the rule

$$\frac{F_1 = F_2 \qquad G_1 = G_2}{F_1 \circ G_1 = F_2 \circ G_2}, \circ \in \{\cdot, +\},$$

then there is a proof of $[F_1 \circ G_1 = F_2 \circ G_2]$, of size $c_1 m_0$ and depth $c_2 k_0$, from the equations $[F_1] = [G_1]$ and $[F_2] = [G_2]$. This is again an application of Lemma 20.

Altogether, we obtain a proof of $[F] = [G]$ of size at most $c_1 s' m_0$ and depth $c_2 k_0$.

Part (ii). Using (i), it is sufficient to prove the following:

CLAIM 22. *If F is a circuit with depth t, syntactic degree d and size s, then $F = [F]$ has a $\mathbb{P}_c$ proof of size poly(s, d) and depth at most $O(t + \log s \cdot \log d + \log^2 d)$.*

Using Lemma 20, this claim can be easily proved by induction on s. QED

THEOREM 23 (THEOREM 6 RESTATED). *Assume that F, G are formulas of syntactic degree at most d such that $F = G$ has a $\mathbb{P}_c$ proof of size s. Then $F = G$ has a $\mathbb{P}_f$ proof of size $(s(d+1))^{O(\log d)}$.*

PROOF. Recall the definition of the formula $F^\bullet$ from Remark 3. It is not hard to show the following:

CLAIM 24. *If $H_1 = H_2$ has a $\mathbb{P}_c$ proof with p proof lines and depth k, then $H_1^\bullet = H_2^\bullet$ has a $\mathbb{P}_f$ proof of size $O(p2^k)$.*

Let F and G be as in the assumption. The previous theorem and Claim 24 give a $\mathbb{P}_f$ proof of

$$[F]^\bullet = [G]^\bullet$$

of size $s \cdot 2^{O(\log s \cdot \log d + \log^2 d)} = (s(d+1))^{O(\log d)}$.

To complete the proof, it is sufficient to show that:

CLAIM 25. *If H is a formula of size s and syntactic degree d, then $[H]^\bullet = H$ has a $\mathbb{P}_f$ proof of size $(s(d+1))^{O(\log d)}$.*

This is proved by induction on s using Lemma 20. QED

Proof of Lemma 20

It is sufficient to prove the statement under the assumption that $F_1 \oplus F_2$ and $F_1 \otimes F_2$ are syntactically homogeneous. This is because of the following: assume that the lemma holds for syntactically homogeneous circuits. First, note that for any circuit of syntactic degree d,

$$[F] = \left[F^{(0)}\right] + \left[F^{(1)}\right] + \cdots + \left[F^{(d)}\right]$$

has a proof of size poly(s, d) and depth $O(\log d \cdot \log s + \log^2 d)$: if F is not syntactically homogeneous, then this stems from the definition of $[F]$; otherwise, F is syntactically homogeneous, and so $[F^{(k)}]$ is the circuit 0 whenever $k < d$ and it is sufficient to construct the proof of $[F] = [F^{(d)}]$, which can be done by induction on the size of F. Second, if for example $F_1 \oplus F_2$ is not syntactically homogenous, then by definition of $[\cdot]$, we have

$$[F_1 \oplus F_2] = \sum_{k=0}^{d} \left[(F_1 \oplus F_2)^{(k)}\right] \ ,$$

where $d = \deg(F_1 \oplus F_2)$. By the definition of $F^{(k)}$, $(F_1 \oplus F_2)^{(k)}$ is a syntactically homogeneous circuit which is either of the form $F_1^{(k)} \oplus F_2^{(k)}$, or it is of the form $F_e^{(k)}$, if $F_{e'}^{(k)} = 0$, $\{e, e'\} = \{1, 2\}$. In both cases we obtain a proof of $[(F_1 + F_2)^{(k)}] = [F_1^{(k)}] + [F_2^{(k)}]$, of small size and depth. This gives a $\mathbb{P}_c$ proof of

$$\begin{aligned}\sum_{k=0}^{d} \left[(F_1 \oplus F_2)^{(k)}\right] &= \sum_{k=0}^{d} \left[(F_1)^{(k)}\right] + \left[(F_2)^{(k)}\right] \\ &= \sum_{k=0}^{d} \left[(F_1)^{(k)}\right] + \sum_{k=0}^{d} \left[(F_2)^{(k)}\right] \ .\end{aligned}$$

We thus consider the syntactically homogeneous case. Let $m(s, d)$ and $r(s, d)$ be functions such that for any circuit F of syntactic degree d and size s, $[F]$ has depth at most $r(s, d)$ and size at most $m(s, d)$. By Lemma 19, we can choose

$$m(s, d) = \text{poly}(s, d) \quad \text{and} \quad r(s, d) = O(\log^2 d + \log d \cdot \log s).$$

Notation: In the following, $[F_v]$ and $[\partial w F_v]$ will denote *circuits*: $[F_v]$ and $[\partial w F_v]$ are the subcircuits of $[F]$ with output nodes $[F_v]$ and $[\partial w F_v]$, respectively; the defining relations between the nodes of $[F]$ (see the definition of $[F]$ above) translate to equalities between the corresponding circuits. For example, if v and m are as in Case 2, part (I) of the definition of $[F]$, then, using just the axioms C1 and C2, we can prove

$$[F_v] = \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_v] \cdot [F_{t_1}] \cdot [F_{t_2}] \, . \tag{8}$$

Here, the left hand side is understood as the circuit $[F_v]$ in which $[\partial t F_v], [F_{t_1}], [F_{t_2}]$ appear as *subcircuits*, and so can share common nodes, while on the right hand side the circuits have *disjoint nodes*. Also, note that if F has size s and degree d, the proof of (8) has size $O(s^2 m(s, d))$ and has depth $O(r(s, d))$. We shall use these kind of identities in the current proof.

The following statement suffices to conclude the lemma. The recurrence (9) implies $\lambda(s, d) = \text{poly}(s, d)$ and it is enough to take F in the statement as either $F_1 \oplus F_2$ or $F_1 \otimes F_2$, and v as the root of F.

Statement.

Let F be a syntactically homogenous circuit of syntactic degree d and size s, and let $i = 0, \ldots, \lceil \log d \rceil$. There exists a function $\lambda(s, i)$ not depending on F with

$$\lambda(s, 0) = O(s^4) \quad \text{and} \quad \lambda(s, i) \leq O(s^4 \cdot m(s, d)) + \lambda(s, i-1), \tag{9}$$

and a $\mathbb{P}_c$ proof-sequence Ψ_i of size at most $\lambda(s, i)$ and depth at most $O(r(s, d))$, such that the following hold:

Part (I): For every node $v \in F$ with

$$\deg(v) \leq 2^i, \tag{10}$$

Ψ_i contains the following equations:

$$[F_v] = [F_{v_1}] + [F_{v_2}]\ , \quad \text{in case } v = v_1 + v_2, \quad \text{and} \tag{11}$$

$$[F_v] = [F_{v_1}] \cdot [F_{v_2}]\ , \quad \text{in case } v = v_1 \cdot v_2. \tag{12}$$

Part (II): For every pair of nodes $w \neq v \in F$, where $w \in F_v$, and with

$$\deg(v) - deg(w) \leq 2^i \quad \text{and} \tag{13}$$

$$2\deg(w) > \deg(v), \tag{14}$$

Ψ_i contains the following equations:

$$[\partial w F_v] = [\partial w F_{v_1}] + [\partial w F_{v_2}], \quad \text{in case } v = v_1 + v_2; \tag{15}$$

$$[\partial w F_v] = [\partial w F_{v_1}] \cdot [F_{v_2}], \tag{16}$$

in case $v = v_1 \cdot v_2$ and $\deg(v_1) \geq \deg(v_2)$
or $v = v_2 \cdot v_1$ and $\deg(v_1) > \deg(v_2)$.

We proceed to construct the sequence Ψ_i by induction on i.

Base case: $i = 0$. We need to devise the proof sequence Ψ_0.

Part (I).

Let $\deg(v) \leq 2^0$. By definition, $[F_v] = \sum_{i=1}^n a_i x_i + b$, where a_i's and b are field elements. If $v = v_1 + v_2$, we have also $[F_{v_e}] = \sum_{i=1}^n a_i^{(e)} x_i + b^{(e)}$, for $e = 1, 2$. Hence the equation $[F_v] = [F_{v_1}] + [F_{v_2}]$ is the (true) identity:

$$\sum_{i=1}^n a_i x_i + b = \sum_{i=1}^n a_i^{(1)} x_i + b^{(1)} + \sum_{i=1}^n a_i^{(2)} x_i + b^{(2)}\ ,$$

which has a proof of size $O(s^2)$ and depth $O(\log s)$ (we assume without loss of generality that $n \leq s$).

In case $v = v_1 \cdot v_2$, either $\deg(v_1) = 0$ or $\deg(v_2) = 0$ and the proof of $[F_v] = [F_{v_1}] \cdot [F_{v_2}]$ is similar.

Part (II).

Since $\deg(v) - \deg(w) \leq 1$, we have $[\partial w F_v] = \sum_{i=1}^n a_i x_i + b$, for some field elements a_i's and b.

In case $v = v_1 + v_2$, we have $\deg(v_e) - \deg(w) \leq 1$ and so $[\partial w F_{v_e}] = \sum_{i=1}^n a_i^{(e)} x_i + b^{(e)}$, where $e = 1, 2$. The assumption $w \neq v$ and Lemma 19, guarantee that $[\partial w F_v] = [\partial w F_{v_1}] + [\partial w F_{v_2}]$ is a correct identity, and we can thus proceed as the base case of Part (I) above.

In case $v = v_1 \cdot v_2$, assume without loss of generality that $\deg(v_1) \geq \deg(v_2)$. Again, we have $[\partial w F_{v_1}] = \sum_{i=1}^n a_i^{(1)} x_i + b^{(1)}$. From the assumptions, we have that $w \in F_{v_1}$, which implies $\deg(v_1) \geq \deg(w)$ and so $\deg(v_2) \leq 1$. Hence $[F_{v_2}] = \sum_{i=1}^n a_i^{(2)} x_i + b^{(2)}$. (One can note that at least one of $[\partial w F_{v_1}]$ or $[F_{v_2}]$ is constant). Thus we can prove the (correct, by virtue of the assumption $w \neq v$) identity $[\partial w F_v] = [\partial w F_{v_1}] \cdot [F_{v_2}]$ with a $\mathbb{P}_c(\mathbb{F})$ proof of size $O(s^2)$ and depth $O(\log s)$.

Overall, Ψ_0 will be the union of all the above proofs, so that Ψ_0 contains all equations (11), (12) (for all nodes v satisfying (10)), and all equations (15) and (16) (for all nodes v, w satisfying (13) and (14)). The proof sequence Ψ_0 has size $\lambda(s, 0) = O(s^4)$ and is and depth $O(\log s)$.

Induction step: We wish to construct the proof-sequence Ψ_{i+1}.

Part (I).

Let v be any node in F such that

$$2^i < \deg(v) \leq 2^{i+1}.$$

Case 1: Assume that $v = v_1 + v_2$. We show how to construct the proof of $[F_v] = [F_{v_1}] + [F_{v_2}]$. Let $m = 2^i$. From the definition of $[\cdot]$ we have:

$$[F_v] = [F_{v_1+v_2}] = \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t(F_{v_1+v_2})]\ . \tag{17}$$

Since $\deg(v_1) = \deg(v_2) = \deg(v)$, we also have

$$[F_{v_e}] = \sum_{t \in \mathcal{B}_m(F_{v_e})} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t(F_{v_e})], \quad \text{for } e \in \{0, 1\}\ . \tag{18}$$

If $t \in \mathcal{B}_m(F_v)$ then $\deg(t) > m = 2^i$. Therefore, for any $t \in \mathcal{B}_m(F_v)$, since $\deg(v) \leq 2^{i+1}$, we have $\deg(v) - \deg(t) < 2^i$ and $2\deg(t) > \deg(v)$ and $t \neq v$ (since t is a product gate). Thus, by induction hypothesis, the proof-sequence Ψ_i contains, for any $t \in \mathcal{B}_m(F_v)$, the equations

$$[\partial t(F_{v_1+v_2})] = [\partial t F_{v_1}] + [\partial t F_{v_2}].$$

Therefore, having Ψ_i as a premise, we can prove that (17) equals:

$$\sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot ([\partial t F_{v_1}] + [\partial t F_{v_2}])$$

$$= \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_1}] + \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_2}]. \tag{19}$$

If $t \in \mathcal{B}_m(F_v)$ and $t \notin F_{v_1}$ then $[\partial t F_{v_1}] = 0$. Similarly, if $t \in \mathcal{B}_m(F_v)$ and $t \notin F_{v_2}$ then $[\partial t F_{v_2}] = 0$. Hence we can prove

$$\sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_{v_e}] = \sum_{t \in \mathcal{B}_m(F_{v_e})} [\partial t F_{v_e}], \quad \text{for } e = 1, 2. \tag{20}$$

Thus, using (18) we have that (19) equals:

$$\sum_{t \in \mathcal{B}_m(F_{v_1})} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_1}] + \sum_{t \in \mathcal{B}_m(F_{v_2})} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_2}]$$

$$= [F_{v_1}] + [F_{v_2}]. \tag{21}$$

The above proof of (21) from Ψ_i has size $O(s^2 \cdot m(s, d))$ and depth $O(r(s, d))$.

Case 2: Assume that $v = v_1 \cdot v_2$. We wish to prove $[F_v] = [F_{v_1}] \cdot [F_{v_2}]$. Let $m = 2^i$. We assume without loss of generality that $\deg(v_1) \geq \deg(v_2)$. By the definition of $[\cdot]$, we have:

$$[F_v] = [F_{v_1 \cdot v_2}] = \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_v].$$

If $v \in \mathcal{B}_m(F_v)$, then $\mathcal{B}_m = \{v\}$ and we have $[F_v] = [F_{v_1}] \cdot [F_{v_2}] \cdot [\partial_v F_v]$. Since $[\partial_v F_v] = 1$, this gives $[F_v] = [F_{v_1}] \cdot [F_{v_2}]$, and we are done.

Otherwise, assume $v \notin \mathcal{B}_m(F_v)$. Then $m = 2^i < \deg(v_1)$ (since, if $\deg(v_1) \leq m$, then also $\deg(v_2) \leq m$ and so by definition $v \in \mathcal{B}_m(F_v)$). Because, moreover, $\deg(v_1) \leq 2^{i+1}$, we have

$$[F_{v_1}] = \sum_{t \in \mathcal{B}_m(F_{v_1})} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_1}]\ . \tag{22}$$

Since $\deg(v) \leq 2^{i+1}$ and $\deg(t) > m = 2^i$, for any $t \in \mathcal{B}_m(F_v)$, we have

$$\deg(v) - \deg(t) \leq 2^i \quad \text{and} \quad 2\deg(t) > \deg(v).$$

Since $v \neq t$, by induction hypothesis, Ψ_i contains, for any $t \in \mathcal{B}_m(F_v)$, the equation:

$$[\partial t(F_{v_1 \cdot v_2})] = [\partial t F_{v_1}] \cdot [F_{v_2}]. \tag{23}$$

Using (23) for all $t \in \mathcal{B}_m(F_v)$, we can prove the following with a $\mathbb{P}_c(\mathbb{F})$ proof of size $O(s^2 \cdot m(s,d))$ and depth $O(r(s,d))$:

$$\begin{aligned}\sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_v] &= \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t(F_{v_1 \cdot v_2})] \\ &= \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot ([\partial t F_{v_1}] \cdot [F_{v_2}]) \\ &= [F_{v_2}] \cdot \sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_1}].\end{aligned} \tag{24}$$

Since $\mathcal{B}_m(F_{v_1}) \subseteq \mathcal{B}_m(F_v)$, we can conclude as in (20) that

$$\sum_{t \in \mathcal{B}_m(F_v)} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_1}] = \sum_{t \in \mathcal{B}_m(F_{v_1})} [F_{t_1}] \cdot [F_{t_2}] \cdot [\partial t F_{v_1}].$$

Using (22), (24) equals $[F_{v_2}] \cdot [F_{v_1}]$. The above proof-sequence (using Ψ_i as a premise) has size $O(s^2 \cdot m(s,d))$ and depth $O(r(s,d))$.

We now append Ψ_i with all proof-sequences of $[F_v] = [F_{v_1}] + [F_{v_2}]$ for every v from Case 1, and all proof-sequences of $[F_v] = [F_{v_1}] \cdot [F_{v_2}]$ for every v from Case 2. We obtain a proof-sequence Ψ'_{i+1} of size

$$\lambda(s, i+1) \leq O(s^3 \cdot m(s,d)) + \lambda(s,i),$$

and depth $O(r(s,d))$.

In Part (II), we extend Ψ'_{i+1} with more proof-sequences to obtain the final Ψ_{i+1}.

Part (II).

Let $v \neq w$ be a pair of nodes in F such that $w \in F_v$ and assume that

$$2^i < \deg(v) - \deg(w) \leq 2^{i+1} \quad \text{and} \quad 2\deg(w) > \deg(v).$$

Let

$$m = 2^i + \deg(w).$$

Case 1: Suppose that $v = v_1 + v_2$. We need to prove

$$[\partial w F_v] = [\partial w F_{v_1}] + [\partial w F_{v_2}] \tag{25}$$

based on Ψ_i as a premise. By construction of $[\partial w F_v]$,

$$\begin{aligned}[\partial w F_v] &= \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_v] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}] \\ &= \sum_{t \in \mathcal{B}_m(F_v)} [\partial t(F_{v_1+v_2})] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}].\end{aligned} \tag{26}$$

Since $\deg(v_1) = \deg(v_2) = \deg(v)$, we also have

$$[\partial w F_{v_e}] = \sum_{t \in \mathcal{B}_m(F_{v_e})} [\partial t F_{v_e}] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}], \quad \text{for } e = 1, 2. \tag{27}$$

Since $m = 2^i + \deg(w)$, we have $\deg(t) > 2^i + \deg(w)$, for any $t \in \mathcal{B}_m(F_v)$. Thus, by $\deg(v) - \deg(w) \leq 2^{i+1}$, we get that for any $t \in \mathcal{B}_m(F_v)$:

$$\deg(v) - \deg(t) \leq 2^i \quad \text{and} \quad 2\deg(t) > \deg(v), \quad \text{and} \quad t \neq v \text{ (since } t \text{ is a product gate).}$$

Therefore, by induction hypothesis, for any $t \in \mathcal{B}_m(F_v)$, Ψ_i contains the equation

$$[\partial t(F_{v_1+v_2})] = [\partial t F_{v_1}] + [\partial t F_{v_2}].$$

Thus, based on Ψ_i, we can prove that (26) equals:

$$\begin{aligned}&\sum_{t \in \mathcal{B}_m(F_v)} ([\partial t F_{v_1}] + [\partial t F_{v_2}])[\partial w F_{t_1}][F_{t_2}] \\ &= \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_{v_1}][\partial w F_{t_1}][F_{t_2}] + \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_{v_2}][\partial w F_{t_1}][F_{t_2}].\end{aligned} \tag{28}$$

As in (20), using (27) we can derive the following from (28):

$$\begin{aligned}\sum_{t \in \mathcal{B}_m(F_{v_1})} [\partial t F_{v_1}][\partial w F_{t_1}][F_{t_2}] + \sum_{t \in \mathcal{B}_m(F_{v_2})} [\partial t F_{v_2}][\partial w F_{t_1}][F_{t_2}] \\ = [\partial w F_{v_1}] + [\partial w F_{v_2}].\end{aligned}$$

The proof of (25) from Ψ_i shown above has size $O(s^2 \cdot m(s,d))$ and depth $O(r(s,d))$.

Case 2: Suppose that $v = v_1 \cdot v_2$. We assume without loss of generality that $\deg(v_1) \geq \deg(v_2)$ and show how to prove

$$[\partial w F_v] = [\partial w F_{v_1}] \cdot [F_{v_2}]. \tag{29}$$

By construction of $[\partial w F_v]$:

$$\begin{aligned}[\partial w F_v] &= \sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_v] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}] \\ &= \sum_{t \in \mathcal{B}_m(F_v)} [\partial t(F_{v_1 \cdot v_2})] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}].\end{aligned} \tag{30}$$

Similar to the previous case, for any $t \in \mathcal{B}_m(F_v)$ we have

$$\deg(v) - \deg(t) < 2^i \quad \text{and} \quad 2\deg(t) > \deg(v).$$

If $v \in \mathcal{B}_m(F_v)$ then $\mathcal{B}_m(F_v) = \{v\}$ and so (30) is simply $\partial v F_v \cdot [\partial w F_{v_1}] \cdot [F_{v_2}] = [\partial w F_{v_1}] \cdot [F_{v_2}]$ as required. Otherwise, assume that $v \notin \mathcal{B}_m(F_v)$. By induction hypothesis, Ψ_i contains the following equation, for any $t \in \mathcal{B}_m(F_v)$:

$$[\partial t(F_{v_1 \cdot v_2})] = [\partial t F_{v_1}] \cdot [F_{v_2}].$$

Using Ψ_i as a premise, we can then prove that (30) equals:

$$\begin{aligned}&\sum_{t \in \mathcal{B}_m(F_v)} ([\partial t F_{v_1}] \cdot [F_{v_2}]) \cdot [\partial w F_{t_1}] \cdot [F_{t_2}] = \\ &\left(\sum\nolimits_{t \in \mathcal{B}_m(F_v)} [\partial t F_{v_1}] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}]\right) \cdot [F_{v_2}].\end{aligned} \tag{31}$$

As in (20), we have $\sum_{t \in \mathcal{B}_m(F_v)} [\partial t F_{v_1}] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}] = \sum_{t \in \mathcal{B}_m(F_{v_1})} [\partial t F_{v_1}] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}]$. Also, since $v_1 \cdot v_2 = v \notin \mathcal{B}_m(F_v)$, we have $\deg(v_1) > m = 2^i + \deg(w)$, and so

$$[\partial w F_{v_1}] = \sum_{t \in \mathcal{B}_m(F_{v_1})} [\partial t F_{v_1}] \cdot [\partial w F_{t_1}] \cdot [F_{t_2}]. \tag{32}$$

Hence by (32), (31) equals $[\partial_w F_{v_1}] \cdot [F_{v_2}]$.

The above proof of (29) from Ψ_i has size $O(s^2 \cdot m(s,d))$ and depth $O(r(s,d))$.

We now append Ψ'_i from Part (I) (which also contains Ψ_i) with all proof-sequences of $[\partial w F_v] = [\partial w F_{v_1}] + [\partial w F_{v_2}]$ in Case 1 and all proof sequences $[\partial w F_v] = [\partial w F_{v_1}] \cdot [F_{v_2}]$ in Case 2, above. We obtain the proof-sequence Ψ_{i+1} of size

$$\lambda(s, i+1) \leq O(s^4 \cdot m(s,d)) + \lambda(s,i),$$

and depth $O(r(s,d))$, as required.

5. PROOFS WITH DIVISION

In this section, we investigate proofs with divisions (as defined in Section 2.3), and prove Theorem 9.

Let us first turn the reader's attention to some peculiarities of the system $\mathbb{P}_c^{-1}$:

- We must be careful not to divide by zero in $\mathbb{P}_c^{-1}$. Hence $\mathbb{P}_c^{-1}$ proofs are *not closed under substitution*. It may happen that $F(z) = G(z)$ has a $\mathbb{P}_c^{-1}$ proof S, $F(0) = G(0)$ is defined, but substituting z by 0 throughout S is not a correct proof.

- Whereas $\mathbb{P}_c^{-1}$ is sound with respect to polynomial identities, it behaves erratically if one considers *proofs from assumptions*. For example, $\mathbb{P}_c^{-1}$ augmented with the axiom $x^2 - x = 0$ proves that $1 = 0$.

- Prima facie, it is not clear whether a $\mathbb{P}_c^{-1}$ proof of equation $F = G$ can be transformed to a proof of $F = G$ which contains only the variables contained in F and G. See Remark 29.

In the sequel, we will consider substitution instances of equations we prove in $\mathbb{P}_c^{-1}$. For instance, we will need to substitute 0 for some variables in the matrix X, when proving equations involving the circuit $\mathrm{DET}(X)$, and we have to guarantee that our proofs remain correct after such a substitution.

There are two *general* ways how to securely handle substitutions in $\mathbb{P}_c^{-1}$ proofs. The first one is to substitute only *algebraically independent elements*: replacing variables $z_1, \ldots, z_k$ with circuits $H_1, \ldots, H_k$ can never produce an undefined proof, if the circuits compute algebraically independent rational functions. The second way is offered in Corollary 33. This corollary allows one to construct a new proof of $F(0) = G(0)$ from the proof of $F(z) = G(z)$. Note, however, that in Corollary 33 the new proof will be polynomial only if the syntactic degree of F and G is polynomial.

Since the determinant circuit DET has an exponential syntactic degree (see Section 7), the second approach to substitution is not suitable for the DET identities. The first approach, which substitutes algebraically independent elements, often cannot cannot be used either, because we need to substitute variables by field elements. Therefore we must sometimes simply make sure that the specific substitutions used do not make the proofs undefined. To this end, we use the following terminology: let $\overline{x} = (x_1, \ldots, x_k)$ be a list of variables and $U = (U_1, \ldots, U_k)$ a list of circuits with divisions. We say that a circuit $F(\overline{x})$ with divisions is *defined* for $\overline{x} = U$, if no divisions by zero occur in $F(U)$; likewise, we say that a $\mathbb{P}_c^{-1}$ proof S is defined for $\overline{x} = U$ (or simply defined, if the context is clear), if every circuit in S is defined for $\overline{x} = U$.

5.1 Eliminating division gates over large enough fields

We first prove Theorem 9 under the assumption that the underlying field $\mathbb{F}$ is large. To eliminate division gates from proofs, we follow the construction of Strassen [16], in which an inverse gate is replaced by a truncated power series. In order to eliminate division gates over small fields, additional work will be needed (see Section 6).

Let F be a circuit with divisions. We say that F is a circuit *with simple divisions*, if for every inverse gate v^{-1} in F the circuit F_v does not contain inverse gates. A size s circuit with division F can be converted to a size $O(s)$ circuit of the form $F_1 \cdot F_2^{-1}$, where F_1, F_2 do not contain inverse gates, as follows.

For every node v introduce two nodes $\mathrm{Den}(v)$ and $\mathrm{Num}(v)$ which will compute the numerator and denominator of the rational function computed by v, respectively, as follows:

(i) If v is an input node of F, let $\mathrm{Num}(v) := v$ and $\mathrm{Den}(v) = 1$.

(ii) If $v = u^{-1}$, let $\mathrm{Num}(v) := \mathrm{Den}(u)$ and $\mathrm{Den}(v) := \mathrm{Num}(u)$.

(iii) If $v = u_1 \cdot u_2$, let $\mathrm{Num}(v) := \mathrm{Num}(v_1) \cdot \mathrm{Num}(v_2)$ and $\mathrm{Den}(v) := \mathrm{Den}(v_1) \cdot \mathrm{Den}(v_2)$.

(iv) If $v = u_1 + u_2$, let $\mathrm{Num}(v) := \mathrm{Num}(u_1) \cdot \mathrm{Den}(u_2) + \mathrm{Num}(u_2) \cdot \mathrm{Den}(u_1)$ and $\mathrm{Den}(v) := \mathrm{Den}(u_1) \cdot \mathrm{Den}(u_2)$.

Let $\mathrm{Num}(F)$ and $\mathrm{Den}(F)$ be the circuits with the output node $\mathrm{Num}(w)$ and $\mathrm{Den}(w)$, respectively, where w is the output node of F. The following lemma will be used in Proposition 28:

LEMMA 26. *Let $\mathbb{F}$ be any field.*

(i). If F is a size s circuit with division, then

$$F = Num(F) \cdot Den(F)^{-1}$$

has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size $O(s)$. The proof is defined whenever F is defined.

(ii). Let F, G be circuits with division. Assume that $F = G$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size s. Then $Num(F) \cdot Den(F)^{-1} = Num(G) \cdot Den(G)^{-1}$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size $O(s)$ such that every circuit in the proof is a circuit with simple divisions.

PROOF. Part (i) is proved by straightforward induction on the size of F and part (ii) by induction on the number of proof lines. We omit the details. QED

Let k be a fixed natural number and define $\mathrm{pow}_k(1-z)$ to be the circuit

$$\mathrm{pow}_k(1-z) := 1 + z + \cdots + z^k .$$

In other words, $\mathrm{pow}_k(1-z)$ is the first $k+1$ terms of the power series expansion of $1/(1-z)$ at $z = 0$.

Let F be a division-free circuit and let $a := \widehat{F^{(0)}}$. Assume that $a \neq 0$, that is, the polynomial computed by F has a nonzero constant term, and let $\mathrm{Inv}_k(F)$ denote the circuit

$$\mathrm{Inv}_k(F) := a^{-1} \cdot \mathrm{pow}_k(a^{-1}F)$$
$$= a^{-1} \cdot \left(1 + (1 - a^{-1}F) + (1 - a^{-1}F)^2 + \ldots + (1 - a^{-1}F)^k\right) .$$

Note that a^{-1} is a field element and hence $\mathrm{Inv}_k(F)$ is a circuit *without division*. The following lemma shows that $\mathrm{Inv}_k(F)$ can *provably* serve as the inverse polynomial of F "up to the k power":

LEMMA 27. *Let $\mathbb{F}$ be any field and let F be a size s circuit without division such that $\widehat{F^{(0)}} \neq 0$. Then the following have $\mathbb{P}_c(\mathbb{F})$ proofs of size $s \cdot poly(k)$:*

$$(F \cdot Inv_k F)^{(0)} = 1 \quad (33)$$

$$(F \cdot Inv_k F)^{(i)} = 0, \text{ for } 1 \leq i \leq k. \quad (34)$$

PROOF. Let z abbreviate the circuit $1 - a^{-1}F$. Then $F = a(1-z)$ and $\mathrm{Inv}_k(F) = a^{-1}(1 + z + z^2 + \cdots + z^k)$. By elementary rearrangement, we can prove

$$F \cdot \mathrm{Inv}_k(F) = (1-z)(1 + z + z^2 + \ldots z^k) = 1 - z^{k+1}.$$

By Lemma 15, $(F \cdot \mathrm{Inv}_k(F))^{(0)} = 1 - (z^{k+1})^{(0)}$ and $(F \cdot \mathrm{Inv}_k(F))^{(i)} = (z^{k+1})^{(i)}$, for $i > 0$. It is therefore sufficient to prove for every $i \leq k$, $(z^{k+1})^{(i)} = 0$. This follows by induction using Lemma 15 and the fact that $z^{(0)} = 0$. QED

The dependency on the field comes from the following fact, which follows from the Schwartz-Zippel lemma [9, 19]:

FACT 1. *Let $f_1, \ldots, f_s \in \mathbb{F}[X]$ be non-zero polynomials of degree $\leq d$, where $X = \{x_1, \ldots x_n\}$. Assume that $|\mathbb{F}| > sd$. Then there exists $\bar{a} \in \mathbb{F}^n$ such that $f_i(\bar{a}) \neq 0$ for every $i \in \{1, \ldots, s\}$.*

PROPOSITION 28. *There exists a polynomial p such that the following holds. Let F, G be circuits without division of syntactic degree at most d. Assume that $F = G$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof with divisions of size at most s and suppose that $|\mathbb{F}| > 2^{\Omega(s)}$. Then $F = G$ has a $\mathbb{P}_c(\mathbb{F})$ proof of size $s \cdot p(d)$.*

PROOF. Let S be a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of $F = G$ of size s. By Lemma 26, we can assume that the proof contains only simple divisions. Let $\mathcal{C}$ be the set of circuits H such that S contains a circuit with a node u^{-1} such that u computes H. Then $|\mathcal{C}| \leq s$ and $\deg H \leq 2^{\Omega(s)}$ for every $H \in \mathcal{C}$, since H has size at most s. By the Fact above, there exists a point $b \in \mathbb{F}^n$ such that $\widehat{H}(b) \neq 0$ for every $H \in \mathcal{C}$, where n is the number of variables in S.

Without loss of generality, we can assume that $b = \langle 0, \ldots, 0 \rangle$. Let S' be the sequence of equations obtained by replacing every circuit $(H)^{-1}$ in S by $\mathrm{Inv}_k(H)$. The sequence S' does not contain divisions, but is not yet a correct proof, since the translation $F \cdot \mathrm{Inv}_k(F) = 1$ of the axiom D is not satisfied. However, we claim that for every equation $F_1 = F_2$ in S' and every $k \leq d$, $F_1^{(k)} = G_1^{(k)}$ has a $\mathbb{P}_c$ proof of size $s \cdot p(d)$ for a suitable polynomial p. The proof is constructed by induction on the length of S', as in Proposition 7. The case of the axiom D follows from Lemma 27: $(F \cdot \mathrm{Inv}_k(F))^{(0)} = 1 = 1^{(0)}$ and $(F \cdot \mathrm{Inv}_k(F))^{(j)} = 0 = 1^{(j)}$, if $j > 0$. Consequently, we obtain proofs of $F^{(k)} = G^{(k)}$, for every $k \leq d$. By Lemma 16, we have $\mathbb{P}_c(\mathbb{F})$ proofs of $F = \sum_{k \leq d} F^{(k)}$, $G = \sum_{k \leq d} G^{(k)}$. This gives $\mathbb{P}_c(\mathbb{F})$ proofs of $F = G$ with the correct size. QED

Another application of Schwartz-Zippel lemma is the following:

REMARK 29. *Let $\mathbb{F}$ be an arbitrary field and assume that $F = G$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size s. Then there exists a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of $F = G$ of size $O(s^2)$ which contains only the variables appearing in F or G.*

PROOF. Let S be a proof of $F = G$ of size s which contains variables $z_1, \ldots, z_m$ not appearing in F or G. Assume that F or G actually contain at least one variable x, otherwise the statement is clear. It is sufficient to find a substitution $z_1 = H_1, \ldots, z_m = H_m$ for which the proof S is defined and $H_1, \ldots, H_m$ are circuits of size $O(s)$ in the variable x only. We will choose the substitution from the set $M = \{x^1, x^2, x^3 \ldots, x^{2^{cs}}\}$, where c is a sufficiently large constant. Note that x^p can be computed by a circuit of size $\log_2 p + 2$, and so every circuit in M has size $O(s)$. That such a substitution exists can be shown as in Proposition 28, when we consider M as a subset of the field of rational functions. QED

5.2 Taylor series

For a later application, we need to introduce the basic notion of a power series. Let $F = F(\overline{x}, z)$ be a circuit with division. We will define $\Delta_{z^k}(F)$ as a circuit in the variables $\overline{x}$, computing the coefficient of z^k in F, when F is written as a power series at $z = 0$. This is done as follows:

Case 1: Assume first that no division gates in F contain the variable z. Then we define $\Delta_{z^k}(F)$ by the following rules (the definition is similar to that of $F^{(k)}$ in Section 3, and so we will be less formal here):

(i) $\Delta_z(z) := 1$ and $\Delta_{z^k}(z) := 0$, if $k > 1$.

(ii) If F does not contain z, then $\Delta_{z^0}(F) := F$ and $\Delta_{z^k}(F) := 0$, if $k > 0$.

(iii) $\Delta_{z^k}(F + G) = \Delta_{z^k}(F) + \Delta_{z^k}(G)$.

(iv) $\Delta_{z^k}(F \cdot G) = \sum_{i=0}^{k} \Delta_{z^i}(F) \cdot \Delta_{z^{k-i}}(G)$.

Case 2: Assume now that some division gate in F contains z. We let:

$$F_0 := ((\mathrm{Den}(F))(z/0)^\sharp,$$

where, given a circuit G, $G^\sharp$ is the non-redundant version of G (see definition in Section 3) and $G(z/0)$ is obtained by substituting in G all occurrences of z by the constant 0.

In case $\widehat{F_0} \neq 0$, we define:

$$\Delta_{z^k}(F) := F_0^{-1} \cdot \Delta_{z^k}\left(\mathrm{Num}(F) \cdot \mathrm{pow}_k\left(F_0^{-1} \cdot \mathrm{Den}(F)\right)\right).$$

Note that z does not occur in any division gate inside $\mathrm{Num}(F) \cdot \mathrm{pow}_k\left(F_0^{-1} \cdot \mathrm{Den}(F)\right)$, and so $\Delta_{z^k} F$ is well-defined.

We summarize the main properties of Δ_{z^k} as follows:

PROPOSITION 30. *(i). If F is a circuit without divisions of syntactic degree $\leq d$ and size s then $F = \sum_{i=0}^{d} \Delta_{z^i}(F) \cdot z^i$ has a $\mathbb{P}_c$ proof of size $s \cdot poly(d)$.*

(ii). If $F_0, \ldots, F_k$ are circuits with divisions not containing the variable z, then $\Delta_{z^j}(\sum_{i=0}^{k} F_i z^i) = F_j$ has a polynomial size $\mathbb{P}_c^{-1}$ proof, for every $j \leq k$.

(iii). Assume that F, G are circuits with divisions such that $F = G$ has a $\mathbb{P}_c^{-1}$ proof of size s which is defined for $z = 0$. Then

$$\Delta_{z^k}(F) = \Delta_{z^k}(G)$$

has a $\mathbb{P}_c^{-1}$ proof of size $s \cdot poly(k)$.

The proofs are almost identical to those of Proposition 7 and Proposition 28. We omit the details.

6. SIMULATING LARGE FIELDS IN SMALL ONES

Recall the notation on matrices given in Section 1.1. Mainly, matrices are understood as matrices whose entries are circuits and operations on matrices are operations on circuits.

LEMMA 31. *Let X, Y, Z be $n \times n$ matrices of distinct variables and I_n the identity matrix. Then the following identities have polynomial-size $\mathbb{P}_c(\mathbb{F})$ proofs:*

$$\begin{array}{ll} X+Y=Y+X & X+(Y+Z)=(X+Y)+Z \\ X\cdot(Y+Z)=X\cdot Y+X\cdot Z & (Y+Z)\cdot X=Y\cdot X+Z\cdot X \\ X\cdot(Y\cdot Z)=(X\cdot Y)\cdot Z & X\cdot I_n=I_n\cdot X=X. \end{array}$$

Similarly for non-square matrices of appropriate dimension.

PROOF. Each of the equalities is a set of n^2 correct equations with degree ≤ 3 and size $O(n)$. Every such equation has a $\mathbb{P}_c$-proof of size $O(n^3)$. QED

Let $\mathbb{F}_1 = GF(p)$ and $\mathbb{F}_2 = GF(p^n)$, where p is a prime power. We will show how to simulate proofs in $\mathbb{P}_c(\mathbb{F}_2)$ by proofs in $\mathbb{P}_c(\mathbb{F}_1)$. Recall that $\mathbb{F}_2$ can be represented by $n \times n$ matrices with elements from $\mathbb{F}_1$, that is, there is an isomorphism θ between $\mathbb{F}_2$ and a subset of $GL_n(\mathbb{F}_1)$. We can also assume that $\theta(a) = aI_n$ if $a \in \mathbb{F}_1 \subseteq \mathbb{F}_2$. This allows one to treat a polynomial f over $\mathbb{F}_2$ as a matrix of n^2 polynomials over $\mathbb{F}_1$. Similarly, we can define a translation of circuits: let F be a circuit with coefficients from $\mathbb{F}_2$. Let $\overline{F}$ be an $n \times n$ matrix of circuits $\{\overline{F}_{ij}\}$, $i, j \in [n]$ with coefficients from $\mathbb{F}_1$, defined as follows: for every gate u in F, introduce n^2 gates $\bar{u} = \{\bar{u}_{ij}\}_{i,j\in[n]}$, and let:

(i). If $u \in \mathbb{F}_2$ is a constant, let $\bar{u} := \theta(u)$.

(ii). If u is a variable, let $\bar{u} := u \cdot I_n$.

(iii). If $u = v + w$, let $\bar{u} := \bar{v} + \bar{w}$, and if $u = v \cdot w$, let $\bar{u} := \bar{v} \cdot \bar{w}$

Then $\overline{F}$ is the matrix computed by $\bar{w}$ where w is the output of F.

Here, $\bar{v} + \bar{w}$, $(\bar{v} \cdot \bar{w})$ and $u \cdot I_n$ are understood as the corresponding matrix operations on circuit nodes.

LEMMA 32. *Let F, G be circuits of size $\leq s$ with coefficients from $\mathbb{F}_2$. Then*

$$\overline{F \oplus G} = \overline{F} + \overline{G}, \quad \overline{F \otimes G} = \overline{F} \cdot \overline{G}, \tag{35}$$

$$\overline{F} \cdot \overline{G} = \overline{G} \cdot \overline{F} \tag{36}$$

have $\mathbb{P}_c(\mathbb{F}_1)$ proofs of size $s \cdot poly(n)$

PROOF. Identities (35) follow from the definition of $\overline{F}$ by means of axioms C1, C2.

Identity (36) follows by induction on the circuit sizes of F and G. We first need to construct the proof of

$$\overline{z_1} \cdot \overline{z_2} = \overline{z_2} \cdot \overline{z_1},$$

where each z_1, z_2 is either a variable or an element of $\mathbb{F}_2$. So assume that z_1 is a variable. Then $\overline{z_1} = z_1 \cdot I_n$. This gives $\overline{z_1} \cdot \overline{z_2} = z_1 \cdot \overline{z_2}$. But $\overline{z_2}$ is a matrix for which each entry commutes with z_1, which gives a proof of $z_1 \cdot \overline{z_2} = \overline{z_2} \cdot z_1 = \overline{z_2} \cdot \overline{z_1}$. The case of z_2 being a variable is similar. If both $z_1, z_2 \in \mathbb{F}_2$, we are supposed to prove $\theta(z_1) \cdot \theta(z_2) = \theta(z_2) \cdot \theta(z_1)$. But this is a set of n^2 true equations of size $O(n)$ which contain only elements of $\mathbb{F}_1$, and hence it has a proof of size $O(n^3)$. In the inductive step, use (35) and Lemma 31 to construct proofs of $(\overline{F_1} + \overline{F_2}) \cdot \overline{G} = \overline{G}(\overline{F_1} + \overline{F_2})$ and of $(\overline{F_1} \cdot \overline{F_2}) \cdot \overline{G} = \overline{G}(\overline{F_1} \cdot \overline{F_2})$ from the proofs of $\overline{F_1} \cdot \overline{G} = \overline{G} \cdot \overline{F_1}$ and $\overline{F_2} \cdot \overline{G} = \overline{G} \cdot \overline{F_2}$. QED

PROOF PROOF OF THEOREM 10. Let F, G be circuits with coefficients from $\mathbb{F}_2$ such that $F = G$ has a $\mathbb{P}_c(\mathbb{F}_2)$ proof of size s. We wish to show that $\overline{F} = \overline{G}$ have proofs of size $s \cdot \text{poly}(n)$ in $\mathbb{P}_c(\mathbb{F}_1)$. This implies Theorem 10, for if F, G contain only coefficients from $\mathbb{F}_1$ then $\overline{F}_{11} = F$ and $\overline{G}_{11} = G$.

The proof is constructed by induction on the number of lines. Axioms C1, C2 follow from equations (35) in Lemma 32, and A4 from equation (36). A9 is a set of n^2 true constant equations. The rest of the axioms are application of Lemma 31. The rules R1, R2 are immediate, and R3, R4 are given by Lemma 32. QED

PROOF OF THEOREM 9. Theorem 9 follows from Theorem 10 and Proposition 28. QED

For a circuit with division F, define its syntactic degree by

$$\deg F := \deg(\text{Num}F) + \deg(\text{Den}F).$$

COROLLARY 33. *Let $\mathbb{F}$ be any field and let F, G, H be circuits with divisions. Assume that $\deg(F)$ and $\deg(G)$ is at most d and that H has size s_1. Suppose that $F = G$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size s_2 and that $F(z/H)$, $G(z/H)$ are defined. Then $F(z/H) = G(z/H)$ has a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of size $s_1 s_2 \cdot poly(d)$.*

PROOF. We aim to construct a proof of $F = G$ of size $s_2 \cdot \text{poly}(d)$ such that the proof is defined for $z = H$. We can then substitute H for z throughout the proof to obtain a proof of $F(z/H) = G(z/H)$ of the required size. By Lemma 26, we have proofs of

$$F = \text{Num}(F) \cdot \text{Den}(F)^{-1} \qquad G = \text{Num}(G) \cdot \text{Den}(G)^{-1}. \tag{37}$$

This and $F = G$ gives a $\mathbb{P}_c^{-1}(\mathbb{F})$ proof of

$$\text{Num}(F) \cdot \text{Den}(G) = \text{Num}(G) \cdot Den(F),$$

of size $O(s_2)$. The last equation does not contain division gates, and so it has a $\mathbb{P}_c(\mathbb{F})$ proof of size $s_2 \cdot \text{poly}(d)$ by Theorem 9. This proof is defined for $z = H$ because it does not contain division gates. By Lemma 26, the proofs of (37) are defined for $z = H$ (because $F(z/H)$ and $G(z/H)$ are defined by assumption). In particular, both $\text{Den}(F)(z/H)$ and $\text{Den}(G)(z/H)$ are nonzero, and we have a proof of

$$\text{Num}(F) \cdot \text{Den}(F)^{-1} = \text{Num}(G) \cdot \text{Den}(G)^{-1}$$

which is defined for $z = H$. Using (37) we obtain a proof of $F = G$ of size $s_2 \cdot \text{poly}(d)$ which is defined for $z = H$. QED

7. COMPUTING THE DETERMINANT

We are now done proving the structural properties of $\mathbb{P}_c$ and $\mathbb{P}_f$ and we proceed to construct proofs of the properties of the determinant. We first compute the determinant as a rational function.

7.1 The determinant as a rational function

The definition of X^{-1} and $\mathrm{DET}(X)$

Let $X = \{x_{ij}\}_{i,j\in[n]}$ be a matrix consisting of n^2 distinct variables. Recursively, we define $n \times n$ matrix X^{-1} whose entries are circuits with divisions.

(i). If $n = 1$, let $X^{-1} := (x_{11}^{-1})$.

(ii). If $n > 1$, partition X as

$$X = \begin{pmatrix} X_1 & v_1^t \\ v_2 & x_{nn} \end{pmatrix}, \tag{38}$$

where $X_1 = \{x_{ij}\}_{i,j\in[n-1]}$, $v_1 = (x_{1n}, \ldots, x_{(n-1)(n-1)})$ and $v_2 = (x_{n1}, \ldots, x_{n(n-1)})$. Assuming we have constructed X_1^{-1}, let

$$\delta(X) := x_{nn} - v_2 X_1^{-1} v_1^t . \tag{39}$$

$\delta(X)$ computes a single non-zero rational function and so $\delta(X)^{-1}$ is defined. Finally, let

$$X^{-1} := \tag{40}$$
$$\begin{pmatrix} X_1^{-1}(I_{n-1}+\delta(X)^{-1}v_1^t v_2 X_1^{-1}) & -\delta(X)^{-1}X_1^{-1}v_1^t \\ -\delta(X)^{-1}v_2 X_1^{-1} & \delta(X)^{-1} \end{pmatrix}.$$

$\mathrm{DET}(X)$ is defined is follows.

(i). If $n = 1$, let $\mathrm{DET}(X) := x_{11}$.

(ii). If $n > 1$, partition X as in (38) and let $\delta(X)$ be as in (39). Let

$$\mathrm{DET}(X) := \mathrm{DET}(X_1)\cdot\delta(X) = \mathrm{DET}(X_1)(x_{nn} - v_2 X_1^{-1} v_1^t).$$

The definition in (40) should be understood as a circuit with n^2 outputs which takes $X_1^{-1}, v_1, v_2, x_{nn}$ as inputs and moreover, such that the inputs from X_1^{-1} occur exactly once (so we slightly deviate from earlier notation). Altogether, we obtain polynomial size circuits for X^{-1} and $\mathrm{DET}(X)$. The fact that $\mathrm{DET}(X)$ indeed computes the determinant (as a rational function) is a consequence Proposition 38 below, where we show that $\mathbb{P}_c^{-1}$ can prove the two identities which characterize the determinant. That X^{-1} computes the matrix inverse is proved in Proposition 34.

It should be emphasized that both X^{-1} and $\mathrm{DET}(X)$ are circuits with divisions and hence not always defined when substituting for X. Let $A := \{a_{ij}\}_{i,j\in[n]}$ be an $n \times n$ matrix whose entries are circuits with division. We will say that A is *invertible* if the circuit A^{-1} is defined—that is, when we substitute the entries of A into X^{-1}, the circuit does not use divisions by zero. Note that A^{-1} may be undefined even if A has inverse "in the real world". For example, if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then both A^{-1} and $\mathrm{DET}(A)$ are undefined, and so A is not invertible in our sense. Moreover, note that $\mathrm{DET}(X)$ has an *exponential* syntactic degree which, in view of Corollary 33, further obscures substitution in $\mathbb{P}_c^{-1}$ proofs.

On the other hand, let us state the basic cases when the determinant and matrix inverse are defined.

(i). If A is invertible (meaning the circuit A^{-1} is defined) then $\mathrm{DET}(A)$ is defined.

(ii). If the entries of A compute algebraically independent rational functions then A^{-1} is defined.

(iii). If A is a triangular matrix with $a_{11}, \ldots, a_{nn}$ on the diagonal such that $a_{11}^{-1}, \ldots, a_{nn}^{-1}$ are defined then A is invertible.

(iv). Setting $A[k] := \{a_{ij}\}_{i,j\in[k]}$, the matrix A is invertible if and only if $A[1], \ldots, A[n-1]$ are invertible and $\delta(A)^{-1}$ is defined.

Properties of matrix inverse

PROPOSITION 34. *Let $X = \{x_{ij}\}_{i,j\in[n]}$ be a matrix with n^2 distinct variables. Then both*

$$X \cdot X^{-1} = I_n \quad \textit{and} \quad X^{-1} \cdot X = I_n$$

have a polynomial-size $\mathbb{P}_c^{-1}$ proof. The proof is defined for $X = A$, whenever A is invertible.

PROOF. Let us construct the proofs of $X \cdot X^{-1} = I_n$ and $X^{-1} \cdot X = I_n$ by induction on n. If $n = 1$, we have $x_{11} \cdot x_{11}^{-1} = x_{11}^{-1} \cdot x_{11} = 1$ which is a $\mathbb{P}_c^{-1}$ axiom. Otherwise let $n > 1$ and X be as in (38). We want to construct a polynomial size proof of $X \cdot X^{-1} = I_n$ from the assumption $X_1 X_1^{-1} = I_{n-1}$. This implies that $X \cdot X^{-1} = I_n$ has a polynomial size proof.

For brevity, let $a := \delta(X)$. Using some rearrangements, and the definition of a, we have:

$$\begin{aligned} &X \cdot X^{-1} = \\ &\begin{pmatrix} X_1 & v_1^t \\ v_2 & x_{nn} \end{pmatrix}\cdot\begin{pmatrix} X_1^{-1}(I_{n-1}+a^{-1}v_1^t v_2 X_1^{-1}) & -a^{-1}X_1^{-1}v_1^t \\ -a^{-1}v_2 X_1^{-1} & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} + a^{-1}v_1^t v_2 X_1^{-1} - a^{-1}v_1^t v_2 X_1^{-1} & -a^{-1}v_1^t + a^{-1}v_1^t \\ v_2 X_1^{-1} + a^{-1}(v_2 X_1^{-1} v_1^t - x_{nn})v_2 X_1^{-1} & a^{-1}(-v_2 X_1^{-1} v_1^t + x_{nn}) \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} & 0 \\ v_2 X_1^{-1} - a^{-1}a v_2 X_1^{-1} & a^{-1}a \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Here we use the fact that basic properties of matrix addition and multiplication have feasible proofs (see Lemma 31).

The proof of $X^{-1} \cdot X = I_n$ is constructed in a similar fashion (where we use the assumption $X_1^{-1} X_1 = I_{n-1}$ instead). Moreover, if A is an $n \times n$ matrix such that A^{-1} is defined, the proofs of $A \cdot A^{-1} = A^{-1} \cdot A = I_n$ are defined. (This is because they employ only the inverse gates appearing already in the definition of X^{-1}.) QED

COROLLARY 35. *The identity $(XY)^{-1} = Y^{-1}X^{-1}$ has a polynomial-size proof in $\mathbb{P}_c^{-1}$. The proof is defined for $X = A, Y = B$ whenever A, B and AB are invertible.*

Beware that invertibility of A and B does not guarantee invertibility of AB.

PROOF. Let $Z := (XY)^{-1}$. Then $(Z(XY))Y^{-1}X^{-1} = Y^{-1}X^{-1}$. On the other hand, $(Z(XY))Y^{-1}X^{-1} = Z(X(YY^{-1})X^{-1} = Z$ and so $Z = Y^{-1}X^{-1}$. QED

An application of the Corollary is the following technical observation. Let X be as in (38) and similarly $Y = \begin{pmatrix} Y_1 & u_1^t \\ u_2 & y_{nn} \end{pmatrix}$. Comparing the entries in the bottom right corners of $(XY)^{-1}$ and $Y^{-1}X^{-1}$, we obtain that

$$\delta(Y)\delta(X) = \delta(XY)(1 + u_2 Y_1^{-1} X_1^{-1} v_1^t), \tag{41}$$

has a polynomial size $\mathbb{P}_c^{-1}$ proof (the proof is defined for $X, Y = A, B$ whenever A, B and AB are invertible).

It is often easier to argue about triangular matrices. We summarize their useful properties in the following:

PROPOSITION 36. *(i). Let A, L, U be $n \times n$ matrices with L lower triangular and U upper triangular. If A, L, U are invertible then so are LA and AU.*

(ii). Let X be a $n \times n$ matrix of distinct variables. Then there exists a lower triangular matrix $L(X)$ and an upper triangular matrix $U(X)$ such that $X = L(X) \cdot U(X)$ has a polynomial size $\mathbb{P}_c^{-1}$ proof. The proof is defined for $X = A$ whenever A is invertible, and then also $L(A), U(A)$ are invertible.

PROOF. Part (i) follows from the fact that $(LA)[k] = L[k]A[k]$ and $\delta((LA)[k]) = \delta(L[k])\delta(A[k])$ for every $k \in \{1, \ldots, n\}$ (where $A[k]$ is defined as $\{a_{ij}\}_{i,j\in[k]}$.)

In part (ii), the matrices $L(X), U(X)$, as well as the $\mathbb{P}_c^{-1}$ proof, are constructed by induction on n. If $n = 1$, let $L(x_{11}) = x_{11}$ and $U(x_{11}) = 1$. If $n > 1$, write X as in (38). Assuming we have $X_1 = L(X_1)U(X_1)$, we have

$$\begin{pmatrix} X_1 & v_1^t \\ v_2 & x_{nn} \end{pmatrix} = \begin{pmatrix} L(X_1) & 0 \\ v_2 U(X_1)^{-1} & x_{nn} - v_2 X_1^{-1} v_1^t \end{pmatrix} \cdot \begin{pmatrix} U(X_1) & L(X_1)^{-1} v_1^t \\ 0 & 1 \end{pmatrix}.$$

Verifying that the proof is defined for an invertible A, and that $L(A), U(A)$ are invertible, is straightforward. QED

Properties of DET

We now want to prove Proposition 38 which is a $\mathbb{P}_c^{-1}$ analogy of Theorem 4. We first prove the following lemma.

LEMMA 37. *Let A be an invertible $n \times n$ matrix and let v_1, v_2 be $n \times 1$ vectors such that $A + v_1^t v_2$ is invertible. Then*

$$DET(A + v_1^t v_2) = DET(A)(1 + v_2 A^{-1} v_1^t) \qquad (42)$$

has a polynomial size $\mathbb{P}_c^{-1}$ proof.

PROOF. The proof is constructed by induction on n. If $n = 1$, the identity is immediate. If $n > 1$, partition A and $A + v_1^t v_2$ as in (38), i.e.,

$$A = \begin{pmatrix} A_1 & w_1^t \\ w_2 & a_{nn} \end{pmatrix} \text{ and } A + v_1^t v_2 = \begin{pmatrix} A_1 + u_1^t u_2 & w_1^t + c_2 u_1^t \\ w_2 + c_1 u_2 & a_{nn} + c_1 c_2 \end{pmatrix},$$

where $v_1 = (u_1, c_1)$ and $v_2 = (u_2, c_2)$. We want to construct a polynomial size proof of (42) from the assumption $\mathrm{DET}(A_1 + u_1^t u_2) = \mathrm{DET}(A_1)(1 + u_2 A_1^{-1} u_1^t)$. This implies that (42) has a polynomial size proof.

By the definition of DET, we have

$$\mathrm{DET}(A) = \mathrm{DET}(A_1)\delta(A), \text{ and}$$
$$\mathrm{DET}(A + v_1^t v_2) = \mathrm{DET}(A_1 + u_1^t u_2)\delta(A + v_1^t v_2).$$

By the assumption, $\mathrm{DET}(A_1 + u_1^t u_2) = \mathrm{DET}(A_1)(1 + u_2 A_1^{-1} u_1^t)$ and so (42) is equivalent to

$$\mathrm{DET}(A_1)(1 + u_2 A_1^{-1} u_1^t)\delta(A + v_1^t v_2) = \mathrm{DET}(A_1)\delta(A)(1 + v_2 A^{-1} v_1^t).$$

Hence in order to prove (42), it is sufficient to prove

$$(1 + u_2 A_1^{-1} u_1^t)\delta(A + v_1^t v_2) = \delta(A)(1 + v_2 A^{-1} v_1^t). \qquad (43)$$

In order to prove (43), we first prove its special case

$$(1 + \bar{u}_2 \bar{u}_1^t)\delta(I_n + \bar{v}_1^t \bar{v}_2) = (1 + \bar{v}_2 \bar{v}_1^t). \qquad (44)$$

where $\bar{v}_1 = (\bar{u}_1, \bar{c}_1)$ and $\bar{v}_2 = (\bar{u}_2, \bar{c}_2)$ are vectors such that $I_n + \bar{v}_1^t \bar{v}_2$ is invertible. Let $\alpha := \bar{u}_2 \bar{u}_1^t$. By the definition of δ

$$\delta(I_n + \bar{v}_1^t \bar{v}_2) = (1 + \bar{c}_1\bar{c}_2 - \bar{c}_1\bar{c}_2\bar{u}_2(I_{n-1} + \bar{u}_1^t \bar{u}_2)^{-1}\bar{u}_1^t)$$

and it is easy to verify that

$$(I_{n-1} + \bar{u}_1^t \bar{u}_2)^{-1} = I_{n-1} - (1 + \alpha)^{-1}\bar{u}_1^t \bar{u}_2.$$

Hence we obtain

$$\begin{aligned}
&(1 + \bar{u}_2 \bar{u}_1^t)\delta(I_n + \bar{v}_1^t \bar{v}_2) \\
&= (1 + \alpha)(1 + \bar{c}_1\bar{c}_2 - \bar{c}_1\bar{c}_2\bar{u}_2(I_{n-1} - (1 + \alpha)^{-1}\bar{u}_1^t \bar{u}_2)\bar{u}_1^t \\
&= (1 + \alpha)(1 + \bar{c}_1\bar{c}_2 - \bar{c}_1\bar{c}_2\bar{u}_2\bar{u}_1^t - \bar{c}_1\bar{c}_2(1 + \alpha)^{-1}(\bar{u}_2\bar{u}_1^t)^2) \\
&= (1 + \alpha)(1 + \bar{c}_1\bar{c}_2 - \bar{c}_1\bar{c}_2\alpha - \bar{c}_1 c_2(1 + \alpha)^{-1}\alpha^2) \\
&= 1 + \bar{c}_1\bar{c}_2 + \alpha = 1 + \bar{v}_2\bar{v}_1^t
\end{aligned}$$

which proves (44).

In order to conclude (43), let $L := L(A)$ and $U := U(A)$ be the matrices from Proposition 36. That is, L and U is an invertible lower resp. upper triangular matrix so that $A = LU$, and hence also $A^{-1} = U^{-1}L^{-1}$, has a polynomial size proof. Let

$$\bar{v}_1^t := L^{-1} v_1^t \text{ and } \bar{v}_2 := v_2 U^{-1}.$$

The definition guarantees that

$$\bar{u}_2 \bar{u}_1^t = u_2 A_1^{-1} u_1^t, \ \bar{v}_2 \bar{v}_1^t = v_2 A^{-1} v_1^t \qquad (45)$$

have polynomial size proof. Moreover, $A + v_1^t v_2 = L(I_n + \bar{v}_1^t \bar{v}_2)U$, which also shows that $I_n + \bar{v}_1^t \bar{v}_2$ is invertible. (41) implies that $\delta(LB) = \delta(L)\delta(B)$ and $\delta(BU) = \delta(B)\delta(U)$ has a polynomial proof (for any invertible B). Hence

$$\delta(A + v_1^t v_2) = \delta(L)\delta(U)\delta(I_n + \bar{v}_1^t \bar{v}_2) = \delta(A)\delta(I_n + \bar{v}_1^t \bar{v}_2).$$

This and (45) gives (43) from (44). QED

PROPOSITION 38. *(i). Let U be an (upper or lower) triangular matrix with $u_1, \ldots u_n$ on the diagonal. If $u_1^{-1}, \ldots, u_n^{-1}$ are defined then*

$$DET(U) = u_1 \cdots u_n$$

has a polynomial-size $\mathbb{P}_c^{-1}$ proof.

(ii). Let X and Y be $n \times n$ matrices, each consisting of pairwise distinct variables. Then

$$DET(X \cdot Y) = DET(X) \cdot DET(Y) \qquad (46)$$

has a polynomial-size $\mathbb{P}_c^{-1}$ proof. The proof is defined for $X = A, Y = B$ provided[7] $A[k], B[k]$ and $A[k]B[k]$ are invertible for every $k \in \{1, \ldots, n\}$.

PROOF. Part (i) readily follows from the definition of DET.

Part (ii) is proved by induction on n. If $n = 1$, it is immediate. Assume that $n > 1$. Let

$$X = \begin{pmatrix} X_1 & v_1^t \\ v_2 & x_{nn} \end{pmatrix}, \ Y = \begin{pmatrix} Y_1 & u_1^t \\ u_2 & y_{nn} \end{pmatrix}.$$

[7] $A[k]$ is defined as $\{a_{ij}\}_{i,j\in[k]}$.

We want to construct a polynomial size proof of $\mathrm{DET}(XY) = \mathrm{DET}(X)\mathrm{DET}(Y)$ from the assumption $\mathrm{DET}(X_1Y_1) = \mathrm{DET}(X_1)\mathrm{DET}(Y_1)$. This implies that $\mathrm{DET}(XY) = \mathrm{DET}(X)\mathrm{DET}(Y)$ has a polynomial size proof.

By the definition of DET, we have

$$\mathrm{DET}(X) = \mathrm{DET}(X_1)\delta(X)\,,\ \mathrm{DET}(Y) = \mathrm{DET}(Y_1)\delta(Y)$$

and $\mathrm{DET}(XY) = \mathrm{DET}(X_1Y_1 + v_1^t u_2)\delta(XY)$, and we are supposed to prove:

$$\mathrm{DET}(X_1Y_1+v_1^t u_2)\delta(XY) = \mathrm{DET}(X_1)\delta(X)\cdot\mathrm{DET}(Y_1)\delta(Y)\,. \tag{47}$$

By the previous lemma, we have $\mathrm{DET}(X_1Y_1 + v_1^t u_2) = \mathrm{DET}(X_1Y_1)(1 + u_2(X_1Y_1)^{-1}v_1^t)$. By the assumption $\mathrm{DET}(X_1Y_1) = \mathrm{DET}(X_1)\mathrm{DET}(Y_1)$, this yields

$$\mathrm{DET}(X_1Y_1+v_1^t u_2) = \mathrm{DET}(X_1)\mathrm{DET}(Y_1)(1+u_2Y_1^{-1}X_1^{-1}v_1^t)\,.$$

Hence in order to prove (47), it is sufficient to prove

$$(1 + u_2Y_1^{-1}X_1^{-1}v_1^t)\delta(XY) = \delta(X)\delta(Y)\,.$$

But this follows from (41).

On the inductive step, we have assumed invertibility of X, Y, XY, X_1, Y_1 and X_1Y_1, as well as invertibility of $X_1Y_1 + v_1^t u_2$. The latter follows from invertibility of XY since $(X_1Y_1 + v_1^t u_2)^{-1}$ is used to define XY^{-1}. Since $X_1 = X[n-1]$, $Y_1 = Y[n-1]$, the proof altogether assumes invertibility of $X[k], Y[k]$ and $X[k]Y[k]$ for every $k \in \{1,\ldots,n\}$. QED

Let us explicitly state the important cases when the proof of $\mathrm{DET}(AB) = \mathrm{DET}(A)\mathrm{DET}(B)$ is defined. This is so, if A and B are invertible and if

(i). the entries of A, B compute algebraically independent rational functions, or

(ii). A is lower triangular or B is upper triangular, or

(iii). the entries of A are field elements and the entries of B are algebraically independent, or vice versa.

The following lemma shows that elementary Gaussian operations are well-behaved with respect to DET.

LEMMA 39. *Let $X = \{x_{ij}\}_{i,j\in[n]}$ be an $n \times n$ matrix of distinct variables. Then the following have polynomial-size $\mathbb{P}_c^{-1}$ proofs:*

(i). $DET(X) = -DET(X')$, where X' is a matrix obtained from X by interchanging two rows or columns.

(ii). $DET(X'') = uDET(X)$, where X'' is obtained by multiplying a row in X by u, such that u^{-1} is defined (and similarly for a column).

(iii). $DET(X) = DET(X''')$, where X''' is obtained by adding a row to a different row in X (and similarly for columns).

(iv). $DET(X) = x_{nn}DET(X_1 - x_{nn}^{-1}v_1^t v_2)$, where X_1, v_1 and v_2 are from the decomposition (38).

PROOF. Items (ii) and (iii) follow from Proposition 38 and the fact that $X'' = AX$ and $X''' = A'X$, where A, A' are suitable triangular matrices. We cannot infer (i) directly from Proposition 38, since $X' = TX$ implies only that T is a transposition matrix and hence not invertible in our sense. However, we can write $T = A_1A_2$, where A_1, A_2 are invertible with $\mathrm{DET}(A_1)\mathrm{DET}(A_2) = -1$: note that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Since X is a matrix of distinct variables, the following is defined:

$$\begin{aligned}\mathrm{DET}(A_1A_2X) &= \mathrm{DET}(A_1)\mathrm{DET}(A_2X)\\ &= \mathrm{DET}(A_1)\mathrm{DET}(A_2)\mathrm{DET}(X)\,.\end{aligned}$$

Part (iv) follows from Lemma 37. QED

7.2 The determinant as a polynomial

Note that we cannot yet apply Theorem 9 to obtain Theorem 4, because DET itself contains division gates. For our purpose it will suffice to compute the determinant by a circuit without division, denoted $\det(X)$, and construct a proof of $\det(X) = \mathrm{DET}(X)$ in $\mathbb{P}_c^{-1}$. In order to do that, we will define $\det(X)$ as the nth term of the Taylor expansion of $\mathrm{DET}(I + zX)$ at $z = 0$, as follows: using notation from Section 5.2, let

$$\det(X) := \Delta_{z^n}\left(\mathrm{DET}(I + zX)\right). \tag{48}$$

Let us note that

(i). $\det(X)$ indeed computes the determinant of X,

(ii). $\det(X)$ is a circuit without divisions of syntactic degree n.

This is because every variable from X in the circuit $\mathrm{DET}(I+zX)$ occurs in a product with z. Hence $\Delta_{z^n}(\mathrm{DET}(I + zX))$ is the n-th homogeneous part of the determinant of $I + X$ - the determinant of X. By the definition of Δ_{z^n}, $\Delta_{z^n}(\mathrm{DET}(I + zX))$ contains exactly one inverse gate, namely the inverse of $\mathrm{Den}(\mathrm{DET}(I+zX))$ at the point $z = 0$. But $a := (\mathrm{Den}(\mathrm{DET}(I + zX)))(z/0)^\sharp$ is a constant circuit computing a non-zero field element, and we can identify a^{-1} with the field constant it computes.

LEMMA 40. *Let X be an $n \times n$ matrix of distinct variables. There exist circuits with divisions $P_0, \ldots, P_{n-1}$ not containing the variable z, such that*

$$DET(zI_n + X) = z^n + P_{n-1}z^{n-1} + \cdots + P_0$$

has a polynomial-size $\mathbb{P}_c^{-1}(\mathbb{F})$ proof. Moreover, this proof is defined for $z = 0$.

PROOF. Let F be a circuit in which z does not occur in the scope of any inverse gate. Then, we define the *z-degree of F* as the syntactic-degree of F considered as a circuit computing a univariate polynomial in z (so that all other variables are treated as constants).

By induction, we will construct matrices $A_1, \ldots, A_n$ with the following properties:

1. $A_1 = X + zI_n$,

2. Every A_k is an $(n-k+1) \times (n-k+1)$ matrix of the form

$$\begin{pmatrix} z^k + f & w \\ v^t & zI_{n-k} + Q \end{pmatrix}$$

where all the entries are circuits with division in which z does not occur in the scope of any division gate, v, w are $1 \times (n-k)$ vectors and moreover: f as well as every entry of w have z-degree less than k and v, Q do not contain the variable z.

3. The identity $\mathrm{DET}(A_k) = \mathrm{DET}(A_{k+1})$ has a polynomial-size proof.

4. The entries of A_k are algebraically independent (this is to guarantee that divisions are defined).

Assume that A_k is given, and let us partition it as

$$A_k = \begin{pmatrix} z^k + f_1 & w & f_2 \\ u_1^t & zI_m + Q & u_2^t \\ a_1 & v & z + a_2 \end{pmatrix}$$

where $m = (n - k - 1)$ and we allow the alternative that $m = 0$. Only f_1, w, f_2 can depend on z and have z-degree $< k$. By Lemma 39 part (iv), we can switch the first and last column to obtain a $\mathbb{P}_c^{-1}$ proof of

$$\mathrm{DET}(A_k) = -\mathrm{DET}\begin{pmatrix} f_2 & w & z^k + f_1 \\ u_2^t & zI_m + Q & u_1^t \\ z + a_2 & v & a_1 \end{pmatrix}.$$

By Lemma 39 part (iv), we have

$$\begin{aligned} &\mathrm{DET}(A_k) = \\ &-a_1\mathrm{DET}\begin{pmatrix} f_2 - a_1^{-1}(z^k+f_1)(z+a_2) & w - a_1^{-1}(z^k+f_1)v \\ u_2^t - a_1^{-1}u_1^t(z+a_2) & zI_m + Q - a_1^{-1}u_1^t v \end{pmatrix} \\ &= \mathrm{DET}\begin{pmatrix} (z^k+f_1)(z+a_2) - a_1 f_2 & a_1 w - (z^k+f_1)v \\ u_2^t - a_1^{-1}u_1^t(z+a_2) & zI_m + Q - a_1^{-1}u_1^t v \end{pmatrix}. \end{aligned}$$

We can write $(z^k + f)(z + a_2) = z^{k+1} + (fz + a_2 z^k + f a_2)$, where the z-degree of $(fz + a_2 z^k + f a_2)$ as well as of every entry of $a_1 w - (z^k + f_1)v$ is at most k. Hence the matrix is of the right form, apart from the occurrence of $z u_1^t$ in the first column. This can be remedied by multiplying by $\begin{pmatrix} 1 & 0 \\ a_1^{-1}u_1^t & I_m \end{pmatrix}$ from the right to obtain A_{k+1} of the required form.

This indicates that, given a circuit computing A_k, we can compute A_{k+1} using polynomially many additional gates. Altogether, every A_k has a polynomial size circuit. The proof of $\mathrm{DET}(A_k) = \mathrm{DET}(A_{k+1})$ has a polynomial number of lines and, as it involves polynomial size circuits, also polynomial size.

Finally, we obtain a polynomial size proof of $\mathrm{DET}(A_n) = \mathrm{DET}(A_1) = z^n + f$, where f is a circuit with z-degree $< n$ in which z is not in the scope of any division gate. Writing f as $\sum_{i=0}^{n-1} P_i z^i$ concludes the lemma. QED

PROPOSITION 41. *(i). If U is a triangular matrix with $u_1, \ldots, u_n$ on the diagonal then $\det(U) = u_1 \cdots u_n$ has a polynomial size $\mathbb{P}_c^{-1}$ proof.*

(ii). Let X be an $n \times n$ matrix of distinct variables. Then $DET(X) = \det(X)$ has a polynomial-size $\mathbb{P}_c^{-1}$ proof.

PROOF. Part (i) follows from Proposition 38. For we have $\mathrm{DET}(I_n + zU) = (1 + zu_1) \cdots (1 + zu_n)$, and the proof is defined for $z = 0$. By Proposition 30, $\det(U) = \Delta_{z^n}((1 + zu_1) \cdots (1 + zu_n)) = u_1 \cdots u_n$ has a polynomial size $\mathbb{P}_c^{-1}$ proof.

(ii) follows from the previous lemma. We obtain polynomial-size $\mathbb{P}_c^{-1}$ proof of the following substitution instance:

$$\mathrm{DET}(zI_n + X^{-1}) = z^n + Q_{n-1}z^{n-1} + \cdots + Q_0, \qquad (49)$$

where the Q_i's are circuits with divisions which do not contain the variable z and the proof is defined for $z = 0$.

By Proposition 38 we have a polynomial-size $\mathbb{P}_c^{-1}$ proof of

$$\mathrm{DET}(I_n + zX) = \mathrm{DET}(zI_n + X^{-1}) \cdot \mathrm{DET}(X).$$

The proof is defined for $z = 0$ (as is witnessed by letting $X := I_n$). From equation (49) we get a polynomial-size proof of

$$\mathrm{DET}(I_n + zX) = z^n \mathrm{DET}(X) + z^{n-1}Q'_{n-1} + \cdots + Q'_0,$$

where $Q'_{n-1}, \ldots, Q'_0$ do not contain z. The proof is defined for $z = 0$ and so Proposition 30gives a polynomial-size $\mathbb{P}_c^{-1}$ proof of

$$\begin{aligned} &\Delta_{z^n}(\mathrm{DET}(I_n + zX)) = \\ &\qquad \Delta_{z^n}(z^n \mathrm{DET}(X) + z^{n-1}Q'_{n-1} + \cdots + Q'_0). \end{aligned}$$

But by the definition of $\det(X)$, $\Delta_{z^n}(\mathrm{DET}(I + zX))$ is $\det(X)$ and by the definition of Δ_{z^n}, $\Delta_{z^n}(z^n\mathrm{DET}(X) + z^{n-1}Q'_{n-1} + \cdots + Q'_0)$ is $\mathrm{DET}(X)$, and we are done. QED

8. CONCLUDING THE MAIN THEOREM

We can now finally prove Theorem 4 (Main Theorem), which we rephrase as follows:

PROPOSITION 42 (THEOREM 4, REPHRASED). *Let X, Y, Z be $n \times n$ matrices such that X, Y consist of different variables and Z is a triangular matrix with $z_{11}, \ldots, z_{nn}$ on the diagonal. Then there exist an arithmetic circuit $\det_c$ and a formula $\det_f$ such that:*

(i). The identity $\det_c(XY) = \det_c(X) \cdot \det_c(Y)$ and $\det_c(Z) = z_{11} \cdots z_{nn}$ have polynomial-size $O(\log^2 n)$ depth proofs in $\mathbb{P}_c$.

(ii). The identity $\det_f(XY) = \det_f(X) \cdot \det_f(Y)$ and $\det_f(Z) = z_{11} \cdots z_{nn}$ have $\mathbb{P}_f$ proofs of size $n^{O(\log n)}$.

PROOF. Let $\det(X) = \Delta_{z^n}\mathrm{DET}(I + zX)$ be the circuit defined in (48). Lemma 41 part (ii) and Proposition 38 imply that the equations

$$\det(XY) = \det(X) \cdot \det(Y) \qquad \text{and} \qquad \det(Z) = z_{11} \cdots z_{nn} \qquad (50)$$

have polynomial-size $\mathbb{P}_c^{-1}$ proofs. By definition, the syntactic degree of $\det(X)$ is at most n. Hence, by Theorem 9 the identities in (50) have polynomial-size $\mathbb{P}_c$ proofs. This almost concludes part (i), except for the bound on the depth. To bound the depth, let

$$\det_c(X) := [\det(X)],$$

where $[F]$ is the balancing operator as defined in Section 4. Thus, Theorem 5 implies that $[\det(XY)] = [\det(X) \cdot \det(Y)]$ and $[\det(Z)] = [z_{11} \cdots z_{nn}]$ have $\mathbb{P}_c$ proofs of polynomial-size and depth $O(\log^2 n)$. By means of Lemma 20, we have such proofs also for

$$[\det(X) \cdot \det(Y)] = [\det(X)] \cdot [\det(Y)] = \det_c(X) \cdot \det_c(Y)$$

and $[\det(Z)] = z_{11} \cdots z_{nn}$. Hence it is sufficient to construct (polynomial-size and $O(\log^2 n)$ depth proofs) of

$$[\det(XY)] = \det_c(XY) \qquad \text{and} \qquad [\det(Z)] = \det_c(Z)$$

(note that defining $\det_c(X)$ as $[\det(X)]$ does not imply that $[\det(XY)] = \det_c(XY)$). This follows from the following more general claim:

CLAIM 43. *Let $F(x_1/g_1, \ldots, x_n/g_n)$ be a circuit of size s and syntactic degree d. Then*

$$[F(x_1/g_1,\ldots,x_n/g_n)] = [F(x_1,\ldots,x_n)]\,(x_1/[g_1],\ldots,x_n/[g_n])$$

has a $\mathbb{P}_c$ proof of size poly(n,d) and depth $O(\log d \log s + \log^2 d)$.

PROOF. This follows by induction using Lemma 20. We omit the details. QED

To prove part (ii), recall the definition of $F^\bullet$ from Remark 3. Let $\det_f(X) := (\det_c(X))^\bullet$. Then the statement follows from part (i) and Claim 24 of the proof of Theorem 23. QED

We should note that in the $\mathbb{P}_c$-proof of the equation $\det(XY) = \det(X)\cdot\det(Y)$ no divisions occur and so it is defined for any substitution. In particular,

$$\det(AX) = \det(A)\cdot\det(X) = a\det(X)$$

has a short $\mathbb{P}_c$ proof for any matrix A of field elements whose determinant is $a \in \mathbb{F}$. Similarly, the elementary Gaussian operations stated in Lemma 39 carry over to polynomial-size $\mathbb{P}_c$ proofs of the corresponding properties of det.

9. APPLICATIONS

In this section, we prove Propositions 11 and 12. First, one should show that the cofactor expansion of the determinant has short proofs. For an $n \times n$ matrix X and $i,j \in [n]$, let $X_{i,j}$ denote the $(n-1)\times(n-1)$-matrix obtained by removing the ith row and jth column from X. Let $\mathrm{Adj}(X)$ be the $n\times n$ matrix whose (i,j)-th entry is $(-1)^{i+j}\det_c(X_{j,i})$ (where $\det_c$ is the circuit from Proposition 42).

PROPOSITION 44 (COFACTOR EXPANSION). *Let $X = \{x_{ij}\}_{i,j\in[n]}$ be an $n\times n$ matrix. Then the following identities have polynomial-size $O(\log^2 n)$-depth $\mathbb{P}_c$ proofs:*

(i) $\det_c(X) = \sum_{j=1}^{n}(-1)^{i+j}x_{ij}\det_c(X_{i,j})$, *for any $i \in [n]$;*

(ii) $X\cdot Adj(X) = Adj(X)\cdot X = \det_c(X)\cdot I$.

The proof of this proposition appears in the full version of this article.

PROPOSITION 45 (PROPOSITION 12 RESTATED). *The identities $YX = I_n$ have polynomial-size and $O(\log^2 n)$-depth $\mathbb{P}_c$ proofs from the equations $XY = I_n$. In the case of $\mathbb{P}_f$, the proofs have quasipolynomial-size.*

PROOF. Note that we are dealing with a $\mathbb{P}_c$ proof from assumptions, and hence we are not allowed to use division gates. The proof is constructed as follows. Assume $XY = I_n$. By Proposition 42, this gives $\det_c(X)\det_c(Y) = 1$. By Proposition 44, we can multiply from left both sides of $XY = I_n$ by $\mathrm{Adj}(X)$, to obtain $\det_c(X)Y = \mathrm{Adj}(X)$. Hence,

$$\det_c(X)YX = \mathrm{Adj}(X)X = \det_c(X)I_n,$$

and so

$$\det_c(Y)\det_c(X)YX = \det_c(Y)\det_c(X)I_n,$$

which, using $\det_c(X)\det_c(Y) = 1$ gives $YX = I_n$. The $\mathbb{P}_f$ proof is identical, except that the steps involving the determinant require a quasipolynomial size. QED

PROOF OF PROPOSITION 11. The proof proceeds via a simulation of the construction in [17]. Due to lack of space, we refer the reader to the full version of this article for more details. QED

10. REFERENCES

[1] Paul Beame and Toniann Pitassi. Propositional proof complexity: past, present, and future. *Bull. Eur. Assoc. Theor. Comput. Sci. EATCS*, (65):66–89, 1998.

[2] Stuart J. Berkowitz. On computing the determinant in small parallel time using a small number of processors. *Inf. Process. Lett.*, 18:147–150, 1984.

[3] Maria Luisa Bonet, Samuel R. Buss, and Toniann Pitassi. Are there hard examples for Frege systems? In *Feasible mathematics, II*, volume 13 of *Progr. Comput. Sci. Appl. Logic*, pages 30–56. Birkhäuser, 1995.

[4] Pavel Hrubeš and Iddo Tzameret. The proof complexity of polynomial identities. In *Proceedings of the 24th IEEE Conference on Computational Complexity*, pages 41–51, 2009.

[5] Laurent Hyafil. On the parallel evaluation of multivariate polynomials. *SIAM J. Comput.*, 8(2):120–123, 1979.

[6] Emil Jeřábek. Dual weak pigeonhole principle, Boolean complexity, and derandomization. *Ann. Pure Appl. Logic*, 129(1-3):1–37, 2004.

[7] Jan Krajíček. *Bounded arithmetic, propositional logic, and complexity theory*, volume 60 of *Encyclopedia of Mathematics and its Applications*. Cambridge, 1995.

[8] Ran Raz and Amir Yehudayoff. Balancing syntactically multilinear arithmetic circuits. *Comput. Complexity*, 17:515–535, 2008.

[9] Jacob T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *Journal of the ACM*, 27(4):701–717, 1980.

[10] Nathan Segerlind. The complexity of propositional proofs. *Bull. Symbolic Logic*, 13(4):417–481, 2007.

[11] Amir Shpilka and Amir Yehudayoff. Arithmetic circuits: A survey of recent results and open questions. *Foundations and Trends in Theoretical Computer Science*, 5(3-4):207–388, 2010.

[12] Michael Soltys. *The complexity of derivations of matrix identities*. PhD thesis, University of Toronto, 2001.

[13] Michael Soltys. Feasible proofs of matrix properties with csanky's algorithm. In *19th International Workshop on Comp. Sci. Log.*, pages 493–508, 2005.

[14] Michael Soltys and Stephen Cook. The proof complexity of linear algebra. *Ann. Pure Appl. Logic*, 130(1-3):277–323, 2004.

[15] Michael Soltys and Alasdair Urquhart. Matrix identities and the pigeonhole principle. *Arch. Math. Logic*, 43(3):351–357, 2004.

[16] Volker Strassen. Vermeidung von divisionen. *J. Reine Angew. Math.*, 264:182–202, 1973. (in German).

[17] Leslie G. Valiant. Completeness classes in algebra. In *Proceedings of the 11th Annual ACM Symposium on the Theory of Computing*, pages 249–261. ACM, 1979.

[18] Leslie G. Valiant, Sven Skyum, S. Berkowitz, and Charles Rackoff. Fast parallel computation of polynomials using few processors. *SIAM J. Comput.*, 12(4):641–644, 1983.

[19] Richard Zippel. Probabilistic algorithms for sparse polynomials. In *Proceedings of the International Symposiumon on Symbolic and Algebraic Computation*, pages 216–226. Springer-Verlag, 1979.