A Simplicial Approach for Discrete Fixed Point Theorems
(Extended Abstract)

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Abstract. We present a new discrete fixed point theorem based on a novel definition of direction-preserving maps over simplicial structures. We show that the result is more general and simpler than the two recent discrete fixed point theorems by deriving both of them from ours. The simplicial approach applied in the development of the new theorem reveals a clear structural comparison with the classical approach for the continuous case.

1 Introduction

There has recently been a sequence of works related to fixed point theorems in a discrete disguise, started with the seminal work of Iimura [13] which introduced a crucial concept of direction-preserving maps. Iimura, Murota and Tamura [14] corrected the proof of Iimura for the definition domains of the maps. With a different technique, Chen and Deng introduced another discrete fixed point theorem in order to achieve the optimal algorithmic bound for finding a discrete fixed point for all finite dimensions [2]. In [15], Laan, Talman and Yang designed an iterative algorithm for the discrete zero point problem. Based on Sperner's lemma which is fundamental for deriving Brouwer's fixed point theorem, Friedl, Ivanyosy, Santha and Verhoeven defined the black-box Sperner problems. They also obtained a $\sqrt{n}$ upper bound for the two-dimensional case [11], which is also a matching bound when combined with the lower bound of Crescenzi and Silvestri [8] (mirroring an early result of Hirsch, Papadimitriou and Vavasis on the computation of 2D approximate fixed points [12]). On the other hand, Chen and Deng [6] showed that the two theorems, that of Iimura, Murota and Tamura [14], as well as that of Chen and Deng [2], cannot directly derive each other.

In this article, we derive a new discrete fixed point theorem based on simplicial structures and a novel definition of direction-preserving maps. We show that both previous discrete fixed point theorems can be derived from this simpler one.

The simplicial structure, together with Sperner's Lemma, has played an important role in establishing various continuous fixed point theorems. Our focus

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on the simplicial structure in the study of the discrete version will help us gain a full and clear understanding of the mathematical structures and properties related to discrete fixed point theorems. Furthermore, even for continuous fixed point theorems, discrete structural propositions are needed to derive them. Our study would provide a unified view of the fixed point theorem, both discrete and continuous, instead of treating them with ad hoc techniques. Our simplicial approach unveils the mystery behind the recent results on discrete fixed points and settles them under the same mathematical foundation as the classical continuous fixed point theorems.

The discrete nature of the fixed point theorem has been noticed previously, mainly due to the proof techniques of Sperner's lemma [16]. The recent effort in direct formulation of the discrete version of the fixed point theorem would be especially useful in the complexity analysis of related problems. The recent work in characterizing the complexity of Nash Equilibria, by Daskalakis, Goldberg, Papadimitriou [9], Chen and Deng [3], Daskalakis and Papadimitriou [10], Chen and Deng [4], has been based on another innovative formulation of the 2D (or 3D) discrete fixed point problem, where a fixed point is a collection of four [7] (or eight [9]) corners of a unit square (or cube). It's difficult to generalize such a formulation to high dimensional spaces, since a hypercube has an exponential number of corners, which is computationally infeasible. Instead, a simplicial definition has been necessary in extending those results to a non-approximability work obtained recently [10].

We first introduce notations and definitions with a review of previous works of Murota, Iimura and Tamura [14], as well as Chen and Deng [2]. The simplicial model is then introduced in section 3 and the fundamental discrete fixed point theorem is proved in section 4. In section 5, we present the discrete Brouwer's fixed point theorem for simplicial direction-preserving maps, with the theorem of Murota, Iimura and Tamura [14] derived as a simple corollary. In Section 6, we give an explicit explanation for the definition of bad cubes in [2] and show that, the theorem of Chen and Deng is a special case of the fundamental fixed point theorem. Finally, we conclude in section 7.

2 Preliminaries

2.1 Notations and Definitions

Informally speaking, map $\mathcal{F}$ (or function f) is hypercubic direction-preserving on a finite set $X \subset \mathbb{Z}^d$ if for every two neighboring points in X, their directions given by $\mathcal{F}$ (or f) are not opposite. The neighborhood relation considered here is defined by the infinity norm.

Definition 1 (Hypercubic Direction-Preserving Maps). *Let X be a finite subset of $\mathbb{Z}^d$. Map $\mathcal{F}$ from X to $\mathbb{R}^d$ is said to be hypercubic direction-preserving* *on X if for every two points $r^1, r^2 \in X$ with $||r^1 - r^2||_\infty \leq 1$, we have $(\mathcal{F}_i(r^1) - r^1_i)(\mathcal{F}_i(r^2) - r^2_i) \geq 0$, for all $i : 1 \leq i \leq d$.*

Definition 2 (Hypercubic Direction-Preserving Functions). *Let X be a finite subset of $\mathbb{Z}^d$. Function f from set X to $\{0, \pm e^1, \pm e^2 \ldots \pm e^{d-1}, \pm e^d\}$ is said to be hypercubic direction-preserving if for every two points r^1, $r^2 \in X$ such that $||r^1 - r^2||_\infty \leq 1$, we have $||f(r^1) - f(r^2)||_\infty \leq 1$.*

Point $r \in X$ is called a fixed point of $\mathcal{F}$ (or f) if $\mathcal{F}(r) = r$ (or $f(r) = 0$).

2.2 The Fixed Point Theorem of Murota, Iimura and Tamura

Murota, Iimura and Tamura proved in [14] that every hypercubic direction-preserving map from an integrally convex set X to $\overline{X}$ must have a fixed point. Here we use $\overline{X}$ to denote the convex hull of finite set $X \subset \mathbb{Z}^d$.

Definition 3 (Integrally Convex Sets). *Finite set $X \subset \mathbb{Z}^d$ is integrally convex if for all $x \in \overline{X}$, $x \in \overline{X \cap N(x)}$ where $N(x) = \{ r \in \mathbb{Z}^d \mid ||r - x||_\infty < 1 \}$.*

Theorem 1 ([14]). *Let X be an integrally convex set in $\mathbb{Z}^d$, then every hypercubic direction-preserving map $\mathcal{F}$ from X to $\overline{X}$ has a fixed point in X.*

2.3 The Fixed Point Theorem of Chen and Deng

Given a hypercubic direction-preserving function f on a lattice set $C_{a,b} \subset \mathbb{Z}^d$, Chen and Deng proved in [2] that if the number of bad $(d-1)$-cubes on the boundary of $C_{a,b}$ is odd, then f must have a fixed point in $C_{a,b}$.

Definition 4. *Lattice set $C_{a,b} \subset \mathbb{Z}^d$ is defined as $C_{a,b} = \{ r \in \mathbb{Z}^d \mid \forall\ 1 \leq i \leq d,\ a \leq r_i \leq b \}$. For every $r \in \mathbb{Z}^d$ and $S \subset \{1, 2 \ldots d\}$ with $|S| = d - t$, the t-cube $C^t \subset \mathbb{Z}^d$ which is centered at r and perpendicular to S is defined as $C^t = \{ p \in \mathbb{Z}^d \bigm| \forall\ 1 \leq i \leq d,$ if $i \in S$, then $p_i = r_i$. Otherwise, $p_i = r_i$ or $r_i + 1 \}$.*

Definition 5 (Bad Cubes). *A 0-cube $C^0 \subset \mathbb{Z}^d$ is bad relative to function f if $f(C^0) = \{e^1\}$. For $1 \leq t \leq d-1$, a t-cube $C^t \subset \mathbb{Z}^d$ is bad relative to f if:*

1. *$f(C^t) = \{ e^1, e^2 \ldots e^{t+1} \}$;*
2. *the number of bad $(t-1)$-cubes in C^t is odd.*

Theorem 2 ([2]). *Let f be a hypercubic direction-preserving function on $C_{a,b} \subset \mathbb{Z}^d$, if N_B, i.e. the number of bad $(d-1)$-cubes on the boundary of $C_{a,b}$ is odd, then f must have a fixed point r in $C_{a,b}$ such that $f(r) = 0$.*

Although the theorem itself is succinct, the definition of bad cubes seems a little mysterious and lacks a satisfactory explanation. In section 6, we will use the fundamental discrete fixed point theorem for the simplicial model to resolve this puzzle.

3 Simplicial Direction-Preserving Maps and Functions

In this section, we introduce simplicial direction-preserving maps and functions based on simplicial structures. Let X be a finite set in $\mathbb{R}^d$. Here we only consider nondegenerate cases where $\overline{X} \subset \mathbb{R}^d$ is a convex d-polytope. For standard definitions concerning polytopes, readers are referred to [17] for details.

Definition 6. *A simplicial decomposition $\mathcal{S}$ of $C \subset \mathbb{R}^d$ is a collection of simplices satisfying:* **1)**. $C = \cup_{S \in \mathcal{S}} S$; **2)**. *For any $S \in \mathcal{S}$, if S' is a face of S, then $S' \in \mathcal{S}$;* **3)**. *For every two simplices $S_1, S_2 \in \mathcal{S}$, if $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2$ is a face of both S_1 and S_2.*

Definition 7. *Let X be a finite set in $\mathbb{R}^d$. A simplicial decomposition $\mathcal{S}$ of set X is a simplicial decomposition of $\overline{X}$ such that for every $S \in \mathcal{S}$, $V_S \subset X$, where V_S is the vertex set of simplex S.*

Given a simplicial decomposition $\mathcal{S}$ of X, we use $F_{\mathcal{S}}$ to denote the set of $(d-1)$-simplices on the boundary of $\overline{X}$, and B_X to denote the set of points on the boundary of $\overline{X}$: $F_{\mathcal{S}} = \{\, (d-1)\text{-simplex } S \in \mathcal{S} \mid S \subset F \text{ and } F \text{ is a facet of } \overline{X} \,\}$, and $B_X = \{\, r \in X \mid r \in F \text{ and } F \text{ is a facet of } \overline{X} \,\}$.

Definition 8 (Simplicial Direction-Preserving Maps). *A simplicial direction-preserving map is a triple $M = (\mathcal{F}, X, \mathcal{S})$. Here X is a finite set in $\mathbb{R}^d$ and $\mathcal{S}$ is a simplicial decomposition of X. Map $\mathcal{F}$ from X to $\mathbb{R}^d$ satisfies for every two points $r^1, r^2 \in X$, if there exists a simplex $S \in \mathcal{S}$ such that $r^1, r^2 \in V_S$, then $(\mathcal{F}_i(r^1) - r^1_i)(\mathcal{F}_i(r^2) - r^2_i) \geq 0$, for all $i : 1 \leq i \leq d$.*

Definition 9 (Simplicial Direction-Preserving Functions). *A triple $G = (f, X, \mathcal{S})$ is said to be a simplicial direction-preserving function if X is a finite set in $\mathbb{R}^d$, $\mathcal{S}$ is a simplicial decomposition of X, and function f from set X to $\{\, 0, \pm e^1, \ldots \pm e^d \,\}$ satisfies for every two points $r^1, r^2 \in X$, if there exists $S \in \mathcal{S}$ such that $r^1, r^2 \in V_S$, then $||\, f(r^1) - f(r^2) \,||_\infty \leq 1$.*

In other words, for every two neighboring points in X, their directions given by map $\mathcal{F}$ (or function f) can't be opposite. The only difference with the hypercubic model is that the neighborhood relation is now defined by simplices in the simplicial decomposition $\mathcal{S}$ instead of unit d-cubes in $\mathbb{Z}^d$.

4 The Fundamental Discrete Fixed Point Theorem

In this section, we present the fundamental discrete fixed point theorem which is both simple and powerful. Any simplicial direction-preserving function which satisfies the boundary condition of the theorem must have a fixed point.

Definition 10 (Bad Simplices). *Let $G = (f, X, \mathcal{S})$ be a simplicial direction-preserving function, where $X \subset \mathbb{Z}^d$. A t-simplex $S \in \mathcal{S}$ where $0 \leq t \leq d$ is said to be bad (relative to function G) if $f(V_S) = \{\, e^1, e^2, \ldots e^{t+1} \,\}$, where V_S is the vertex set of S. We use N_G to denote the number of bad $(d-1)$-simplices in $F_{\mathcal{S}}$.*

Lemma 1. *For any simplicial direction-preserving function $G = (f, X, \mathcal{S})$, if there exists no fixed point in X, then N_G is even.*

Proof. Firstly, one can show that for every $(d-1)$-simplex $S \in \mathcal{S}$, if $S \in F_{\mathcal{S}}$, then there exists exactly one d-simplex in $\mathcal{S}$ containing S. Otherwise, there are exactly two such simplices. Using this property, the parity of N_G is same as the one of the following summation:

$$\sum_{d\text{-simplex } S^d \in \mathcal{S}} \Big| \left\{ \text{ bad } (d-1)\text{-simplices in } S^d \right\} \Big|.$$

As G is direction-preserving and has no fixed point, the number of bad $(d-1)$-simplices in S^d is either 0 or 2. Therefore, the summation above must be even.

We now get the fundamental theorem as a simple corollary of Lemma 1.

Theorem 3 (The Fundamental Discrete Fixed Point Theorem). *Let $G = (f, X, \mathcal{S})$ be a simplicial direction-preserving function. If N_G, i.e. the number of bad $(d-1)$-simplices on the boundary is odd, then G must have a fixed point $r \in X$ such that $f(r) = 0$.*

5 The Discrete Brouwer's Fixed Point Theorem

In this section, the fundamental discrete fixed point theorem will be employed to prove a fixed point theorem concerning simplicial direction-preserving maps. It can be recognized as a discrete version of Brouwer's fixed point theorem. It states that for any simplicial direction-preserving map from some finite set to its convex hull, there must exist a fixed point in the definition domain.

We will also derive the theorem of Murota, Iimura and Tamura as a simple corollary. Actually, the one derived here is much stronger than theirs.

5.1 Preliminaries

We use e^k to denote the kth unit vector of $\mathbb{Z}^d$ where $e^k_k = 1$ and $e^k_i = 0$ for all $i : 1 \le i \ne k \le d$.

Definition 11. *For every $(d-1)$-simplex $S \in F_{\mathcal{S}}$, we let e_S be the unit vector which is outgoing and perpendicular to S. For all $r \in \overline{X}$ and $r_S \in S$, we have $e_S \cdot (r - r_S) \le 0$.*

$S \in F_{\mathcal{S}}$ is visible from point $r \notin \overline{X}$ if $e_S \cdot (r - r_S) > 0$ for some $r_S \in S$.

Construction 1 (Extension of Simplicial Decomposition). *Let $X \subset \mathbb{R}^d$ be a finite set and $\mathcal{S}$ be a simplicial decomposition of X. For every point $r \notin \overline{X}$, we can add new simplices into $\mathcal{S}$ and build a simplicial decomposition $\mathcal{S}'$ of set $X' = X \cup \{r\}$ as follows. For every $(d-1)$-simplex $S \in F_{\mathcal{S}}$ visible from r, we add d-simplex* $\text{conv}(S, r)$ *and all its faces into $\mathcal{S}$. One can check that $\mathcal{S}'$ is a simplicial decomposition of X', and $\mathcal{S} \subset \mathcal{S}'$.*

Given a simplicial direction-preserving map $M = (\mathcal{F}, X, \mathcal{S})$, we can convert it into a direction-preserving function $G = (f, X, \mathcal{S})$ as follows.

Construction 2. *Given a simplicial direction-preserving map $M = (\mathcal{F}, X, \mathcal{S})$, we can build a simplicial direction-preserving function $G = (f, X, \mathcal{S})$ as follows. For every $r \in X$, if $\mathcal{F}(r) = r$, then $f(r) = 0$. Otherwise, let $i : 1 \le i \le d$ be the smallest integer such that $\mathcal{F}_i(r) - r_i \neq 0$, then $f(r) = \text{sign}(\mathcal{F}_i(r) - r_i)e^i$.*

5.2 The Key Lemma

Lemma 2. *Let $M = (\mathcal{F}, X, \mathcal{S})$ be a simplicial direction-preserving map where $\mathcal{F}$ is from X to $\overline{X}$, and $G = (f, X, \mathcal{S})$ be the function constructed above, then either f has a fixed point in B_X or N_G is odd.*

Proof (Proof Sketch). Let $n = \max_{r \in X, 1 \le i \le d} |r_i|$, then we can scale down X to be $X' \subset (-1, 1)^d$ where $X' = \{ r/(n+1),\ r \in X \}$. We also get a simplicial decomposition $\mathcal{S}'$ of X' from $\mathcal{S}$ using the one-to-one correspondence between X and X', and a map $\mathcal{F}'$ from X' to $\overline{X'}$ where $\mathcal{F}'(r) = \mathcal{F}((n+1)r) \big/ n+1$.

Let G' be the function constructed from map $M' = (\mathcal{F}', X', \mathcal{S}')$, then it is easy to check that $N_G = N_{G'}$. Therefore, we only need to prove the lemma for maps $M = (\mathcal{F}, X, \mathcal{S})$ with $X \subset (-1, 1)^d$. From now on, we always assume that $X \subset (-1, 1)^d$.

If f has a fixed point in set B_X, then the lemma is proven. Otherwise, we extend (by applying Construction 1 for d times) $G = (f, X, \mathcal{S})$ to be a new function $G^* = (f^*, X^*, \mathcal{S}^*)$ such that $X \subset X^*$, $\overline{X^*} = [-1, 1]^d$ and $\mathcal{S} \subset \mathcal{S}^*$. After proving that G^* is simplicial direction-preserving, we show the following two properties of G and G^*:

Property 1. N_{G^*} is odd;

Property 2. $N_G \equiv N_{G^*} \pmod 2$,

and the lemma is proven.

Details of the proof can be found in the full version [1].

5.3 The Discrete Brouwer's Fixed Point Theorem

From Construction 2, every fixed point of function f is also a fixed point of map $\mathcal{F}$. By Theorem 3 and Lemma 2, we get the following theorem immediately.

Theorem 4 (The Discrete Brouwer's Fixed Point Theorem). *For every simplicial direction-preserving map $M = (\mathcal{F}, X, \mathcal{S})$ such that $\mathcal{F}$ maps X to $\overline{X}$, there must exist a fixed point in X.*

Now we prove the fixed point theorem of Murota, Iimura and Tamura [14] as a direct corollary of Theorem 4.

Lemma 3 (Property of Integrally Convex Sets [14]). *For every integrally convex set X, there exists a simplicial decomposition $\mathcal{S}$ of $\overline{X}$, which satisfies for every $x \in \overline{X}$, letting $S_x \in \mathcal{S}$ be the smallest simplex containing x, then all of its vertices belong to $N(x) = \{ r \in \mathbb{Z}^d \mid ||r - x||_\infty < 1 \}$.*

Let $\mathcal{F}$ be a hypercubic direction-preserving map from integrally convex set $X \subset \mathbb{Z}^d$ to $\overline{X}$, and $\mathcal{S}$ be a simplicial decomposition of X which satisfies the condition in Lemma 3, then one can check that $M = (\mathcal{F}, X, \mathcal{S})$ is a simplicial direction-preserving map from X to $\overline{X}$. By Theorem 4, we know that there is a fixed point of $\mathcal{F}$ in X.

Moreover, the argument above shows that the theorem of Murota, Iimura and Tamura can be greatly strengthened. Actually, map $\mathcal{F}$ is not necessary to be hypercubic direction-preserving. Being simplicial direction-preserving relative to some simplicial decomposition of X is sufficient to ensure the existence of a fixed point in X.

6 An Explanation for the Definition of Bad Cubes

Chen and Deng [2] defined the badness of $(d-1)$-cubes relative to hypercubic direction-preserving functions in d-dimensional space, and showed that for any hypercubic direction-preserving function f on $C_{a,b} \subset \mathbb{Z}^d$, if the number of bad $(d-1)$-cubes on the boundary is odd, then f must have a fixed point in $C_{a,b}$. While the theorem itself is succinct, the definition of bad cubes seems a little mysterious and lacks a satisfactory explanation. In this section, we will use the simplicial model developed in section 3 and 4 to resolve this puzzle.

First, we add extra points into the lattice set $C_{a,b} \subset \mathbb{Z}^d$ and construct a simplicial decomposition for the new set $D_{a,b}$, where $\overline{C_{a,b}} = \overline{D_{a,b}}$. Then, we extend the hypercubic direction-preserving function f on $C_{a,b}$ to be a simplicial direction-preserving function on $D_{a,b}$. Finally, we prove that the parity of N_B is same as N_G, where N_B is the number of bad $(d-1)$-cubes and N_G is the number of bad $(d-1)$-simplices on the boundary. In this way, we show that Chen and Deng's theorem [2] is a special case of the fundamental discrete fixed point theorem.

6.1 Preliminaries

Definition 12. *A convex subdivision $\mathcal{P}$ of a finite set $X \subset \mathbb{R}^d$ is a collection of convex d-polytopes such that:* **1)**. $\overline{X} = \cup_{P \in \mathcal{P}} P$, *and for every polytope $P \in \mathcal{P}$, all of its vertices are drawn from X;* **2)**. *For every two polytopes $P_1, P_2 \in \mathcal{P}$, if $P_1 \cap P_2 \neq \emptyset$, then $P_1 \cap P_2$ is a face of both P_1 and P_2.*

Definition 13. *Let P be a convex t-polytope in $\mathbb{R}^d$ and V_P be its vertex set. The center point c_P of polytope P is defined as $c_P = \sum_{r \in V_P} (r/|V_P|)$. Obviously, we have $c_P \in P$ and $c_P \notin V_P$.*

For example, let $C \subset \mathbb{Z}^d$ be a t-cube centered at $r \in \mathbb{Z}^d$ and perpendicular to T, then the center point c of $\overline{C}$ satisfies that $c_k = r_k$ for every $k \in T$, and $c_k = r_k + 1/2$ for every $k \notin T$.

Let $\mathcal{P}$ be a convex subdivision of set X in $\mathbb{R}^d$. We now add extra points $r \in \overline{X}$ into X and construct a simplicial decomposition $\mathcal{S}'$ for the new set X'. Details of the construction are described by the algorithm in Figure 1.

$\mathcal{S}' = \{\, \{r\} \mid r \in X \,\}$ and $X' = X$
for any t **from** 1 **to** d **do**
  for any F that is a t-face of some d-polytope in $\mathcal{P}$ **do**
    add the center point c_F of F into X'
    for any $(t-1)$-simplex $S \in \mathcal{S}'$ and $S \subset F$ **do**
      add every face of t-simplex $\mathrm{conv}(S, c_F)$ into $\mathcal{S}'$

Fig. 1. The Construction of $\mathcal{S}'$ and X'

Every lattice set $C_{a,b} \subset \mathbb{R}^d$ has a natural convex subdivision $\mathcal{P}$ where $\mathcal{P} = \{\, \overline{C} \mid d\text{-cube } C \subset C_{a,b} \,\}$. Using Figure 1, we get a simplicial decomposition $\mathcal{S}$ of

$$D_{a,b} = \left\{\, r \in \mathbb{R}^d \;\middle|\; \forall\, 1 \le i \le d,\ a \le r_i \le b \text{ and } \exists\, r' \in \mathbb{Z}^d,\ r = r'/2 \,\right\}.$$

6.2 Extension of Hypercubic Direction-Preserving Functions

Let f be a hypercubic direction-preserving function on $C_{a,b}$, we now extend it onto set $D_{a,b}$ as follows. For every $r \in D_{a,b} - C_{a,b}$, assume it is the center point of t-cube $C \subset C_{a,b}$. If $0 \in f(C)$, then $f(r) = 0$. Otherwise, let $1 \le t \le d$ be the largest integer such that $f(C) \cap \{\pm e^t\} \neq \emptyset$, then $f(r) = e^t$ if $e^t \in f(C)$ and $f(r) = -e^t$ if $-e^t \in f(C)$. One can prove the following two properties.

Property 1. *Let f be a hypercubic direction-preserving function on $C_{a,b} \subset \mathbb{Z}^d$, then $G = (f, D_{a,b}, \mathcal{S})$ is a simplicial direction-preserving function.*

Property 2. *If the extended function $G = (f, D_{a,b}, \mathcal{S})$ has a fixed point in $D_{a,b}$, then the original function must have a fixed point in $C_{a,b}$.*

6.3 The Nature of Bad Cubes

We are ready to give an explicit explanation for the definition of bad cubes.

Lemma 4. *Let f be a hypercubic direction-preserving function on $C_{a,b} \subset \mathbb{Z}^d$ and $G = (f, D_{a,b}, \mathcal{S})$ be the extend function. For every t-cube C^t in $C_{a,b}$ where $0 \le t \le d-1$, it is bad relative to f iff the cardinality of the following set is odd:*

$$S_{C^t} = \left\{\, t\text{-simplex } S \in \mathcal{S} \text{ is bad relative to } G \;\middle|\; S \subset \overline{C^t} \,\right\}.$$

Proof. We use induction on t. The base case for $t = 0$ is trivial.

For $t > 0$, we assume the lemma is true for case $t-1$. Let c be the center point of $\overline{C^t}$, then the way we build simplicial decomposition $\mathcal{S}$ implies that

$$\left\{\, t\text{-simplex } S^t \subset \overline{C^t} \,\right\} = \left\{\, \mathrm{conv}(S^{t-1}, c),\ S^{t-1} \in \mathcal{S} \text{ is on the boundary of } \overline{C^t} \,\right\}$$

where S^{t-1} is used to denote $(t-1)$-simplices in $\mathcal{S}$.

Firstly, we prove that, if t-cube $C^t \subset \mathbb{Z}^d$ is <u>not</u> bad, then $|S_{C^t}|$ is even. If $0 \in f(C^t)$, then $f(c) = 0$. As each t-simplex in $\overline{C^t}$ has c as one of its vertices,

$S_{C^t} = \emptyset$ and we are done. Similarly, we can prove if $f(C^t) \cap \{\pm e^k\} \neq \emptyset$ where $k > t+1$, then $S_{C^t} = \emptyset$. If $e^k \notin f(C^t)$ where $1 \leq k \leq t+1$, then for every t-simplex $S \subset \overline{C^t}$, $e^k \notin f(V_S)$, and thus, $S_{C^t} = \emptyset$. Otherwise, we have $f(C^t) = \{ e^1, ..., e^{t+1} \}$, and thus $f(c) = e^{t+1}$. Because C^t is not bad, the number of bad $(t-1)$-cubes on the boundary of C^t is even. Using the induction hypothesis on $t-1$, a $(t-1)$-cube is bad iff the number of bad $(t-1)$-simplices in it is odd. As a result, the number of bad $(t-1)$-simplices on the boundary of C^t is even. Using the equation in the first paragraph, we know $|S_{C^t}|$ is even too.

On the other hand, we prove if C^t is bad, then $|S_{C^t}|$ is odd. Since $f(C^t) = \{ e^1, ..., e^{t+1} \}$, we have $f(c) = e^{t+1}$. As the number of bad $(t-1)$-cubes on the boundary of C^t is odd, the number of bad $(t-1)$-simplices on the boundary of C^t is also odd, according to the induction hypothesis on case $t-1$. Using the equation in the first paragraph again, we know $|\, S_{C^t} \,|$ is odd.

We now get Lemma 5 as a direct corollary of Lemma 4.

Lemma 5. *The parity of N_B (the number of bad $(d-1)$-cubes on the boundary of $C_{a,b}$) is same as the one of N_G (the number of bad $(d-1)$-simplices on the boundary of $D_{a,b}$).*

With Property 1, 2 and Lemma 5 above, Chen and Deng's theorem can be immediately derived from the fundamental discrete fixed point theorem.

7 Concluding Remarks

In this paper, we generalize the concept of direction-preserving maps and characterize a new class of discrete maps over simplicial structures. The fundamental discrete fixed point theorem is then proposed, which is based on the counting of bad $(d-1)$-simplices on the boundary. The power of this theorem is demonstrated in two ways. First, it is applied to prove the discrete Brouwer's fixed point theorem which is much more general than the one of Murota, Iimura and Tamura. Second, we resolve the puzzle of bad cubes, and show that the boundary condition of Chen and Deng's theorem is exactly equivalent to the one of the fundamental theorem.

Our work would immediately imply the corresponding discrete concept of *degree*. It would be an especially interesting problem to study the case when the fixed point is defined in the recent model of a set of points. An immediate follow-up research direction is to understand other concepts and theorems related to *degree*. A clear understanding would definitely advance the state of art of the numerical computation of related problems, such as the case of discrete fixed points versus approximate fixed points [2].

References

1. X. Chen and X. Deng. A Simplicial Approach for Discrete Fixed Point Theorems. *full version, available at* http://www.cs.cityu.edu.hk/~deng/.
2. X. Chen and X. Deng. On Algorithms for Discrete and Approximate Brouwer Fixed Points. In *STOC 2005*, pages 323–330.

3. X. Chen and X. Deng. 3-Nash is PPAD-complete. *ECCC, TR05-134*, 2005.
4. X. Chen and X. Deng. Settling the Complexity of 2-Player Nash-Equilibrium. *ECCC, TR05-140*, 2005.
5. X. Chen, X. Deng, and S-H Teng. Computing Nash Equilibria: Approximation and Smoothed Complexity. *ECCC, TR06-023*, 2006.
6. X. Chen and X. Deng. Lattice Embedding of Direction-Preserving Correspondence Over Integrally Convex Set. *Accepted by AAIM*, 2006.
7. X. Chen and X. Deng. On the Complexity of 2D Discrete Fixed Point Problem. *Accepted by ICALP*, 2006.
8. P. Crescenzi and R. Silvestri. Sperner's lemma and robust machines. *Comput. Complexity*, 7(2):163–173, 1998.
9. C. Daskalakis, P.W. Goldberg, and C.H. Papadimitriou. The Complexity of Computing a Nash Equilibrium. *ECCC, TR05-115*, 2005.
10. C. Daskalakis and C.H. Papadimitriou. Three-player games are hard. *ECCC, TR05-139*.
11. K. Friedl, G. Ivanyos, M. Santha, and F. Verhoeven. On the black-box complexity of Sperner's Lemma. In *FCT 2005*.
12. M.D. Hirsch, C.H. Papadimitriou, and S. Vavasis. Exponential lower bounds for finding Brouwer fixed points. *J.Complexity*, 5:379–416, 1989.
13. T. Iimura. A discrete fixed point theorem and its applications. *Journal of Mathematical Economics*, 39(7):725–742, 2003.
14. T. Iimura, K. Murota, and A. Tamura. Discrete Fixed Point Theorem Reconsidered. *Journal of Mathematical Economics*, to appear.
15. G. Laan, D. Talman, and Z. Yang. Solving discrete zero point problems. *Tinbergen Institute Discussion Papers*, 2004.
16. E. Sperner. Neuer Beweis fur die Invarianz der Dimensionszahl und des Gebietes.
17. G.M. Ziegler. *Lectures on Polytopes*. Springer-Verlag, New York, 1982.