Massively Parallel A* Search on a GPU: Appendix

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A1 Pseudocode of Traditional A* Search

Algorithm A1 Sequential A* graph search algorithm

1: procedure A*(s, t)  \(\triangleright\) find the shortest path from s to t
2:     Let Q be a priority queue  \(\triangleright\) Q is the open list
3:     Push(Q, s)
4:     Let H be a linked hash table  \(\triangleright\) H is the close list
5:     \(H \leftarrow \emptyset\)
6:     while Q is not empty do
7:         \(q \leftarrow {\text{Extract}}(Q)\)
8:         if \(q = t\) then
9:             return the path found
10:     end if
11:     \(R \leftarrow \text{Expand}(q)\)
12:     \(S \leftarrow R - H\)  \(\triangleright\) deduplication (needless for A* tree search)
13:     for \(s \in S\) do
14:         \(\text{Push-Back-Or-Update}(Q, s, f(s))\)  \(\triangleright\) f is the heuristic function
15:     end for
16:     \(H \leftarrow H \cup \{q\}\)
17: end while
18: end procedure

Algorithm A1 gives the pseudocode of the traditional A* search algorithm. In each round, we extract the state with the minimum \(f\) value from the open list (Line 7). Then we expand its outer neighbors and check for node duplication (Line 12). The step for node deduplication is not required for tree search. Finally, we calculate the \(f\) values of the new expanded states and push them back to the open list (Line 14). If the nodes of some states are already in the open list, we only update their \(f\) values for the open list.

In order to find the global optimal solution with an admissible heuristic function rather than a consistent one, in Line 12 we need to keep the state that has a shorter path from the starting node to the duplicated node (Russell and Norvig 2009).
A2 Node Duplication Detection

A2.1 Parallel Cuckoo Hashing

The lookup operations of traditional cuckoo hashing can be performed in constant time in the worst case. We only need to check whether the query node exists in any of the tables or not. Below is a sequential implementation of the insertion procedure in the cuckoo hashing scheme using two tables with hash functions $h_1(x)$ and $h_2(x)$, respectively:

1. Let $z \in \{1, 2\}$. Initially, $z = 1$;
2. Let $H[h_z(x)]$ denote the bucket indexed by the hash function $h_z(x)$. If any of the buckets $H[h_1(x)]$ or $H[h_2(x)]$ is empty, we place node $x$ into that bucket and terminate the procedure;
3. Otherwise, we displace the node currently occupying position $H[h_z(x)]$ and place $x$ into $H[h_z(x)]$;
4. Alternate $z$ between 1 and 2; Go back to Step 2 again to insert the node displaced.

There is a small possibility that the above procedure runs into a cycle and never ends. A result in closed form of this possibility was derived in (Kutzelnigg, 2006) when $d = 2$ (i.e., using two hash tables). In practice, we can solve this problem by terminating the procedure after a certain number of iterations and rebuilding the hash tables using a new set of hash functions.

Algorithm A2 describes the pseudocode of parallel cuckoo hashing, in which Cuckoo-Hash-Insert stands for the partition procedure and Insert stands for the parallel insert procedure. The parameter $H$ of Cuckoo-Hash-Insert is an array of cuckoo hashing instance, and $H[i]$ stands for the cuckoo hashing instance on block $i$. Parameter $T$ is a list of nodes that need to be inserted. The third parameter $k$ stands for the total number of the blocks available on the GPU platform. Lines 2-7 assign nodes to different blocks according to $h(x)$ and stores the partitioned node temporarily in $T[h(x)]$. Lines 8-12 launch the parallel insertion process.

Parameter $H$ in the procedure Insert is the actual cuckoo hashing instance for the block of current thread and $t$ is the node to be inserted. The variable $d$ is the number of hash tables in cuckoo hashing. Lines 17-23 scan for an empty bucket to insert the current node. The thread synchronization operation in Line 24 is a common operation in a GPU algorithm to wait until all other threads in the same block have reached that line. The check in Line 25 is necessary because other threads may write to the same bucket in Line 19. Finally, if we fail to find a suitable bucket to write, we evict a node randomly from these valid buckets and place the current node into its position in Lines 28-29. The function Atomic-Swap swaps two parameters in the memory with one machine instruction to avoid the race hazard. After that, we go back to Line 16 and try to insert the evicted node.

For simplicity, the procedure of rebuilding the hash tables is not included in Algorithm A2. Also, we need to guarantee that the input $T$ of Cuckoo-Hash-Insert does not contain two identical nodes. This can be done by sorting all elements in $T$ and then removing the same adjacent nodes to deduplicate locally in advance. The GPU radix sort (Sintorn and Assarsson, 2008) and the GPU scan (Sengupta et al., 2007) can be used in this local deduplication stage.
A2.2 Parallel Hashing with Replacement

Algorithm A2 Parallel cuckoo hashing on a GPU

```plaintext
1: procedure Cuckoo-Hash-Insert(H, T, k)
2:   for i ← 0 to k − 1 do
3:     T[i] ← ∅
4:   end for
5:   for all t ∈ T do ▷ dispatch nodes according to \( h(x) \)
6:       Push-Back(\( T[h(t)] \), t)
7:   end for
8:   for i ← 0 to k − 1 in parallel do
9:     for j ← 1 to |T[i]| in parallel do
10:        call Insert(H[i], T[i][j]) on block i
11:   end for
12: end procedure

13: procedure Insert(H, t)
14:   z ← −1
15:   while true do
16:     for i ← 0 to d − 1 do
17:       if \( H[h_i(t)] \) is empty then
18:         \( H[h_i(t)] \) ← t
19:         z ← i
20:         break
21:       end if
22:     end for
23:     thread synchronization
24:     if \( z \neq -1 \) and \( H[h_z(t)] = t \) then
25:       return
26:     end if
27:   end while
28: end procedure
```

A2.2 Parallel Hashing with Replacement

Algorithm A2 gives the pseudocode of parallel hashing with replacement. To increase the occupancy factor, multiple hash functions are also used. As no synchronization operation is required, we no longer need multiple hash table instances. The input \( H \) is the hash table and \( T \) is the list of nodes to be inserted. The procedure Hash-with-Replacement-Deduplicate does two jobs. First, it inserts all the nodes from \( T \) to \( H \). Second, it detects the duplication and returns a new list \( T' \) after removing the redundant nodes in \( T \).

Lines 2-10 scan for an empty bucket for inserting \( T[i] \). Lines 11-12 use an atomic swap operation to assign the current node to the hash table. If there are duplicated nodes \( t_0 \) in \( T \),
A3 Analysis of GA*

A3.1 Correctness of GA*

Here we re-state the pseudocode of GA* (as given in Algorithm A4) and then provide its theoretical analysis.

Lemma A3.1. Let $h'(x)$ denote the optimal distance or cost from current state $x$ to the target state. If the given heuristic function is admissible, i.e., $h(x) \leq h'(x)$ for each state $x$, there exists a state $t'$ in one of the priority queues such that $f(t') \leq f(t)$ before executing Line 19 in Algorithm A4, where $t$ is an optimal target state.

Proof. Let $d(x, y)$ denote the optimal distance or cost from state $x$ to state $y$. For each state $t'$
Algorithm A4 GA*: Parallel A* search on a GPU

1: procedure GA*(s, t, k) \(\triangleright\) find the shortest path from \(s\) to \(t\) with \(k\) queues
2: Let \(\{Q_i\}_{i=1}^k\) be the priority queues of the open list
3: Let \(H\) be the close list
4: Push\((Q_1, s)\)
5: \(m \leftarrow \text{nil}\) \(\triangleright m\) stores the best target state
6: while \(Q\) is not empty do
7: Let \(S\) be an empty list
8: for \(i \leftarrow 1\) to \(k\) in parallel do
9: if \(Q_i\) is empty then
10: continue
11: \(q_i \leftarrow \text{Extract}(Q_i)\)
12: if \(q_i.\node = t\) then
13: if \(m = \text{nil}\) or \(f(q_i) < f(m)\) then
14: \(m \leftarrow q_i\)
15: continue
16: end if
17: end if
18: \(S \leftarrow S + \text{Expand}(q_i)\)
19: end for
20: if \(m \neq \text{nil}\) and \(f(m) \leq \min_{q \in Q} f(q)\) then
21: return the path generated from \(m\)
22: end if
23: \(T \leftarrow S\)
24: for \(i \leftarrow 1\) to |\(S\)| in parallel do
25: \(s \leftarrow S[i]\)
26: if \(s.\node \in H\) and \(H[s.\node].g < s.g\) then
27: remove \(s\) from \(T\)
28: end if
29: end for
30: \(T \leftarrow S\)
31: for \(i \leftarrow 1\) to |\(T\)| in parallel do
32: \(t \leftarrow T[i]\)
33: \(t.f \leftarrow f(t)\)
34: Push \(t\) to one of priority queue \(H[t.\node] \leftarrow t\)
35: end for
36: end while
37: end procedure
that is on an optimal path from $s$ to $t$, we have

$$f(t') = g(t') + h(t')$$
$$\leq d(s, t') + h'(t')$$
$$= d(s, t') + d(t', t)$$
$$= d(s, t)$$
$$= f(t).$$

Thus it is sufficient to prove that there always exists a state $t'$ in one of the priority queues along an optimal path from $s$ to $t$. At the beginning, the state with starting node $s$ satisfies such a condition. At any time, if Line 12 in Algorithm A4 pops such a state $t'$, Line 35 will generate another state that is also on an optimal path from $s$ to $t$, which is then pushed back into the queues. Thus, such a state always exists in one of the priority queues.

**Theorem A3.2.** Let $h'(x)$ denote the optimal cost from current state $x$ to the target current state. If the given heuristic function is admissible, i.e., $h(x) \leq h'(x)$ for each state $x$, the first solution returned by GA* must be the optimal solution.

**Proof.** We prove this theorem by contradiction. There exists two possible situations that may violate our conclusion:

1. GA* never ends. In this case, suppose that the minimum cost of an edge is $\delta$, then the number of steps from the starting node to the current node with minimum $f$ value in the open list is never greater than $\lceil f(t)/\delta \rceil$, where $t$ is the target state. This is because if such node $t_0$ exists, i.e., $f(t_0) > f(t)$, and according to Lemma A3.1 there exists a state with smaller $f$ value. Let $A$ be the set of all the states that are reachable from the starting state within $f(t)/\delta$ steps. In each expansion round, GA* will extract at least one state in $A$ from one of the priority queues. However, $|A|$ must be finite because the number of paths from the starting state within $\lceil f(t)/\delta \rceil$ steps is finite. Thus this situation is impossible.

2. When GA* terminates, it returns a non-optimal solution. In this case, assume that our algorithm returns a state $t_1$, while the optimal solution is state $t_2$. Because $t_1$ is not optimal, we have $f(t_1) > f(t_2)$. However, according to Lemma A3.1, we have a state $t'$ in one of the priority queues that satisfies $f(t') \leq f(t_2) < f(t_1)$, which violates the condition in Line 21 of Algorithm A4.

The reason why GA* can guarantee the optimality without the requirement of a consistent heuristic function is that GA* keeps the state that has a shorter path from the starting node to the current node during the node deduplication process.
A3.2 Theoretical Bound on the Number of Nodes Expanded by GA*

Lemma A3.3. In the same expansion round of sequential A* (Line 11 of Algorithm A1) and GA* (Line 19 of Algorithm A4) on the same input, the state expanded by A* will also be expanded in the same round or has already been expanded by GA*, provided that the heuristic function is consistent.

Proof. We prove this lemma by contradiction. Let $r$ be the first expansion round that we have such a situation. Suppose that the state $s$ is expanded in the sequential A* algorithm but not in the GA* algorithm. There are two possibilities why $s$ is not expanded in the GA* algorithm:

1. The state $s$ is not in the open list by the GA* algorithm. Let the parent of $s$ be $s'$. Because $r$ is the first round of this situation, $s'$ must have already been expanded by the GA* algorithm. As $s$ is a child of $s'$, it must be in the open list;

2. The state $s$ is in the open list, but has not been extracted by the GA* algorithm. Then, there must exist another node $s''$ in the open list of the GA* algorithm such that $f(s'') < f(s)$. Let us check the expansion situation of $s''$ in the sequential A* algorithm. If $s''$ has already been expanded in the sequential A* algorithm, $s''$ must also have been expanded in the GA* algorithm, because $r$ is the first round that we have such a situation. If $s''$ has not been expanded but exists in the open list of the A* algorithm, the sequential A* algorithm will expand $s''$ rather than $s$ in this round. If $s''$ does not exist in the open list of the sequential A* algorithm, then $f(s'') < f(s)$, which violates the property of consistency that the $f$ values of the nodes extracted from the priority queue are monotone increasing.

Theorem A3.4. The number of expansion rounds of expansion in GA* is less than or equal to that in the sequential A* algorithm on the same input, provided that the heuristic function is consistent.

Proof. Suppose the sequential A* algorithm terminates when expanding the target state $t$ in round $k$. From Lemma A3.3, GA* algorithm will expand in the same round or have already expanded $t$ in round $k$.

Corollary A3.5. The GA* algorithm with $k$ parallel queues on a graph with $N$ nodes will expand at most $kN$ states, provided that the heuristic function is consistent.

Proof. When the heuristic function is consistent, the sequential A* algorithm will terminate in $N$ expansion rounds. According to Theorem A3.4, GA* will also terminate in less than or equal to $N$ expansion rounds. In each expansion round, GA* expands at most $k$ nodes from the open list. Thus, the total number of the nodes expanded by GA* is less than or equal to $kN$. 

7
A4 Failure Rate of Parallel Cuckoo Hashing

In this section, we derive the theoretical analysis on the failure rate of the parallel cuckoo hashing algorithm for the case $d = 2$, i.e., using only two hash functions.

**Lemma A4.1** (Kutzelnigg 2006). The probability that a single cuckoo hashing instance with $n = (1 - \varepsilon)m$ nodes and two hash tables of size $m$ succeeds to construct, where $\varepsilon \in (0, 1)$, is equal to

$$1 - \frac{(2\varepsilon^2 - 5\varepsilon + 5)(1 - \varepsilon)^3}{12(2 - \varepsilon)^2\varepsilon^3} \frac{1}{m} + O\left(\frac{1}{m^2}\right). \quad (1)$$

The proof of this lemma was first given by Kutzelnigg (2006), which was derived using generating functions and other advanced mathematic tools.

**Theorem A4.2.** The probability that the parallel cuckoo hashing algorithm with $k$ cuckoo hashing instances, in which each instance has $n = (1 - \varepsilon)m$ nodes and two hash tables of size $m$, fails to construct is equal to

$$c(\varepsilon)\frac{k^2}{m} + O\left(\frac{1}{m^2}\right), \quad (2)$$

where $\varepsilon \in (0, 1)$ and $c(\varepsilon) = \frac{(2\varepsilon^2 - 5\varepsilon + 5)(1 - \varepsilon)^3}{12(2 - \varepsilon)^2\varepsilon^3}$.

**Proof.** Let $p(m, \varepsilon)$ be the success probability of a single cuckoo hashing instance with $n = (1 - \varepsilon)m$ nodes and two hash tables of size $m$. From Lemma A4.1 we know that $p(m, \varepsilon) = 1 - \frac{c(\varepsilon)}{m} + O\left(\frac{1}{m^2}\right)$. Then the success rate of all cuckoo hashing instances in the parallel cuckoo hashing algorithm is

$$\left[p\left(\frac{m}{k}, \varepsilon\right)\right]^k = \left(1 - \frac{c(\varepsilon)}{m} + O\left(\frac{1}{m^2}\right)\right)^k = \left(1 - \frac{c(\varepsilon)}{m}\right)^k + O\left(\frac{1}{m^2}\right) = \binom{k}{0} - \binom{k}{1} \frac{c(\varepsilon)}{m} + \cdots + O\left(\frac{1}{m^2}\right) = 1 - c(\varepsilon)\frac{k^2}{m} + O\left(\frac{1}{m^2}\right).$$

Thus, the failure rate of the parallel cuckoo hashing is equal to $c(\varepsilon)\frac{k^2}{m} + O\left(\frac{1}{m^2}\right)$. \qed

A5 Analysis of Parallel Hashing with Replacement

In the Analysis Section of the main paper, we have stated Theorem 4.4 about the behaviors of parallel hashing with replacement. Here we first re-describe an alternative statement (i.e., Theorem A5.1) and then provide its proof.
Theorem A5.1. If the parameter $T$ of Algorithm A3 contains duplicated nodes, namely there exists $i_1 < i_2 < \cdots < i_n$ such that $T[i_1] = T[i_2] = \cdots = T[i_n] = t_0$, then the return value $T'$ will contain at most one instance of $t_0$ if there does not exist a number $z \in \{0, 1, \ldots, d-1\}$ and a node $t_1 = T[j]$ such that $h_z(t_0) = h_z(t_1)$.

Proof. Because there does not exist a number $z$ and $t' = T[j]$ such that $h_z(t) = h_z(t')$, we can just focus on the threads dealing with $T[i_1], T[i_2], \ldots, T[i_n]$ without considering the behaviors of other threads. We label these threads as thread 1, thread 2, ..., and thread $n$, respectively.

We first consider the situation in which $t_0$ is in $H$ before Hash-with-Replacement-Deduplicate is called. Threads 1 to $n$ will find this $t_0$ in Line 6. In that case, Line 12 will not modify the hash table and all the $t_0$ nodes will be removed from $T'$.

Now let us consider the second situation in which $t_0$ is not in $H$ before Hash-with-Replacement-Deduplicate is called. We will discuss the following different cases:

1. If any bucket for $t_0$ is still empty, the first thread that execute Line 12 will insert it into an empty bucket $h_u[t_0]$. For this thread, $t = \text{nil}$ and its $t_0$ will be kept in $T'$. For all other threads, when they check the bucket $h_u[t_0]$, it is either filled with $t_0$ or empty. In either case, Line 7 will set $z \leftarrow u$. Because they execute Line 12 later than the first thread, after Line 12 $t$ will be set to $t_0$ and Line 14 will be executed. So there will only one $t_0$ left in $T'$.

2. If none of the buckets for $t_0$ is empty, Line 7 will never be executed in these threads. So $z$ is equal to zero in Line 12. Then the first thread that execute Line 12 will insert $t_0$ into bucket $h_0[t_0]$ and keep its own $t_0$ in $T'$, while the other threads will remove their $t_0$ from $T'$.

\[\square\]

References


