Market Share Analysis with Brand Effect

Zhixuan Fang and Longbo Huang

Abstract— In this paper, we investigate the effect of brand in market competition. Specifically, we propose a variant Hotelling model where companies and customers are represented by points in an Euclidean space, with axes being product features. N companies compete to maximize their own profits by optimally choosing their prices, while each customer in the market, when choosing sellers, considers the sum of product price, discrepancy between product feature and his preference, and a company's brand name. A company's brand name influence is modeled by a function of its market area in the form of $-\beta \cdot (\text{Market Area})^q$, where β is the weight of brand influence and q captures how market share affects the brand. By varying the parameters β and q, we derive existence results of Nash equilibrium and equilibrium market prices and shares. In particular, we prove that pure Nash equilibrium always exists when $q = 0$ for markets with either one and two dominating features, and it always exists in a single dominating feature market when market affects brand name linearly, i.e., $q = 1$. Moreover, we show that at equilibrium, a company's price is proportional to its market area over the competition intensity with its neighbors, a result that quantitatively reconciles the common belief of a company's pricing power. We also study an interesting "wipe out" phenomenon that only appears when $q > 0$, which is similar to the "undercut" phenomenon in the Hotelling model, where companies may suddenly lose the entire market area with a small price increment. Our results offer novel insight into market pricing and positioning under competition with brand effect.

I. INTRODUCTION

According to Gartner's recent report [18], Samsung suffered a significant market share drop in smartphones in year 2014, i.e., dropped from a dominant 32.1% market share in 2013 Q3 to only 24.4% in 2014 Q3. During the same time, Apple gained a mild share growth from 12.1% to 12.7%. While market fluctuations are normal, somewhat surprisingly, this occurred when Samsung decreased the average selling price (ASP) from $235 to $209 of its smartphones (targeting customers who prefer low-end phones) while Apple steadily increased its ASP (targeting customers who prefer high-end phones) [20] [19]. This phenomenon appears to contradict with the common belief that lowering selling prices is an efficient way for boosting market share. Motivated by this phenomenon, we carry out our study on market area equilibrium based on a general model.

In our model, companies are represented by points in an Euclidean space, where each axis denotes a particular product feature. Consumers are located uniformly in the same space and their coordinates specify their feature preference. Each company sets its product price. Consumers determine their choice of companies based on the summation of the product price, discrepancy between product feature and his desire, and the brand name of the company. In this paper, we mainly focus on one-dimensional and two-dimensional Euclidean market space, i.e., one-feature or two-feature, since consumers in the real world often consider only one or two major features when choosing commodities. For instance, the appearance and performance for smartphone customers, or as in the Hotelling model, where two-dimension setting is also enough to describe a real world's transportation. Even so, note that many of our results can adapt to higher dimension. Different from location-price game models such as [15], where location-selection is part of a strategy, our model assumes pre-determined locations (features).[1]

Then, in order to study the effect of brand names, we propose a novel brand effect measure, which has the form of *Brand effect* $= -\beta \cdot S^q$, where S is a company's market area, or market share under normalization, q denotes how the market share contributes to the brand name, and β represents the degree to which customers consider the brand names when making decisions.[2] By varying the values of β and q, we model different markets where market share has different effect on the brand name and where customers can have varying respect for the brand.

Note that our model is not restrictive. Consumers' decisions are often complex and are affected by various factors. Therefore, the exact degree to which brand name affects them has been a subject of continuous investigation, e.g., [3] and [4]. However, we already know that market share is often considered a major valid reflection of the brand's value [2]. Thus, we use the market share, or market area to reflect the brand effect. Our model is motivated by the classic Hotelling model [13] [6], where the total price a consumer considers is the sum of product price and the distance between consumer and company, based on linear or quadratic distance function.

Under our model, we first show the existence of Nash equilibrium among N companies for the zero brand effect case, i.e., $q = 0$ in both single and dual feature market. Then, we prove the existence for the linear brand effect case, i.e.,

[1]Such a model approximates the situation when companies are constrained by various factors and have limited flexibility in choosing product features. Doing so also allows us to focus on the market space structure, pricing and brand effect, three important aspects that have not been jointly studied before in the market equilibrium context.

[2]In practice, different companies may have different β values. Here we use the same β value, so as to simplify the analysis without sacrificing the economic insight.

$q = 1$, in single feature market. We also derive properties of a company's market area and pricing strategy at equilibrium. In particular, we show that for any company j, the relation between its market area and pricing at equilibrium takes the following concise form :

$$\text{Price}_j \propto \frac{\text{Market}_j}{\text{Competition Intensity}}. \quad (1)$$

Moreover, competition intensity increases as β grows. This result gives an explicit characterization of the equilibrium price and market area.

Besides giving the description of the equilibrium, we also discover and analyze an interesting "wipe out" phenomenon that can occur in the market when β exceeds certain threshold value: some companies' market area will immediately shrink to zero with an arbitrarily small price increment. This kind of "sudden death" phenomenon is very similar to the "undercut phenomenon" in the Hotelling model with linear distance function.

The main contributions of this work are summarized as follows.

- To the best of our knowledge, we are the first to connect brand name effect to the market space and analyze the market equilibrium in this case.
- We develop techniques to prove the existence of market equilibrium when agents' utility functions are piecewise continuous. This enables us to handle the situations when neighbor competitors change dynamically.
- We show that one company's pricing power is proportional to its market area at market equilibrium and is inversely proportional to competition intensity.
- We develop the dimensionality reduction reasoning approach that allows us to extend the problem into higher dimension.

The organization of the rest of this paper is as follow: Section II reviews the existing literature. Section III presents our model and setting, and Section IV gives the market struture analysis results. Section V and VI have the results for cases $q = 0$ and $q = 1$. Conclusions come at the last. Due to space limit, some proofs are omitted. Interested readers please find them in our online technical report [1].

A. Notations

Bold symbol $\mathbf{x}$ denotes a vector and capital italic symbol $\mathcal{X}$ denotes a set. $\mathbf{cl}(\mathcal{X})$ and $\mathbf{int}(\mathcal{X})$ denote the closure and the interior of $\mathcal{X}$. We use $X = |\mathcal{X}|$ to denote its volume, e.g., if $\mathcal{S}$ denotes the set of points in a unit circle on a plane, $S = |\mathcal{S}|$ denotes its area. Exclusion of a set is denoted by $\mathcal{X}/\mathcal{A} = \{x \,|\, x \in \mathcal{X}, x \notin \mathcal{A}\}$.

II. RELATED WORK

Market analysis has been a subject with extensive studies. In Hotelling's seminal paper [13], an one-dimensional two-seller market with linear shipping cost was considered. It is concluded in the paper that companies follow the "Principle of Minimum Differentiation" in competition, which states

that companies choose similar strategy in location and pricing. Later, [6] pointed out that location-price equilibrium may not always exist under the linear cost model, and a quadratic distance function was used instead to guarantee the existence of equilibrium. The paper also proved the principle of maximal differentiation holds, which indicates that companies tend to produce products with opposite characteristics.

Since then a lot of works have been done to deepen our understanding about market equilibrium. [5] proved the existence of mixed strategy of this two-stage game, and [16] proved the existence of subgame perfect equilibrium in pure strategy in an one-dimensional torus market. In [21], the market was extended to two-dimensional and they showed that maximal differentiation in one dimension is enough for existence of equilibrium. [15] extended the market to multi-dimensions and showed that firms tend to maximize the difference in the dominant dimension but minimize differentiation in other dimensions. [17] also showed similar results for a duoplay two-dimensional market.

Besides the existence results, properties of market area have also received a lot of attentions. Boundary of market with respect to different distance costs was studied in [9], [14], and [12]. Researchers also studied the brand effect in [23], where it was divided into two types as inertial and cost-based, while [22] and [10] studied the brand choice through repeated learning and Markov process. The main difference between previous studies on brand effects and our work is that we use market area to reflect brand effect, which is by far the first to connect market space and brand name in analyzing market equilibrium, while previous works usually treat the brand name effect and market share separately.

III. MODEL AND PRELIMINARIES

Consider an abstract market modeled by the K-dimensional Euclidean space $\mathcal{M} = \mathbb{R}^K$, where each axis represents one feature of the products in consideration. For example, $\mathcal{M} = \mathbb{R}^2$ can represent the smartphone market, where one coordinate can be the camera quality and the other coordinate can be the CPU speed of the phone. As another example, $\mathcal{M}$ can represent the market in the most traditional way as in the Hotelling model [13], in which case coordinates denote the locations of companies and customers.

We use $\mathcal{N}'$ to denote the set of companies in the market. Each company's coordinates in the market are fixed and are represented by a point $\mathbf{x_k} \in \mathcal{M}, \forall k \in \mathcal{N}'$. Without loss of generality, we assume that companies are placed all over the market.[3] For simplicity, we assume that all companies produce products different only in features we consider, with zero cost and no limit on production capacity. Since there are infinitely many companies in the market, we only focus on some chosen companies among them. Specifically, let $\mathcal{M}_{chosen}(B) = \{\mathbf{x} \,|\, \|\mathbf{x}\|_\infty \leq B/2\}$ be a cube in $\mathcal{M}$ with edge length being B and use $\mathcal{N}$ to denote companies inside

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[3]That is, for all $\mathbf{x_0} \in \mathcal{M}$, $\exists r < \infty$ such that there exists at least one company in the ball $\{\mathbf{x} \,|\, \|\mathbf{x} - \mathbf{x_0}\|_2 < r\}$

$\mathcal{M}_{chosen}(B)$, i.e.,

$$\mathcal{N} = \{k \,|\, \mathbf{x}_k \in \mathcal{M}_{chosen}(B), k \in \mathcal{N}'\}$$

We focus on the companies located inside $\mathcal{M}_{chosen}(B)$[4] and choose B so that $N = |\mathcal{N}| > 0$. In other words, we assume an unbounded market with infinite companies, but discuss only those inside a bounded set. Doing so allows us to eliminate the boundary effect as in [13] while not ignoring the competition between companies reside near the boundaries.[5]

We denote P_i the product price of a company i, which is the price it charges for its product. We assume that $P_i \in [0, P_{upper}]$, where P_{upper} is some sufficiently large but finite price upper bound for every company. To gain accurate result on the chosen objects, in the following, we assume that product prices of companies outside $\mathcal{M}_{chosen}(B)$ always remain constant and we only consider the competition among companies in $\mathcal{M}_{chosen}(B)$. That is,

$$P_j \equiv P_{j0}, \forall j \in \mathcal{N}'/\mathcal{N}.$$

We then use $\mathbf{P} = (P_i, i \in \mathcal{N})$ to denote the price vector for the companies in consideration.

The points in $\mathcal{M}$ denote the set of customers, where each coordinate specifies a customer's desired value of that feature. Then, each customer will choose to purchase one unit product from a company that provides maximum satisfaction. For instance, customers often first care about the product price when choosing companies. Then, they will also take into account the personal preferences, e.g., smartphone buyers may prefer the ones that match his usage requirements, while coffee buyers may prefer shops at closer distances. Furthermore, they may also value the brand name of a company when making decisions. To model this behavior, we denote $P_i\{\mathbf{x}\}$ the *aggregate* price that customer $\mathbf{x}$ sees from company i (its components will be specified later) and assume that customers always purchase at companies with the lowest aggregate price. By choosing different forms of $P_i\{\mathbf{x}\}$, one can take into account various factors.

With the aggregate prices, we define the *ownership* of a customer and the *market area* of a company as follows:

Definition 1: (Ownership) $\mathcal{O}'(\mathbf{x}, \mathbf{P})$ is the set of companies who offer the *lowest* aggregate price at any $\mathbf{x} \in \mathcal{M}$, i.e.,

$$\mathcal{O}'(\mathbf{x}, \mathbf{P}) = \{i \,|\, i \in \mathbf{argmin}\, P_i\{\mathbf{x}\}\}. \quad (2)$$

In other words, customers always choose to purchase goods at companies in the local ownership set.

Definition 2: (Market area) Company i's market area is given by $S_i = |\mathcal{S}_i|$, where $\mathcal{S}_i$ is the set of customers for which i is the *unique* owner, i.e.,

$$\mathcal{S}_i = \{\mathbf{x} \,|\, \mathcal{O}'(\mathbf{x}, \mathbf{P}) = \{i\}\}. \quad (3)$$

[4]Note that there are many ways to delimit the area, we use $\mathcal{M}_{chosen}(B)$ for simplicity in presentation, since it becomes a square in 2D market.

[5]Note that this is important. As an example, suppose we define a city block as $\mathcal{M}_{chosen}(B)$. Then, customers live near the edge will go to the coffee shop at the next block if they find that more worthwhile. Hence, companies at the corners still face competition from those outside the bounded set.

In (3) we only consider customers that strictly prefer company i to others. Later we will see that this is not restrictive, as almost all customers choose only one company. The boundary of company i's market area is denoted by $\mathcal{BR}(i) = \mathbf{cl}(\mathcal{S}_i)/\mathbf{int}(\mathcal{S}_i)$.

Definition 3: (Surviving companies) A company i is called *surviving* if $S_i > 0$.

Similarly, surviving owners of $\mathbf{x}$, denoted by $\mathcal{O}(\mathbf{x}, \mathbf{P})$, is the set of companies who provide the lowest price among all surviving companies at $\mathbf{x}$, i.e.,

$$\mathcal{O}(\mathbf{x}, \mathbf{P}) = \{i \,|\, i \in \mathbf{argmin}_{i \in \{j|S_j > 0\}} P_i\{\mathbf{x}\}\}.$$

In this work, unless otherwise stated, we focus on the following general $P_i\{\mathbf{x}\}$ function:

$$P_i\{\mathbf{x}\} = P_i + D(\mathbf{x}, \mathbf{x_i}) - \beta S_i^q, \quad \beta, q \geq 0. \quad (4)$$

Here the first term in $P_i\{\mathbf{x}\}$ is company i's mill price. The second term $D(\mathbf{x}, \mathbf{x_i})$ is a distance function, which captures customer $\mathbf{x}$'s disutility due to the difference between his preferred product feature profile and that offered by company i. The last term models the *brand effect*, i.e., how much a company's brand name helps in attracting customers. The parameter q captures how brand is related to the market area (or market share), and β denotes how much customers value the brand name when making purchase decisions. By varying the values of q and β, our model captures a wide range of brand effect in a market.[6]

The objective of each company i is to choose its mill price P_i, so as to maximize its revenue, i.e.,

$$W_i = P_i \cdot S_i. \quad (5)$$

For tractability, in this paper, we restrict our attentions mainly to the single-feature market $\mathcal{M} = \mathbb{R}$ (1D) and the dual-feature market $\mathcal{M} = \mathbb{R}^2$ (2D). We also use the common L_2 Euclidean norm as the distance measure, i.e.,

$$D(\mathbf{x}, \mathbf{x_i}) = ||\mathbf{x} - \mathbf{x_i}||_2^2. \quad (6)$$

Note that linear and pure quadratic distance functions are commonly adopted for market analysis, e.g., [13], [6]. We also use $d_{ij} = D(\mathbf{x_i}, \mathbf{x_j})$ to denote the distance between i and j.

We remark here that our model is different from the classic Hotelling model [13], where companies also choose their locations as part of their strategies. It has been shown in [6] that no equilibrium exists under that model, even with two companies with linear distance costs. We instead focus on the case when company locations (features) are predetermined. This models markets where locations cannot be arbitrarily determined by companies.

We now define the following notions that will be used repeatedly.

Definition 4: (Neighbor) Neighbors of i are the surviving owners of $\mathcal{BR}(i)$, i.e., $\mathcal{NR}(i) = \{j \,|\, j \neq i, \exists \mathbf{x} \in \mathcal{BR}(i), \text{ s.t. } j \in \mathcal{O}(\mathbf{x}, \mathbf{P})\}$.

[6]Here we assume that the β value is the same for all customers.

Definition 5: (Nash Equilibrium) Given the company price vector $\mathbf{P}^* = (P_i^*, i \in \mathcal{N}) = (P_i^*, P_{-i}^*)$, we say that the market is at pure strategy Nash equilibrium if

$$P_i^* = \mathbf{argmax}_{P_i} W(P_i, P_{-i}^*), \quad \forall i \in \mathcal{N}.$$

That is, no company can improve its profit by changing its price unilaterally.

Since we only study pure strategy Nash equilibrium, we will use Nash equilibrium for short in the rest of the paper. It is clear that the *Individual Rationality (IR)* condition always holds since $W_i = P_i \cdot S_i \geq 0$ for all $i \in \mathcal{N}$. Besides, when $\beta = 0$, every company can survive by setting a very small price and gaining the customers close to its position. For general positive β values, however, some companies may have zero market area even when they adopt a zero price.

Companies compete directly with their neighbors on market share. Meanwhile, we define the notion of a potential competitor, which will be useful for our later analysis.

Definition 6: (Potential competitor) Company j is called a *potential competitor* of company i if it satisfies any one of the following conditions:
  (a) If $S_j = 0$, there exists $\mathbf{x} \in \mathbf{cl}(\mathcal{S}_i)$, such that $P_j\{\mathbf{x}\} = P_i\{\mathbf{x}\}$.
  (b) If $j \in \mathcal{NR}(i)$, $\frac{|\mathbf{cl}(\mathcal{S}_j) \cap \mathbf{cl}(\mathcal{S}_i)|}{|\mathcal{BR}(i)|} = 0$.
In the case when Condition (a) holds, with an increment in P_i, j may survive and become an actual competitor of i. Condition (b) can appear in the 2D market where j is a neighbor of i, yet the length of their border line is zero, i.e., their market area only intersect at one point. Similarly, i may start facing a direct competition from j if P_i increases. The existence of potential competitors captures the general dynamics of different markets. However, it also greatly complicates the problem and requires a very different analysis from previous work.

Last, we have the following theorem from [11] [7] [8], which will be used later for proving the existence of Nash equilibrium.

Theorem 1 ([11][7][8]): For a game with compact convex strategy space, if each player i's payoff function $W_i(\mathbf{P})$ is continuous in $\mathbf{P}$ and quasiconcave in P_i, there exists a pure strategy Nash equilibrium.

IV. MARKET AREA AND OWNERSHIP ANALYSIS

We now present our results for market area and ownership. Unless otherwise stated, results in this section apply to general K-dimensional markets.

Lemma 1: For any customer $\mathbf{x}$ in the market , there exists at least one surviving company who owns him, i.e., $|\mathcal{O}(\mathbf{x}, \mathbf{P})| > 0$ for all $\mathbf{x} \in \mathcal{M}$.

Lemma 1 shows that $\mathcal{O}(\mathbf{x}, \mathbf{P}) \subseteq \mathcal{O}'(\mathbf{x}, \mathbf{P})$. Thus, each customer has at least one surviving owner.

We now have the first main theorem of the paper:

Theorem 2: For any surviving company i in the market, $\mathcal{S}_i' = \{\mathbf{x} \,|\, i \in \mathcal{O}'(\mathbf{x}, \mathbf{P})\}$ is a convex polyhedron, and $\mathcal{S}_i' = \mathbf{cl}(\mathcal{S}_i)$.

Proof: Consider the surviving companies. We only show the case in 2D, since situations in other dimensions are

similar. First note that the set of market area where company i and j provide the same price is a straight line (hyperplane in high dimension) given by:

$$
\begin{aligned}
\mathcal{L}_{i,j} \quad &= \{(x,y) | P_i + (x - x_i)^2 + (y - y_i)^2 - \beta S_i \\
&\quad - [P_j + (x - x_j)^2 + (y - y_j)^2 - \beta S_j] = 0\} \\
&= \{(x,y) | 2x(x_i - x_j) + 2y(y_i - y_j) - P_i + P_j \\
&\quad - \beta S_j + \beta S_i + x_j^2 + y_j^2 - x_i^2 - x_j^2 = 0\}
\end{aligned}
$$

This line divides the plane into 2 open half space $\mathcal{H}_i(i,j)$, $\mathcal{H}_j(i,j)$, where in $\mathcal{H}_i(i,j)$ we have $P_i\{(x,y)\} < P_j\{(x,y)\}$ and in $\mathcal{H}_j(i,j)$ we have $P_j\{(x,y)\} < P_i\{(x,y)\}$. Thus, for any point $\mathbf{x} = (x,y) \in \mathcal{H}_i(i,j)$, we have $j \notin \mathcal{O}(\mathbf{x})$ and vice versa. Let $\mathcal{U}_i = \cap_{j \in \mathcal{N}, j \neq i} \mathcal{H}_i(i,j)$, since $\mathcal{U}_i$ is the intersection of half spaces, $\mathcal{U}_i$ is a convex polygon.

Now we prove that $\mathcal{U}_i = \mathcal{S}_i$. For any $\mathbf{x} \in \mathcal{S}_i$, we have $\mathcal{O}(\mathbf{x}) = \{i\}$. Hence, $\mathbf{x} \in \mathcal{H}_i(i,j), \forall j \neq i$. Thus, $\mathbf{x} \in \cap_{j \in \mathcal{N}, j \neq i} \mathcal{H}_i(i,j) = \mathcal{U}_i$ and $\mathcal{S}_i \in \mathcal{U}_i$. For any $\mathbf{x} \in \mathcal{U}_i$, we have $j \notin \mathcal{O}(\mathbf{x})$ if $j \neq i$. Since $\mathcal{O}(\mathbf{x}) \neq \emptyset$, $\mathcal{O}(\mathbf{x}) = \{i\}$, $\mathcal{U}_i \in \mathcal{S}_i$ and $\mathcal{U}_i = \mathcal{S}_i$.

It remains to show that $\mathcal{S}_i' = \mathbf{cl}(\mathcal{U}_i)$. For any $\mathbf{x} \in \mathbf{cl}(\mathcal{U}_i)$, we have $P_i\{\mathbf{x}\} \leq P_j\{\mathbf{x}\}, \forall j \neq i$. Hence, $i \in \mathcal{O}(\mathbf{x})$ and $\mathbf{x} \in \mathcal{S}_i'$, which implies $\mathbf{cl}(\mathcal{U}_i) \in \mathcal{S}_i'$. Now consider any $\mathbf{x} \in \mathcal{S}_i'$. Since for all $\mathbf{x}' \in \mathbf{cl}(\mathcal{U}_i^C)$, where $\mathbf{cl}(\mathcal{U}_i^C) = \mathcal{M}/\mathbf{cl}(\mathcal{U}_i)$ is the complementary set of $\mathbf{cl}(\mathcal{U}_i)$, we have $i \notin \mathcal{O}(\mathbf{x}')$. Thus, we must have $\mathbf{x} \in \mathbf{cl}(\mathcal{U}_i)$ and $\mathcal{S}_i' \in \mathbf{cl}(\mathcal{U}_i)$.

In conclusion, $\mathcal{S}_i' = \mathbf{cl}(\mathcal{U}_i)$, and $\mathcal{S}_i' = \mathbf{cl}(\mathcal{S}_i)$. ∎

From Theorem 2 we know that in the 1D market, each surviving company's market consists of only one bounded continuous interval, while in the 2D market, it consists of a polygon.

V. MARKET EQUILIBRIUM WHEN $q = 0$

In this section, we present our results for the 1D and 2D markets with $q = 0$, i.e., brand names do not affect customers. In this case, the aggregate price of company i simplifies to:

$$P_i\{\mathbf{x}\} = P_i + ||\mathbf{x} - \mathbf{x_i}||_2^2 - \beta, \quad \beta \geq 0. \tag{7}$$

A. Single-feature (1D) market

In this case the market is an infinite line and $\mathcal{N}$ denotes the set of companies in some arbitrarily chosen nonempty interval $[-B/2, B/2]$, with finite number of companies $N > 0$. For simplicity, we denote $d_i \triangleq d_{i,i+1} = x_{i+1} - x_i$. From Theorem 2, we can use $\mathcal{S}_i = (L_i, R_i)$ to denote each surviving company i's market area, where $L_i \triangleq \inf\{x : x \in \mathcal{S}_i\}$ and $R_i \triangleq \sup\{x : x \in \mathcal{S}_i\}$ are the boundary points.

Theorem 3: In the 1D market, Nash equilibrium always exists when $q = 0$.

Theorem 3 is proved by showing the utility function of a company is exactly a parabola. Before proving the theorem, we first have the following lemma.

Lemma 2: In the 1D market, let x_i denote the i-th surviving company in increasing order of their coordinates. We have $L_i = R_{i-1}$ for all $i \in \mathcal{N}_S(\mathbf{P})$.

Proof: (Theorem 3) From (7) we know that when $q = 0$, the brand effect factor will be the same to every companies, regardless of the β value. Thus, we consider $\beta = 0$.

For any companies i, other companies' aggregate prices at $\mathbf{x_i} = x_i$ are always positive due to distance. Hence, for any companies i, given P_{-i}, i can always set a price P_i such that $S_i > 0$. Therefore, every company will survive in the market.

For company i, by Theorem 2 and Lemma 2, we know that i's market area will be an interval on the line, with neighbors always being $i-1, i+1$, which satisfies $x_{i-1} < x_i < x_{i+1}$. The right boundary of $i-1$ and i can be calculated as:

$$P_i + (R_{i-1} - x_i)^2 = P_{i-1} + (R_{i-1} - x_{i-1})^2$$
$$P_i + (R_i - x_i)^2 = P_{i+1} + (R_i - x_{i+1})^2$$

Thus, company i's utility W_i is given by:

$$W_i = P_i S_i = P_i(R_i - R_{i-1})$$
$$= -P_i^2 \left(\frac{1}{2d_i} + \frac{1}{2d_{i-1}}\right)$$
$$+ P_i \left(\frac{P_{i+1} + x_{i+1}^2 - x_i^2}{2d_i} - \frac{P_{i-1} - x_i^2 + x_{i-1}^2}{2d_{i-1}}\right). \quad (8)$$

Hence, the profit will be a downward parabola for $P_i \in \left[0, \frac{\frac{P_{i+1} + x_{i+1}^2 - x_i^2}{2d_i} - \frac{P_{i-1} - x_i^2 + x_{i-1}^2}{2d_{i-1}}}{\frac{1}{2d_i} + \frac{1}{2d_{i-1}}}\right]$ and zero otherwise.

The payoff of i is continuous in $\mathbf{P}$, quasiconcave in P_i. From theorem 1, Nash equilibrium always exists. ∎

Theorem 4: In the 1D market with $q = 0$, when market is at equilibrium, we must have:

$$P_i = \frac{2d_i d_{i-1}}{d_i + d_{i-1}} S_i, \forall i \in \mathcal{N}. \quad (9)$$

Proof: When the market is at equilibrium, we have:

$$\frac{dW_i}{dP_i} = S_i + P_i \frac{dS_i}{dP_i} = 0$$

and

$$\frac{dS_i}{dP_i} = \frac{1}{2d_i} + \frac{1}{2d_{i-1}}.$$

Combining these two equations proved the theorem. ∎

Theorem 4 tells us that a company's price is proportional to its market area at equilibrium. The coefficient $\frac{2d_i d_{i-1}}{d_i + d_{i-1}}$ is a constant for one company since their locations are fixed. Denote $\frac{1}{\gamma} = \frac{2d_i d_{i-1}}{d_i + d_{i-1}}$, or $\gamma = \frac{1}{2}(\frac{1}{d_i} + \frac{1}{d_{i-1}})$. Notice that with bigger d_i, d_{i-1} values, we will have a smaller γ, which implies higher equilibrium prices with the same market area. Therefore, γ can be viewed as competition intensity, i.e., farther distance between companies mitigates the competition and increase companies' profits. This is similar to the maximal differentiation principle [6], which means that companies should not choose similar positions in the market, i.e. larger d_i values. The simple form in Theorem 4 that equilibrium price is determined by market area over competition intensity appears to match our intuition that companies with more market share or less competition in products usually have more pricing power.
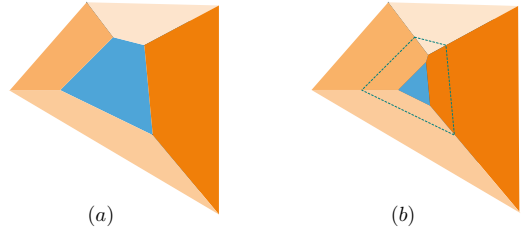

Fig. 1. (a) Company i (the one in the middle with blue colour) has 4 neighbors at price P_i; (b)When i increases its price, its market area shrinks from the dotted quadrangle to the blue triangle, and it has only 3 neighbors now.
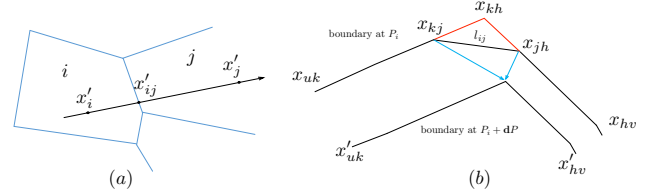

Fig. 2. (a) Set an axis with $\mathbf{x}_i, \mathbf{x}_j$ on it so that we turn the 2D problem into 1D. Here x'_i, x'_j, and x'_{ij} denote the coordinates of $\mathbf{x}_i, \mathbf{x}_j$ and boundary line l_{ij} on the axis, respectively. (b) When i increases its price from P_i to $P_i + dP$, boundary between i and j disappears.

B. Dual-feature (2D) market

We now turn to the 2D case. The biggest problem of analyzing 2D market is that companies' neighbors may change during their price change (as showed in Figure1), while in 1D market, company i's neighbors will always be $i-1$ and $i+1$. Due to the change in neighbors, companies' utility functions will be piecewise, i.e., utility function changes everytime a neighbor comes or goes. Moreover, since companies' locations are given arbitrarily, the shape of an company's market area may be irregular, which makes the analysis more difficult.

Our method to overcome these difficulties is to transform the problem into 1D. Figure 2 shows such an example. We set an axis on two neighboring companies i and j, i.e., they are both on the axis as shown in Figure 2 (a). Since the border line is straight and perpendicular to the axis, the position of the border line can be described in one dimension axis by variable x'_{ij}. The direction of the axis is determined by the way such that $x'_i < x'_{ij} < x'_j$. Note that there is only one way to determine this axis once the direction is set, since d_{ij} is constant. In this case, we have our second main theorem in the paper.

Theorem 5: In the 2D market, Nash equilibrium always exists when $q = 0$.

Proof: (Sketch) We prove Theorem 5 as follows. We first show that $S_i(P_i)$ and $W_i(P_i)$ are continuous and that $W_i(P_i)$ is twice differentiable within each continuous piece. Then, we calculate $\frac{d^2 W_i}{dP_i^2}$, which turns out to be downward parabola in each pieces:

$$\frac{d^2 W_i}{dP_i^2} = -\sum_{j \in \mathcal{NB}(i)} \frac{6\tilde{M}_{ij} P_i^2 + 5\tilde{N}_{ij} P_i + C_{ij,1}^2 + C_{ij,2}^2}{d_{ij} l_{ij}}.$$
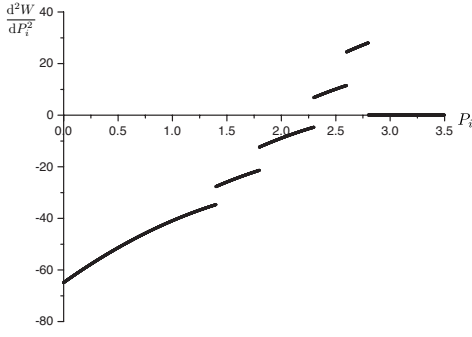
Fig. 3. An example of $\frac{\mathrm{d}^2 W_i}{\mathrm{d}P_i^2}$. Each piece of $\frac{\mathrm{d}^2 W_i}{\mathrm{d}P_i^2}$ is a fragment from an increasing downward parabola, therefore $\frac{\mathrm{d}^2 W_i}{\mathrm{d}P_i^2}$ is a monotonically increasing function on $[0, P_{upper}]$.

Here $\tilde{M}_{ij}$, $\tilde{N}_{ij}$ are related to the location of i and j, $C_{ij,1}$, $C_{ij,2}$ are related to the location of two end points of i and j's border, and l_{ij} denotes the length of border line between i and j (See technical report [1] for more details). We then show its monotonicity, that is, the functions are monotone within each differential interval and monotone at each discontinuous point (Fig. 3 shows an example). Specifically, we first show that each pieces is on the left side of its axis of symmetry and prove monotonicity inside each pieces. For the discontinuous point between two pieces, corresponding situation is shown in Fig. 2 (b): i and j's boundary l_{ij} disappears when i increases its price from P_i to $P_i + \mathrm{d}P$, i.e., i's boundary changes from $\overline{\mathbf{x}_{uk}\mathbf{x}_{kj}\mathbf{x}_{jh}\mathbf{x}_{hv}}$ to $\mathbf{x}'_{uk}\mathbf{x}'_{kh}\mathbf{x}'_{hv}$. We have applied triangle inequality on the triangle $\triangle \mathbf{x}_{kh}\mathbf{x}_{kj}\mathbf{x}_{jh}$ to show that $\frac{\mathrm{d}^2 W_i}{\mathrm{d}P_i^2}$ increases after the change of neighbors. Lastly, we prove that company i's payoff function is quasi-concave in P_i and then prove the existence of equilibrium. ∎

This theorem guarantees the existence of equilibrium in 2D market, regardless of the change in each company's neighbors, or the change in the shape and location of their market area.

Theorem 6: In the 2D market with $q = 0$, when the market is at Nash equilibrium, we have:

$$P_i = \frac{1}{\sum_{j \in \mathcal{NR}(i)} \frac{l_{ij}}{2d_{ij}}} S_i, \quad \forall i \in \mathcal{N}. \tag{10}$$

The factor $\gamma = \sum_{j \in \mathcal{NR}(i)} \frac{l_{ij}}{2d_{ij}}$ represents the competition intensity. For a company i, farther distance to competitors (bigger d_{ij}) can reduce the competition intensity, while longer contiguous border (smaller l_{ij}) increases it.

VI. MARKET EQUILIBRIUM WHEN $q = 1$

In this section, we discuss the situation when $q = 1$, i.e., when the market area has a linear relationship with the brand name. We show that the interesting "wipe out" phenomenon appears when $q > 0$ (In particular, it happens when $q = 1$).

The "wipe out" phenomenon substantially increases the difficulty in analyzing the problem, because in this case a company's market area can suddenly shrinks to zero after

some threshold price. In this case, its neighbors' utility functions are not continuous. This is exactly the same problem as in the classic Hotelling model, where "undercut" destroys the continuity of the utility function, and therefore lead to the non-existence of equilibrium.

Since this case is very different from the previous cases due to the "wipe out" phenomenon, we first discuss it below before presenting the existence and necessary conditions for equilibrium. We focus on the 1D market and $q = 1$.

A. The "Wipe-out" Phenomenon

Fix $P_{-i} = (P_1, ...P_{i-1}, P_{i+1}, ...P_N)$ and consider $P_i = 0$. If $S_i(P_i = 0) = 0$, then $W_i(P_i) = 0$ for all $P_i \in [0, P_{upper}]$ and it is concave. If $S_i(P_i = 0) > 0$, let us gradually increase P_i from zero and consider a price $P_i \in [0, P_{upper}]$ during this process. Denote its surviving left and right neighbors by $i - 1, i + 1$, and denote R_i as the right boundary of i. Recall that in this case, $P_i\{R_i\} = P_i + (R_i - x_i)^2 - \beta S_i$. Thus, we have:

$$\begin{aligned} P_i\{R_i\} &= P_{i+1}\{R_i\} \\ P_i\{R_{i-1}\} &= P_{i-1}\{R_{i-1}\} \end{aligned} \tag{11}$$

Suppose now i changes its price by a small amount $\mathrm{d}P_i$, i.e., $P_i' = P_i + \mathrm{d}P_i$, such that $i - 1, i + 1$ still remain being its neighbors, i.e., they still survive. Denote $\mathrm{d}R_i$ and $\mathrm{d}R_{i-1}$ the deviations of the right and left boundaries of i. The change of market area for i can then be written as $\mathrm{d}S_i = \mathrm{d}R_i - \mathrm{d}R_{i-1}$, and the price of i at new boundary is:

$$\begin{aligned} P_i'\{R_i + \mathrm{d}R_i\} &= P_i + \mathrm{d}P_i + (R_i + \mathrm{d}R_i - x_i)^2 \\ &\quad - \beta(S_i + \mathrm{d}S_i). \end{aligned}$$

Similar to (11), we have two equations for the new boundaries:

$$\begin{aligned} P_i'\{R_i + \mathrm{d}R_i\} &= P_{i+1}'\{R_i + \mathrm{d}R_i\} \\ P_i'\{R_{i-1} + \mathrm{d}R_{i-1}\} &= P_{i-1}'\{R_{i-1} + \mathrm{d}R_{i-1}\}. \end{aligned} \tag{12}$$

Subtracting (11) from (12), we have:

$$\begin{aligned} \mathrm{d}S_i(1 - \frac{\beta}{2d_i} - \frac{\beta}{2d_{i-1}}) &= -\mathrm{d}P_i(\frac{1}{2d_i} + \frac{1}{2d_{i-1}}) \\ &\quad - \beta\frac{\mathrm{d}S_{i+1}}{2d_i} - \beta\frac{\mathrm{d}S_{i-1}}{2d_{i-1}}. \end{aligned} \tag{13}$$

In fact, when we gradually change P_i from 0 to P_{upper}, S_i will decrease. This is because with any small increment in P_i, at least customers at i's boundary will change to purchase at a different company. Due to area decrease, i will lose even more customers. Similar analysis shows that $i + 1$ and $i - 1$ will have a positive market area deviation shown in (13).

Hence, the right hand side of (13) and $\mathrm{d}S_i$ in the left hand side will remain negative when $\mathrm{d}P_i > 0$, implying that $1 - \frac{\beta}{2d_i} - \frac{\beta}{2d_{i-1}} > 0$, or equivalently $\beta < \frac{2d_i d_{i-1}}{d_i + d_{i-1}}$. Since during the price increment, i's area will shrink and i' neighbors will change, once a close neighbor of i with $\frac{2d_i d_{i-1}}{d_i + d_{i-1}} < \beta$ gains a positive market area (the new neighbor is denoted by $i - 1$), i's area will immediately shrink to 0, i.e., "wipe out" happens.

Let us try to understand the meaning of upper bound $\frac{2d_i d_{i-1}}{d_i+d_{i-1}}$ for β. Suppose we hide company i from the market just for the moment, i.e., i is invisible to customers. Since $i-1$ and $i+1$ remain activated, they will share i's previous market area. Let R' denote $i-1$ and $i+1$'s boundary, and $\tilde{P}_{i-1}\{x\}$ denote the aggregate price of $i-1$ at x when i is hidden. Since their price at boundary will be the same, i.e., $\tilde{P}_{i-1}\{R'\} = \tilde{P}_{i+1}\{R'\}$, and note that $x_{i+1} - x_{i-1} = d_i + d_{i-1}$, we have:

$$R' = \frac{P_{i+1} - P_{i-1} - \beta S_{i+1} + \beta S_{i-1}}{2(d_i + d_{i-1})} + \frac{x_{i+1} + x_{i-1}}{2} \quad (14)$$

Now let i come back to the market. If i's price at R' is higher than the price of $i-1$ or $i+1$ here, i.e., $\tilde{P}_{i+1}\{R'\}$, or $\tilde{P}_{i-1}\{R'\}$, then i will have no market area. This is because if $P_i\{R'\} > \tilde{P}_{i-1}\{R'\}$, for any $\delta > 0$, at market point $R' - \delta$, we have $P_i\{R'-\delta\} - \tilde{P}_{i-1}\{R'-\delta\} = P_i\{R'\} - \tilde{P}_{i-1}\{R'\} + 2d_{i-1}\delta > 0$, similarly we always have $P_i\{R' + \delta\} > \tilde{P}_{i+1}\{R'+\delta\}$, which means i will have no market area either at $x < R'$ or $x > R'$.

Consider the price gap between i and $i-1$ at R', by equation (14), we have:

$$\tilde{P}_{i-1}\{R'\} - P_i\{R'\} = \frac{2}{d_i + d_{i-1}}\Psi, \quad (15)$$

where $\Psi \triangleq \frac{P_{i+1}+d_i^2-\beta S_{i+1}-P_i}{2d_i} + \frac{P_{i-1}+d_{i-1}^2-\beta S_{i-1}-P_i}{2d_{i-1}}$. If $\Psi < 0$, which means $P_i\{R'\} > \tilde{P}_{i-1}\{R'\}$, accroding to the analysis above, i will have no market area. Hence, if i survives, we must have $\Psi > 0$. When i is activated in the market, with those two equations in (11), we have

$$(1 - \frac{\beta}{2d_i} - \frac{\beta}{2d_{i-1}})S_i = \frac{P_{i+1} + d_i^2 - \beta S_{i+1} - P_i}{2d_i} \quad (16)$$
$$+ \frac{P_{i-1} + d_{i-1}^2 - \beta S_{i-1} - P_i}{2d_{i-1}}$$
$$= \Psi \quad (17)$$

Interestingly, Ψ from (15) appears in (17) again. From (17), it is now clear that if i survives, we must have $S_i > 0$ and $\Psi > 0$. Hence, we must have $(1 - \frac{\beta}{2d_i} - \frac{\beta}{2d_{i-1}}) > 0$, i.e., $\beta < \frac{2d_i d_{i-1}}{d_i+d_{i-1}}$, if i is not wiped out.

B. Existence of Equilibrium

So far we have introduced the "wipe out" phenomenon and explained the bound for β in two ways. Now we show the existence of equilibrium.

Theorem 7: Nash equilibrium always exists in the 1D market with $q = 1$.

Proof: (Sketch)(a) To prove this theorem, we first show that when there is no "wipe out," i.e., β is small such that $\beta < \frac{2d_i d_{i-1}}{d_i+d_{i-1}}, \forall i$, Nash equilibrium exists. Under the no "wipe out" condition, one can guarantee that the utility function of a company is a piecewise continuous function. The main difficulty in proving the quasi-concavity here is in showing that there exists a threshold price, such that the utility function is non-decreasing before the price exceeds it, and non-increasing after.

(b) For the case when $\beta < \frac{2d_i d_{i-1}}{d_i+d_{i-1}}$ does not hold for all i, we construct an equilibrium with a set of carefully chosen "activated" companies, while the others are considered "hidden." Then, we show that this partial equilibrium among the chosen companies turns out to be exactly the equilibrium for all companies. We can operate in the following way to give at least one equilibrium. We choose some companies to be hidden, which means they can be seen as not existed in the market, while the others being "activated." For simplicity in presentation below, we define a condition among those activated companies:

$$\beta < \frac{2d_i d_{i-1}}{d_i + d_{i-1}} \quad \forall \text{ activated company } i. \quad (18)$$

The way we choose is as follow: (i) Hide all companies first. Then activate companies one by one until we can not activate any one more companies without bringing "wipe out" phenomenon into the market. That is, all activated companies satisfied the condition (18), but activating any one of the hidden companies will violates it. (ii) Among all activation schemes in (i), we can guarantee that there exists at least one scheme such that for any one hidden company, say j, it violates the inequality $\beta < \frac{2d_j d_{j-1}}{d_j+d_{j-1}}$. This is because if $\beta < \frac{2d_j d_{j-1}}{d_j+d_{j-1}}$ is satisfied for j, according to (i), there must exists one or two of its neighbors, say k, such that $\beta > \frac{2d_k d_{k-1}}{d_k+d_{k-1}}$, in that case, we can always activate j and hide k. (iii) Now that all activated companies satisfy the condition (18), while each hidden company is being "wiped out".

Then for activated companies there exists equilibrium as proved in (a). Now let those hidden companies come back to the market with their prices being P_{upper}. From the analysis of equation (15), we can see that hidden company i will never survive by unilateral price change. For any activated company j, since hidden companies are being "wiped out"(at the status of price being P_{upper} and have no market area), they can never survive by j's unilateral action. Hence, j's price will remain unchanged. This setting guarantees at least one equilibrium when $\beta < \frac{2d_i d_{i-1}}{d_i+d_{i-1}}, \forall i$ does not hold. ∎

C. Necessary condition for equilibrium

Theorem 8: When a market with dimension K is at a Nash equilibrium, for any company i, if it has no potential competitors,

$$P_i^* = c_i \cdot S_i^*, \quad \forall i \in \mathcal{N}_S \quad (19)$$

where $c_i = -\frac{dP_i}{dS_i}|_{P_i=P_i^*} > 0$.
Otherwise, we have:

$$c_i' \cdot S_i^* \leq P_i^* \leq c_i'' \cdot S_i^*, \quad \forall i \in \mathcal{N}_S. \quad (20)$$

Here $c_i' = -\frac{dP_i}{dS_i}|_{P_i=P_i^{*+}} > 0$, $c_i'' = -\frac{dP_i}{dS_i}|_{P_i=P_i^{*-}} > 0$. Inequality (20) becomes an equality only when i's potential competitor does not survive at P_i^*.
In particular, when the market is 1D or 2D, and $\beta << d_i$ holds for all i, we have: $c_i = -\frac{dP_i}{dS_i}|_{P_i=P_i^*} \approx \frac{1-\beta \sum_{j\in\mathcal{NR}(i)} \frac{l_{ij}}{2d_{ij}}}{\sum_{j\in\mathcal{NR}(i)} \frac{l_{ij}}{2d_{ij}}(\frac{\beta l_{ij}}{2d_{ij}}+1)}$.

Equation (19) is similar to Theorem 4, where the equilibrium price is proportional to market area. Inequality (20) shows that potential competitors restrict the pricing power of a company, which results in lower selling price and hence lower profit compare to (19). Since potential competitors are closer than actual competitors in market, once they survive, they can cause significant harm to a company's market area and profit, due to the close similarity in product features and high substitutability.

In addition, a company with larger market area can widen the market share gap by expanding the dimension of market, i.e., actively compete with weaker competitors using products different in multi-features. This is because market area gap in high dimension can be much bigger than the cases in 1D market, which in return will result in bigger aggregate price gap.

VII. Conclusion

In this paper, we study equilibrium properties based on a variant Hotelling model, considering brand name effect in the market by including a market area term into aggregate price. We prove the existence of Nash equilibrium in single-feature and dual-feature market, and also derive explicit characterizations of equilibrium prices and market areas. Our results reconcile the common belief that company's pricing power is proportional to its market area over competition intensity, and offer insight into pricing under brand name effect and market positioning.

Specifically, our results offer the following insights: (i) When there is no brand effect or equivalent brand effect, i.e., β or q is zero, the equilibrium price of a company is proportional to its market area (market power) over the competition intensity with its neighbors (boundary over distance). This implies that lowering prices does not necessarily lead to market share increase, as observed in the smartphone market, where the competition intensity may be higher for customers who prefer low-end phones. (ii) When brand name has a positive effect in attracting customers, it is important to lower the price and seize more market area. Meanwhile, companies with big market areas should try to steer customers' shopping habits towards emphasizing on brand reputation (increase brand factor). (iii) New companies should try to avoid markets where the brand factor is large, in which case big companies are at strong advantage, and to avoid positioning at market points where competition is intense (many nearby companies), where due to the "wipe out" phenomenon, small companies may not be able to survive.

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