Stochastic Combinatorial Optimization via Poisson Approximation

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Outline

- Threshold Probability Maximization
- Stochastic Knapsack
- Other Results
Threshold Probability Maximization

- **Deterministic version:**
  - A set of elements \( \{e_i\} \), each associated with a weight \( w_i \)
  - A solution \( S \) is a subset of elements (that satisfies some property)
  - **Goal:** Find a solution \( S \) such that the total weight of the solution \( w(S) = \sum_{i \in S} w_i \) is minimized
  - E.g. shortest path, minimal spanning tree, top-k query, matroid base
Threshold Probability Maximization

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- **Stochastic version:**
  - \( w_i \)'s are independent positive random variables
  - **Goal:** Find a solution \( S \) such that the *threshold probability* 
    \[ \Pr[w(S) \leq 1] \] is maximized.
Related Work

*Studied extensively before:*

- Many heuristics
- **Stochastic shortest path** [Nikolova, Kelner, Brand, Mitzenmacher. ESA’06] [Nikolova. APPROX’10]
- **Fixed set stochastic knapsack** [Kleinberg, Rabani, Tardos. STOC’97] [Goel, Indyk. FOCS’99] [Goyal, Ravi. ORL09][Bhalgat, Goel, Khanna. SODA’11]
- ...
- **Chance-constrained (risk-averse) stochastic optimization problem** [Swamy. SODA’11]
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- …..
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*A common challenge:* How to deal with/ optimize on the distribution of the sum of several random variables.

*Previous techniques:*
- LP [Dean, Goemans, Vondrak. FOCS’04]
- Discretization [Bhalgat, Goel, Khanna. SODA’11],
- Characteristic function [Li, Deshpande. FOCS’11]
Our Result

- If the deterministic problem is “easy”, then for any $\epsilon > 0$, we can find a solution $S$ such that

$$\Pr[w(S) \leq 1 + \epsilon] > OPT - \epsilon$$

“Easy”: there is a PTAS for the multi-dimensional version of the problem: Shortest path, MST, matroid base, matroid intersection, min-cut (strictly generalizing the result in [Li, Deshpande. FOCS’11])

- The above result can be generalized to the expected utility maximization problem:

$$\text{maximize } E[\mu(X(S))] \text{ for Lipschitz utility } \mu$$
Our Algorithm

• Step 1: Discretizing the prob distr
  (Similar to [Bhalgat, Goel, Khanna. SODA’11], but much simpler)

• Step 2: Reducing the problem to the multi-dim problem
Our Algorithm

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The behaviors of $\tilde{X}_i$ and $X_i$ are close:

1. $\Pr[X(S) \leq \beta] \leq \Pr[\tilde{X}(S) \leq \beta + \epsilon] + O(\epsilon)$;
2. $\Pr[\tilde{X}(S) \leq \beta] \leq \Pr[X(S) \leq \beta + \epsilon] + O(\epsilon)$. 
Our Algorithm

- Step 2: Reducing the problem to the multi-dim problem
  - Heavy items: $E[X_i] > \text{poly}(\epsilon)$
    - At most $O(1/\text{poly}(\epsilon))$ many heavy items, so we can afford enumerating them
Our Algorithm

• Step 2: Reducing the problem to the multi-dim problem
  • Heavy items: $E[X_i] > \text{poly}(\epsilon)$
    • At most $O(1/\text{poly}(\epsilon))$ heavy items, so we can afford enumerating them
  • Light items:
    • Each $X_i$ can be represented as a $O(1)$-dim vector $Sg(i)$ (signature)
      \[
      Sg(i) = (\Pr[\tilde{X}_i = \epsilon^4], \Pr[\tilde{X}_i = \epsilon^4 + \epsilon^5], \ldots) \]
    • Enumerating all $O(1)$-dim (budget) vectors $B$
      • Find a set $S$ such that
        \[
        Sg(S) = \sum_{i \in S} Sg(i) \leq B \quad \text{(using the multi-dim PTAS)}
        \]
  • Return $S$ for which $\Pr[w(S) \leq 1 + \epsilon]$ is largest
Poisson Approximation

Well known: Law of small numbers

\( n \) Bernoulli r.v. \( X_i (p, 1-p) \), \( np = \text{const} \)

As \( n \to \infty \), \( \sum X_i \sim \text{Poisson}(np) \)
Poisson Approximation

(Somewhat less well-known)

**Le Cam’s theorem:**

$n$ r.v. $X_i$ (with common support $(0,1,2,3,4,...)$)

$p_i = \Pr[X_i \neq 0], \lambda = \sum p_i, q_j = \sum \Pr[X_i = j]$

$Y_i$ is a r.v. with distr $(0, \frac{q_1}{\lambda}, \frac{q_2}{\lambda}, \frac{q_3}{\lambda}, \frac{q_4}{\lambda}, \ldots)$

$Y$ is a **compound Poisson distr** (*CPD*)

$$\sum_{i=1}^{N} Y_i \text{ where } N \sim \text{Poisson}(\lambda)$$

$$\Delta(\sum X_i, Y) \leq \sum p_i^2$$

Variational distance:

$$\Delta(X, Y) = \sum |\Pr[X = i] - \Pr[Y = i]|$$
Poisson Approximation

- **Le Cam’s theorem:** $\Delta(\sum X_i, Y) \leq \sum p_i^2$

- If $S_1$ and $S_2$ have the same signature, then they correspond to the same CPD

- So if $\sum_{i \in S_1} p_i^2$ and $\sum_{i \in S_2} p_i^2$ are sufficiently small, the distributions of $X(S_1)$ and $X(S_2)$ are close

- Therefore, enumerating the signature of light items suffices
Outline

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Stochastic Knapsack

- A knapsack of capacity C
- A set of items.
- Known: Prior distr of (size, profit) of each item.
- Items arrive one by one
- Irrevocably decide whether to accept the item
- The actual size of the item becomes known after the decision
- Knapsack constraint: The total size of accepted items $\leq C$
- Goal: maximize $E[\text{Profit}]$
Stochastic Knapsack

Previous work

- 5-approx [Dean, Goemans, Vondrak. FOCS’04]
- 3-approx [Dean, Goemans, Vondrak. MOR’08]
- \((1+\epsilon, 1+\epsilon)\)-approx [Bhalgat, Goel, Khanna. SODA’11]
- 2-approx [Bhalgat 12]
- 8-approx (size\&profit correlation, cancellation) [Gupta, Krishnaswamy, Molinaro, Ravi. FOCS’11]

Our result:

\((1+\epsilon, 1+\epsilon)\)-approx (size\&profit correlation, cancellation)
2-approx (size\&profit correlation, cancellation)
Stochastic Knapsack

- Decision Tree

Exponential size!!!! (depth=n)
How to represent such a tree? Compact solution?
Stochastic Knapsack

- By discretization, we make some simplifying assumptions:
  - Support of the size distribution: \((0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots, 1)\).
  - All prob. values are in the form of \(k/M\), \((k<M\) and \(M=\text{poly}(n))\)
  - Profit of each item \(i\) is a fixed value

Still way too many possibilities, how to narrow the search space?
Block Adaptive Policies

- Block Adaptive Policies: Process items block by block

Key Properties:
1. Depth = O(1)
2. Degree = O(1)
So #nodes = O(1)
Note: O(1) depends on $\epsilon$

LEMMA: [Bhalgat, Goel, Khanna. SODA’11] There is a block adaptive policy that is nearly optimal (under capacity $(1 + \epsilon)C$)
Block Adaptive Policies

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1. Depth=$O(1)$
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Still exponential many possibilities, even in a single block

LEMMA: [Bhalgat, Goel, Khanna. SODA'11] There is a block adaptive policy that is nearly optimal (under capacity $(1 + \epsilon)C$)
Poisson Approximation

- Each heavy item consists of a singleton block
- Light items:
  - Recall if two blocks have the same signature, their size distributions are similar
  - So, enumerate Signatures! (instead of enumerating subsets)

\[
S_g = S_g(\text{item2}) + S_g(\text{item3})
\]

\[
\text{CPD}(S_g) \sim \text{size}(\text{item2}) + \text{size}(\text{item3})
\]
Algorithm

- Outline: Enumerate all block structures with a signature associated with each node

- O(1) nodes
- Poly(n) possible signatures for each node
- So total configuration = poly(n)
Algorithm

2. Find an assignment of items to blocks that matches all signatures
   - (this can be done by standard dynamic program)
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On any root-leaf path, we can select one choice for each item
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Other Results

- Incorporating other constraints
  - Size/profit correlation
  - Cancellation
- Bayesian Online Selection Problem with Knapsack Constraint
  - Can see the actually size and profit of an item before the decision
  - \((1+\varepsilon, 1+\varepsilon)\)-approx (against the optimal adaptive policy)

  ✓ Prophet inequalities [Chawla, Hartline, Malec, Sivan. STOC10] [Kleinberg, Weinberg. STOC12]
  ✓ Close relations with Secretary problems
  ✓ Applications in multi-parameter mechanism design

- Stochastic Bin Packing
Conclusion

- Using Poisson approximation, we can often reduce the stochastic optimization problem to a multi-dimensional packing problem
- More applications
Thanks

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