

Universal Work Fluctuations During Shortcuts to Adiabaticity by Counterdiabatic Driving

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Counterdiabatic driving (CD) exploits auxiliary control fields to tailor the nonequilibrium dynamics of a quantum system, making possible the suppression of dissipated work in finite-time thermodynamics and the engineering of optimal thermal machines with no friction. We show that while the mean work done by the auxiliary controls vanishes, CD leads to a broadening of the work distribution. We derive a fundamental inequality that relates nonequilibrium work fluctuations to the operation time and quantifies the thermodynamic cost of CD in both critical and noncritical systems.

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Understanding the far-from-equilibrium dynamics of quantum systems is an open problem at the frontiers of physics. Yet, tailoring such dynamics is a necessity for the advancement of quantum technologies. This challenge is fully embodied within the field of quantum thermodynamics with potential applications in energy science. Our understanding of nonequilibrium behavior of thermodynamic systems has deepened profoundly owing to fluctuation theorems (e.g., the Jarzynski equality) and stochastic thermodynamics [1–6]. For example, the Jarzynski equality [1] has been used to find the equilibrium free energy of a system through measurements of the fluctuating nonequilibrium work [7], and quite recently, its quantum version [4,5] has been tested experimentally in a trapped ion system [8]. In an effort to develop control tools to engineer the dynamics of thermal machines, schemes to suppress excitations and work fluctuations have been put forward [9–11]. In any physical implementation, thermal machines such as quantum heat engines and refrigerators must operate in a finite time to achieve a nonvanishing output power. This motivates the study of finite-time thermodynamics that targets the optimization of the trade-off between efficiency and power [12]. In this context, control techniques known as shortcuts to adiabaticity (STA) have emerged as a disruptive paradigm as they reproduce the quantum adiabatic dynamics of the system by suppressing excitations without the requirement of slow driving [13]. STA have been used to boost the performance of quantum heat engines by enhancing its output power at zero friction [14–16] and to suppress work fluctuations, assisting, for example, the convergence of the Jarzynski equality [10].

Assessing the cost of implementing STA arises as a natural question with both fundamental and practical

implications in nonequilibrium statistical mechanics. Among the different techniques to engineer STA [13], counterdiabatic driving stands out as a unifying framework [17–19]. It relies on the use of auxiliary control fields so that the exact evolution along STA is described by the adiabatic approximation to the dynamics of the (uncontrolled) system, even in arbitrarily fast processes. Since the introduction of STA, it has been understood that the amplitude of the auxiliary control field increases as the duration of the STA is reduced [20–24]. In this Letter, we elucidate the thermodynamic cost of counterdiabatic driving by studying how work fluctuations are modified during STA. We show that the work done by the counterdiabatic fields vanishes on average. However, we find that STA modify the work probability distribution and increase work fluctuations, whose amplitude is set by the quantum geometric tensor of the underlying Hilbert space. Furthermore, we derive a fundamental nonequilibrium inequality that relates work fluctuations to the duration of the process.

Shortcuts to adiabaticity by counterdiabatic driving.— Consider a time-dependent Hamiltonian $\hat{H}_0(\lambda_t)$ with instantaneous eigenvalues $\{\epsilon_n(t)\}$ and eigenstates $\{|n(\lambda_t)\rangle\}$, which depend explicitly on a set of parameters $\lambda_t = [\lambda^1(t), \dots, \lambda^N(t)]$. Here, we fix the initial and final parameters to $\lambda_0 = \lambda_i$ and $\lambda_\tau = \lambda_f$, respectively, where τ is the time required to complete the protocol. We pose the problem of driving an initial state $|n(\lambda_i)\rangle$ to a final state in a given finite time τ , so that the final state is given by $|n(\lambda_f)\rangle$. In the following, we simplify the notation of this protocol dependence, e.g., $|n(\lambda_t)\rangle = |n(t)\rangle$. A technique that achieves this goal is the counterdiabatic driving (CD), also known as transitionless quantum driving [17,19], that will

be our focus in this Letter. CD has inspired several experiments [25,26] and has recently been implemented in systems with discrete [27] and continuous variables [28]. Given a protocol λ_t , whenever $\hat{H}_0(t)$ varies slowly, the dynamics of the n th eigenstate $|n(0)\rangle$ in the adiabatic approximation at time t reads

$$|\chi_n(t)\rangle = e^{-(i/\hbar) \int_0^t dt' \epsilon_n(t') - \int_0^t dt' (d\lambda^\mu/dt') \langle n(t') | \partial_\mu n(t') \rangle} |n(t)\rangle, \quad (1)$$

where $\partial_\mu = (\partial/\partial\lambda^\mu)$, and summation over repeated Greek indices is implicit. The first and second terms in the exponent of Eq. (1) correspond to the dynamical and geometric phases, respectively. The primary goal of CD is to find a Hamiltonian \hat{H}_{CD} for which the adiabatic approximation to $\hat{H}_0(t)$ becomes the exact solution of the time-dependent Schrödinger equation for $\hat{H}_{\text{CD}}(t)$, i.e., $\hat{U}_{\text{CD}}(t, 0)|n(0)\rangle = |\chi_n(t)\rangle$, where $\hat{U}_{\text{CD}}(t, 0) := T \exp[-(i/\hbar) \int_0^t dt' \hat{H}_{\text{CD}}(t')]$ is the time evolution operator with T being the time-ordering operator. Direct construction of the time-evolution operator

$$\hat{U}_{\text{CD}}(t, 0) = \sum_n |\chi_n(t)\rangle \langle n(0)| \quad (2)$$

yields an explicit form of $\hat{H}_{\text{CD}}(t)$ [17–19]:

$$\begin{aligned} \hat{H}_{\text{CD}}(t) &= i\hbar[\partial_t \hat{U}_{\text{CD}}(t, 0)] \hat{U}_{\text{CD}}^\dagger(t, 0) = \hat{H}_0(t) + \hat{H}_1(t), \\ \hat{H}_1(t) &= i\hbar \sum_n (|\partial_t n(t)\rangle \langle n(t)| - \langle n(t)| \partial_t n(t)\rangle |n(t)\rangle \langle n(t)|). \end{aligned} \quad (3)$$

We find that $\hat{H}_1(t)$ is the auxiliary term required to drive the system from $|n(0)\rangle$ to $|n(\tau)\rangle$ for all n in a given time τ , maintaining adiabaticity with respect to \hat{H}_0 . As a result, the CD control Hamiltonian $\hat{H}_1(t)$ differs from similar Hamiltonians that appear in the proof of the adiabatic theorem [29]. We shall assume that the auxiliary term is switched off at the initial and final stages of the process, i.e., $\hat{H}_1(0) = \hat{H}_1(\tau) = 0$. We note that the evolution $|\chi_n(t)\rangle$ is nonadiabatic with respect to the full driving Hamiltonian, whose instantaneous eigenstates satisfy

$$\hat{H}_{\text{CD}}(t)|\Psi_n(t)\rangle = E_n(t)|\Psi_n(t)\rangle. \quad (4)$$

Work fluctuations under counterdiabatic driving.—We next study how work fluctuations along CD are modified with respect to a truly adiabatic process [i.e., the limit of slow driving when $\hat{H}_1(t) \rightarrow 0$]. To do so, we introduce a work cost of the system for a microscopic trajectory of the system. Suppose that we start from the canonical distribution whose occupation probability in the energy

eigenstate $|n(0)\rangle$ is $p_n^0 = \exp[-\beta\epsilon_n(0)]/Z$, where $Z = \sum_n \exp[-\beta\epsilon_n(0)]$ is the partition function. If we observe a trajectory starting from $|n(0)\rangle$ and find the state of the system to be $|\Psi_k(t)\rangle$ at time t , the probability of obtaining that trajectory is given by

$$\begin{aligned} p_n^0 p_{n \rightarrow k}^t &:= p_n^0 |\langle \Psi_k(t) | \hat{U}_{\text{CD}}(t, 0) | n(0) \rangle|^2 \\ &= p_n^0 |\langle \Psi_k(t) | n(t) \rangle|^2, \end{aligned} \quad (5)$$

and the work cost along that trajectory is given by $E_k(t) - \epsilon_n(0)$. Note that we need to perform two projective energy measurements at times $t' = 0$ and t to obtain the probability distribution (5), which is referred to as the two-point measurement scheme [4–6]. The explicit expression for the work probability distribution $P[W(t)]$ reads

$$P[W(t)] := \sum_{k,n} p_n^0 p_{n \rightarrow k}^t \delta\{W(t) - [E_k(t) - \epsilon_n(0)]\}. \quad (6)$$

In the truly adiabatic limit, H_1 vanishes and $\hat{H}_{\text{CD}}(t) = \hat{H}_0(t)$ for all t , and the transition probability becomes the Kronecker delta: $p_{n \rightarrow k}^t = \delta_{k,n}$. As a result, the work probability distribution takes the form

$$P_{\text{ad}}[W(t)] = \sum_n p_n^0 \delta[W(t) - W_{\text{ad}}^{(n)}(t)], \quad (7)$$

where $W_{\text{ad}}^{(n)}(t) := \epsilon_n(t) - \epsilon_n(0)$ is the work cost along the adiabatic trajectory.

Because $\hat{H}_{\text{CD}}(\tau) = \hat{H}_0(\tau)$, we obtain $p_{n \rightarrow k}^\tau = |\langle k(\tau) | n(\tau) \rangle|^2 = \delta_{k,n}$ and thus $P[W(\tau)] = P_{\text{ad}}[W(\tau)]$. In particular, an initial thermal state $\rho(0) = \sum_n p_n^0 |n(0)\rangle \langle n(0)|$ evolves into $\rho(\tau) = \sum_n p_n^0 |n(\tau)\rangle \langle n(\tau)|$. Therefore, at the end of the protocol, all properties about W for CD become equivalent to those for the adiabatic dynamics. However, $P[W(t)]$ and $P_{\text{ad}}[W(t)]$ are different at the intermediate stage. In what follows, we analyze the deviations of the mean and variance of work during CD from those of the adiabatic dynamics for arbitrary $0 \leq t \leq \tau$.

We first show that the average work cost along the CD evolution is always equal to the adiabatic value,

$$\langle W(t) \rangle = \langle W(t) \rangle_{\text{ad}}. \quad (8)$$

We note that from the instantaneous Schrödinger equation for $H_{\text{CD}}(t)$ it is possible to derive

$$[E_m(t) - \epsilon_n(t)] \langle n(t) | \Psi_m(t) \rangle = \langle n(t) | H_1 | \Psi_m(t) \rangle. \quad (9)$$

An explicit evaluation of the right-hand side leads to

$$\langle n(t) | H_1 | \Psi_m(t) \rangle = i\hbar \dot{\lambda}^\mu \sum_{k(\neq n)} \langle n(t) | \partial_\mu k(t) \rangle \langle k(t) | \Psi_m(t) \rangle.$$

Multiplying both sides of this equation by $\langle \Psi_m(t) | n(t) \rangle$ and summing over the quantum number m labeling the instantaneous eigenstate of the full driving Hamiltonian, we find

$$\sum_m P_n^t P_{n \rightarrow m}^t [E_m(t) - \varepsilon_n(t)] = 0, \quad (10)$$

and in particular

$$\sum_{n,m} P_n^0 P_{n \rightarrow m}^t \{ [E_m(t) - E_n(0)] - [\varepsilon_n(t) - \varepsilon_n(0)] \} = 0, \quad (11)$$

which proves the equality between the mean work under CD and the adiabatic work, i.e., Eq. (8). Therefore, we find no difference between CD and the adiabatic dynamics in terms of the mean work. However, as we shall see below, CD alters work fluctuations with respect to the adiabatic dynamics, and we identify this difference as the thermodynamic cost to implement CD.

We next characterize the CD work fluctuations with respect to the adiabatic trajectory. Taking the absolute square of Eq. (9), multiplying by P_n^0 , and summing over n and m , we have

$$\begin{aligned} & \sum_{n,m} P_n^0 P_{n \rightarrow m}^t [E_m(t) - \varepsilon_n(t)]^2 \\ &= \hbar^2 \sum_n P_n^0 \langle \partial_\mu n(t) | P_n^\perp | \partial_\nu n(t) \rangle \dot{\lambda}^\mu \dot{\lambda}^\nu \\ &= \hbar^2 \sum_n P_n^0 g_{\mu\nu}^{(n)} \dot{\lambda}^\mu \dot{\lambda}^\nu, \end{aligned} \quad (12)$$

where $P_n^\perp = 1 - P_n$ is the projector onto the space orthogonal to that spanned by the state $|n(t)\rangle$ with $P_n = |n(t)\rangle\langle n(t)|$ being the projector onto $|n(t)\rangle$. In Eq. (12), we have identified the metric $g_{\mu\nu}^{(n)} = \text{Re} Q_{\mu\nu}^{(n)}$ with the real part of the quantum geometric tensor of the $|n(t)\rangle$ -state manifold introduced by Provost and Vallee [30]

$$Q_{\mu\nu}^{(n)} := \langle \partial_\mu n(t) | P_n^\perp | \partial_\nu n(t) \rangle. \quad (13)$$

Note that Eq. (13) dictates the quadratic decay of the square root of the fidelity between two states $|n(t)\rangle$ and $|n(t + \delta t)\rangle$, i.e., $\sqrt{F(|n(t)\rangle, |n(t + \delta t)\rangle)} := |\langle n(t) | n(t + \delta t) \rangle| = 1 - g_{\mu\nu}^{(n)} \dot{\lambda}^\mu \dot{\lambda}^\nu \delta t^2 / 2 + \mathcal{O}(\delta t^3)$. Multiplying Eq. (10) by $2[\varepsilon_n(t) - \varepsilon_n(0)]$, we have

$$2 \sum_{n,m} P_n^0 P_{n \rightarrow m}^t [\varepsilon_n(t) - \varepsilon_n(0)] [E_m(t) - \varepsilon_n(t)] = 0. \quad (14)$$

Adding the above equation to the left-hand side of Eq. (12), we obtain

$$\begin{aligned} \text{LHS} &= \sum_{n,m} P_n^0 P_{n \rightarrow m}^t \{ [E_m(t) - \varepsilon_n(0)]^2 - [\varepsilon_n(t) - \varepsilon_n(0)]^2 \} \\ &= \langle W^2(t) \rangle - \langle W^2(t) \rangle_{\text{ad}}. \end{aligned} \quad (15)$$

By combining Eqs. (8), (12), and (15), we find that

$$\delta(\Delta W)^2 := \text{Var}[W(t)] - \text{Var}[W(t)]_{\text{ad}} = \hbar^2 \sum_n P_n^0 g_{\mu\nu}^{(n)} \dot{\lambda}^\mu \dot{\lambda}^\nu, \quad (16)$$

where $\text{Var}[W] := \int dWP(W)W^2 - [\int dWP(W)W]^2$. This result is remarkable in that it relates the instantaneous excess of work fluctuations $\delta(\Delta W)^2$ to an exclusively geometric quantity, namely, the quantum geometric tensor $g_{\mu\nu}^{(n)}$ induced by the protocol $\lambda(t)$. The excess of work fluctuations quantifies how the work probability distribution along the protocol at time t is broadened with respect to the adiabatic dynamics as a result of the CD protocol.

We define

$$\ell(\rho(0), \rho(\tau)) := \int_{\lambda_0}^{\lambda_\tau} \sqrt{\sum_n P_n^0 g_{\mu\nu}^{(n)} d\lambda^\mu d\lambda^\nu} \quad (17)$$

as a natural distance between $\rho(0)$ and $\rho(\tau)$ under the CD dynamics, which enforces parallel transport for each eigenmode independently. Combining Eqs. (16) and (17), we obtain

$$\tau = \frac{\hbar \ell(\rho(0), \rho(\tau))}{\langle \delta \Delta W \rangle_\tau}, \quad (18)$$

where $\langle f \rangle_\tau := \tau^{-1} \int_0^\tau f(t) dt$ denotes the time average and $\delta \Delta W := \sqrt{\delta(\Delta W)^2}$. As we show in Ref. [31], we can further bound $\ell(\rho(0), \rho(\tau))$ from below as

$$\ell(\rho(0), \rho(\tau)) \geq \mathcal{L}(\rho(0), \rho(\tau)) \quad (19)$$

in terms of the Bures length between two mixed states $\mathcal{L}(\rho, \sigma) := \arccos \sqrt{F(\rho, \sigma)}$, where $F(\rho, \sigma) := \{\text{tr}[(\sqrt{\rho} \sigma \sqrt{\rho})^{1/2}]\}^2$ is the fidelity [32,33]. On the other hand, as we show in Ref. [31], we can bound the excess of the work fluctuations (16) from above as follows:

$$\delta(\Delta W)^2 \leq (\Delta E_{\text{CD}})^2 := \langle H_{\text{CD}}^2 \rangle - \langle H_{\text{CD}} \rangle^2. \quad (20)$$

Combining Eqs. (18), (19), and (20), we obtain

$$\tau \geq \frac{\hbar \mathcal{L}(\rho(0), \rho(\tau))}{\langle \delta \Delta W \rangle_\tau} \geq \frac{\hbar \mathcal{L}(\rho(0), \rho(\tau))}{\langle \Delta E_{\text{CD}} \rangle_\tau}. \quad (21)$$

This result gives a tighter bound on the duration of time τ compared with the Mandelstam-Tamm time-energy

uncertainty relation [32,34,36,37], i.e., $\tau \geq \hbar \mathcal{L}(\rho(0), \rho(\tau)) / \langle \Delta E \rangle_\tau$, with the role of the time-averaged standard deviation of energy $\langle \Delta E \rangle_\tau$ replaced by the time-averaged excess of the work fluctuations $\langle \delta \Delta W \rangle_\tau$. This quantity $\langle \delta \Delta W \rangle_\tau$ captures the enhancement of the fluctuations in work done on or by the system as the duration of the protocol is shortened. We identify this quantity, dictated by the geometry of the Hilbert space, as the thermodynamic cost to implement CD. Therefore, inequality (21) gives a novel type of the quantum speed limit that provides a quantification of the thermodynamic cost of shortening the protocol time τ of STA assisted by CD, which entails an increase in the excess of work fluctuations with respect to the adiabatic limit.

We note that in the zero-temperature limit and λ_t follows the geodesic connecting the initial and final states, the distance ℓ reduces to the Bures length for pure states [31], i.e., $\ell(|\psi(0)\rangle, |\psi(\tau)\rangle) = \mathcal{L}(|\psi(0)\rangle, |\psi(\tau)\rangle)$ with $|\psi\rangle$ denoting the ground state. As a result, the equality is achieved in Eq. (21) for pure states, i.e.,

$$\tau = \frac{\hbar \mathcal{L}(|\psi(0)\rangle, |\psi(\tau)\rangle)}{\langle \delta \Delta W \rangle_\tau}. \quad (22)$$

Trapped-ion implementation.—We now consider a driven harmonic oscillator that can be implemented by a laser-induced potential in a trapped-ion system. Specifically, we consider a single $^{171}\text{Yb}^+$ ion confined in a linear Paul trap, which is a natural platform for the experimental investigation of nonequilibrium work fluctuations and STA [8,28]. In the interaction picture with respect to the quantum harmonic oscillator $\hat{H}_{\text{HO}} = \hbar(\omega_0 - \nu)[\hat{a}^\dagger \hat{a} + (1/2)]$, the effective Hamiltonian of two distinct Raman processes (see the Supplemental Material [31]) can be written as

$$\hat{H}_{\text{eff}}(t) = \hbar\nu(\hat{a}^\dagger \hat{a} + 1/2) + \frac{\hbar}{2} [\Omega_{\text{eff},1}(t)\hat{a}\hat{a} + \Omega_{\text{eff},1}^*(t)\hat{a}^\dagger \hat{a}^\dagger] - \frac{\hbar\Omega_{\text{eff},2}(t)}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}), \quad (23)$$

where ν is the detuning from the two-photon sideband Raman transitions and the annihilation operator $\hat{a} = \sqrt{m\omega_0/(2\hbar)}[\hat{q} + i\hat{p}/(m\omega_0)]$ is defined on the motional mode with the (time-independent) trap frequency ω_0 , and a similar relation holds for the creation operator \hat{a}^\dagger . The strengths of the laser-induced potentials are characterized by $\Omega_{\text{eff},1}(t)$ and $\Omega_{\text{eff},2}(t)$, which can be controlled by tuning the strengths and phases of the Raman laser beams [31]. Note that $\Omega_{\text{eff},2}(t)$ is real and $\Omega_{\text{eff},1}(t)$ can be complex. Setting $\Omega_{\text{eff},1}(t) = -\Omega(t) + [i\dot{\omega}(t)/2\omega(t)]$ and $\Omega_{\text{eff},2}(t) = \Omega(t)$ with the time-dependent frequency $\omega(t) := \sqrt{\nu[\nu - 2\Omega(t)]}$, the above effective Hamiltonian can be rewritten as a sum of the quantum harmonic-oscillator term $\hat{H}_0(t)$ with the effective mass $m_{\text{eff}} = m\omega_0/\nu$ and the auxiliary counterdiabatic field $\hat{H}_1(t)$ [38–40] as follows:

$$\begin{aligned} \hat{H}_{\text{eff}}(t) &= \hat{H}_0(t) + \hat{H}_1(t), \\ \hat{H}_0(t) &= \frac{\hat{p}^2}{2m_{\text{eff}}} + \frac{1}{2}m_{\text{eff}}\omega(t)^2\hat{q}^2, \\ \hat{H}_1(t) &= -\frac{\dot{\omega}}{4\omega}(\hat{q}\hat{p} + \hat{p}\hat{q}). \end{aligned} \quad (24)$$

The condition $\hat{H}_1 = 0$ at $t = \{0, \tau\}$ leads to the boundary conditions on $\omega(t)$ that can be satisfied by a polynomial ansatz, such as $\omega(t) = \omega_i + 10\delta s^3 - 15\delta s^4 + 6\delta s^5$, where $\omega_f = \omega_i + \delta$ and $s = t/\tau$. We use this driving protocol to analyze the thermodynamic cost of STA engineered via CD. We numerically calculate $\langle W(t) \rangle$, $\text{Var}[W(t)]$, and $\langle \delta \Delta W \rangle_\tau$ as functions of the evolution time $0 \leq t \leq \tau$ along STA. The results are shown in Fig. 1. In the numerical calculation, we choose $\omega_i = 1$, $\omega_f = 3$ and set $m = \hbar = \beta = 1$. The mean work along STA is shown to match exactly the adiabatic value in Fig. 1(a) in agreement with Eq. (8). As a result, the effect of the fast driving along STA is observed only on the work fluctuations and leads to a broadening of the work probability distribution $P[W(t)]$. The instantaneous variance of $W(t)$ surpasses the adiabatic value by a quantity that is directly related to the quantum geometric tensor as shown in Figs. 1(b) and 1(c). Furthermore, the time-averaged work fluctuations with respect to the adiabatic trajectory are bounded by the Bures length between the initial and final states, $\mathcal{L}(\rho(0), \rho(\tau))$, as dictated by Eq. (21). It follows that the thermodynamic cost of implementing the CD scheme is constrained by the geometry of the Hilbert space. This imposes a fundamental work-time uncertainty relation that

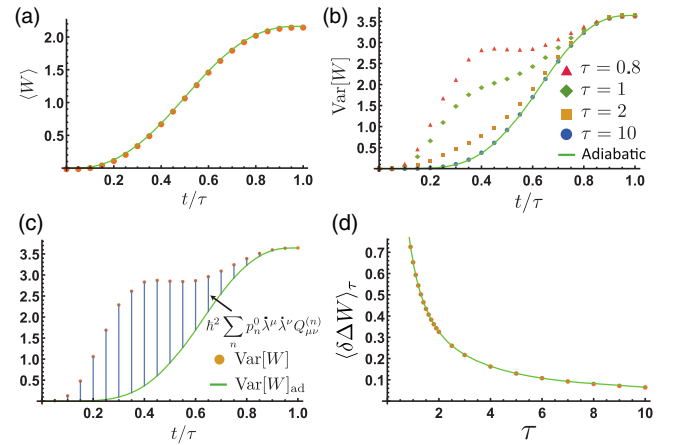


FIG. 1. Thermodynamic cost of counterdiabatic driving (CD). Numerical calculation for a time-dependent harmonic oscillator. (a) Instantaneous value of the average work cost. The orange dots are obtained via CD for $\tau = 0.8$ and the green curve is obtained from the adiabatic protocol. (b) Time dependence of the variance of work for different values of the duration τ of the process. (c) Time dependence of each term in Eq. (16) for $\tau = 0.8$. (d) Time average of the work fluctuations $\langle \delta \Delta W \rangle_\tau$. The green curve is a τ^{-1} fit with the best fit coefficient found to be 0.653, which is larger than the Bures length $\mathcal{L} = 0.476$ consistent with inequality (21).

determines the scaling of work fluctuations with respect to the duration of the process, as shown in Fig. 1(d).

Quantum critical systems.—Equations (16) and (21) have direct implications on the work fluctuations of many-body quantum systems that exhibit quantum phase transitions by varying a parameter λ of the uncontrolled system Hamiltonian $\hat{H}_0(\lambda)$ across a critical value λ_c at which the energy gap between the ground and first-excited state closes. CD can be applied to this situation [21,41–44]. In the neighborhood of the critical point, the emergent conformal symmetry leads to the divergence of the equilibrium correlation length $\xi = \xi_0/|\lambda - \lambda_c|^\nu$, where ν is the correlation-length critical exponent. This power-law behavior sets the scaling of the quantum geometric tensor [45]. As a result, during the STA dynamics induced by CD, the work fluctuations exhibit a universal scaling that can be characterized by both the proximity to the critical point and the system size. While the mean work done by the CD term remains equal to the adiabatic case, work fluctuations diverge in the thermodynamic limit. In particular, when the behavior is dominated by the low-energy excitations, we obtain

$$\delta(\Delta W)^2 \sim \frac{N}{|\lambda - \lambda_c|^{2-\nu d}}, \quad (25)$$

where N denotes the number of particles and d is the dimension of the system. At the critical point, the scaling with the system size reads $\text{Var}[W(\lambda_c)] - \text{Var}[W(\lambda_c)]_{\text{ad}} \sim N^{2/d\nu}$. For the sake of illustration, we consider the quantum Ising chain, which is a prototypical model for quantum phase transitions described by the Hamiltonian

$$\hat{H}_0[\lambda] = - \sum_{n=1}^N (\sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^z), \quad (26)$$

where λ represents an external magnetic field and we consider periodic boundary conditions $\sigma_{N+1}^{x,z} = \sigma_1^{x,z}$. The competition between the two terms in the Hamiltonian leads to a well-known quantum phase transition with $\lambda_c = \pm 1$ between a paramagnetic phase ($|\lambda| > 1$) and a doubly degenerate ferromagnetic phase ($|\lambda| < 1$). The relevant diagonal elements of the quantum geometric tensor for $\hat{H}_0[\lambda]$ have recently been reported in a closed form [46]. We consider the symmetric counterdiabatic driving of the chain, initialized in its ground state, across the critical point $\lambda_c = 1$ with an arbitrary protocol $\lambda(t)$ satisfying $\dot{\lambda} = 0$ at $t = 0$ and τ , and amplitude $2\delta = |\lambda(\tau) - \lambda(0)|$. For example, we take $\lambda(t) = 1 + \delta - 6\delta(t/\tau)^2 + 4\delta(t/\tau)^3$ but the following results hold independently of the concrete form of $\lambda(t)$. During the evolution, the instantaneous work fluctuations shown in Fig. 2 exhibit a pronounced peak in the neighborhood of the critical point, $\lambda \approx \lambda_c$, in agreement with Eq. (25). This scaling leads to a divergence of the time-integrated work fluctuation with the system size N ,

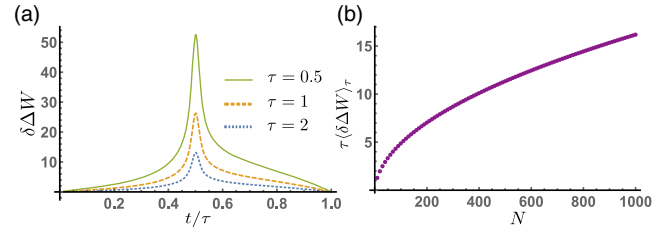


FIG. 2. Work fluctuations across a quantum phase transition. (a) Instantaneous work fluctuations during the CD evolution of a quantum Ising chain as a function of time for different choices of the duration of the protocol $\tau = 0.5, 1$ and 2 with $\delta = 1$. (b) Divergence of the time-integrated work fluctuations with the system size N .

i.e., $\tau \langle \delta\Delta W \rangle_\tau \sim N^\alpha$. A fit to the numerical data leads to the power-law exponent $\alpha = 0.516$, which is consistent with the theoretical value $1/2$. The difference can be attributed to the fact that precisely at the critical point $\lambda = \lambda_c$ the scaling becomes linear in N with $\nu = d = 1$.

Conclusion.—Shortcuts to adiabaticity have recently been proposed as a disruptive paradigm in finite-time thermodynamics to engineer thermal machines that operate at maximum efficiency (zero friction) and arbitrary output power. We have analyzed the thermodynamic cost of implementing the counterdiabatic driving scheme that provides a unifying framework to engineer such shortcuts. In particular, we have shown that the mean work done by the auxiliary counterdiabatic fields vanishes [Eq. (8)], while the work fluctuations are substantially modified [Eq. (16)]. We have derived a fundamental inequality that constrains the enhancement of work fluctuations as a function of the duration of the process [Eq. (21)] and proposed a test with trapped ions to verify our findings. Our work should find broad applications in quantum thermodynamics and potentially in energy science and for the advancement of trapped-ion quantum technology as a test bed for nonequilibrium statistical mechanics.

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