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Article *in* Journal of High Energy Physics · May 2017

DOI: 10.1007/JHEP11(2017)148

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Discrete Gravity on Random Tensor Network and Holographic Rényi Entropy

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In this paper we apply the discrete gravity and Regge calculus to tensor networks and Anti-de Sitter/conformal field theory (AdS/CFT) correspondence. We construct the boundary many-body quantum state $|\Psi\rangle$ using random tensor networks as the holographic mapping, applied to the Wheeler-deWitt wave function of bulk Euclidean discrete gravity in 3 dimensions. The entanglement Rényi entropy of $|\Psi\rangle$ is shown to holographically relate to the on-shell action of Einstein gravity on a branch cover bulk manifold. The resulting Rényi entropy S_n of $|\Psi\rangle$ approximates with high precision the Rényi entropy of ground state in 2-dimensional conformal field theory (CFT). In particular it reproduces the correct n dependence. Our results develop the framework of realizing the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence on random tensor networks, and provide a new proposal to approximate CFT ground state.

PACS numbers:

I. INTRODUCTION

The tensor network is a quantum state of many-body system constructed by contracting tensors according to a network graph with nodes and links (FIG.1). It is originated in condense matter physics because tensor network states efficiently compute the ground states of many-body quantum systems [1, 2]. In addition, tensor networks have wide applications to quantum information theory by its relation to error correction code and quantum entanglement [3], and recently relate to quantum machine learning [4], as well as neuroscience [5].

One of the most fascinating developments of tensor networks is the recent relation to the AdS/CFT correspondence and emergent gravity program [6, 7]. The AdS/CFT correspondence proposes that the quantum gravity theory on d-dimensional Anti-de Sitter (AdS) spacetime is equivalent to a conformal field theory (CFT) living at the $(d-1)$-dimensional boundary of AdS. It offers a dictionary between the observables of the d-dimensional bulk gravity theory and those of the $(d-1)$-dimensional boundary CFT. Properties of the bulk gravity and geometry may be reconstructed or emergent from the boundary CFT, known as the emergent gravity program [8, 9]. As an important ingredient of AdS/CFT, the bulk geometry relates holographically to the entanglement in the boundary CFT, via the Ryu-Takayanagi (RT) formula

$$S_{EE}(A) = \frac{\mathbf{Ar}_{\min}}{4G_N}, \tag{1}$$

which identifies the entanglement entropy $S_{EE}(A)$ of a $(d-1)$-dimensional boundary region A with the area $\mathbf{Ar}_{\min}$ of the bulk $(d-2)$-dimensional minimal surface anchored to A [10–15]. G_N is the Newton constant in d dimensions. $S_{EE}(A)$ satisfying RT formula is referred to as the holographic entanglement entropy (HEE) [16–22]

Tensor networks can be understood as a discrete version of the AdS/CFT correspondence [23–30]. Tensor network states approximate CFT states at the boundary, while the structure of tensor networks emerges an bulk dimension built by layers of tensors. Tensors in the tensor network correspond to local degrees of freedom in the bulk [31–33]. The feature

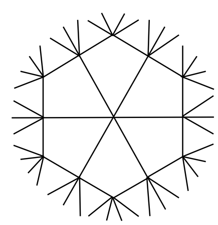

FIG. 1: An example of tensor network with rank-6 tensors. The tensor network state $|\Psi\rangle$ is given by an expansion with certain basis in the Hilbert space of many-body system $|\Psi\rangle = \sum_{\{a_i,\gamma_l\}}\prod_{\mathfrak{p}}(T_{\mathfrak{p}})_{\gamma_1\gamma_2\dots a_i\dots}|a_1, a_2, \cdots a_N\rangle$. The coefficients is constructed by distributing a rank-6 tensor $T_{\mathfrak{p}}$ at each 6-valent node $\mathfrak{p}$, such that each tensor index associates to a link adjacent to $\mathfrak{p}$. Connecting 2 nodes by a link means contracting the corresponding indices γ_l.

of tensor network makes it an interesting tool for realizing the AdS/CFT correspondence constructively from many-body quantum states. Among many recent progress, one of the most interesting results is reproducing HEE on tensor network.

There has been two recent approaches of realizing RT formula on tensor networks, [24] using tensor networks with perfect tensors, and [25] using random tensor networks. Given a boundary region A containing a number of open links of the tensor network, entanglement entropies of tensor networks in both approaches reproduce an analog of RT formula (See e.g. [34–42] for some more recent developments)

$$S_{EE}(A) = \mathrm{Min}(\#_{cut}) \cdot \ln D, \tag{2}$$

where $\mathrm{Min}(\#_{cut})$ is the minimal number of tensor network links cut by a surface anchored to A. D is the bond dimension (range of tensor index). The random tensor network approach has a relation to loop quantum gravity (LQG) and quantum geometry ([43–46] for reviews), which relates Eq.(2) to the geometrical RT formula Eq.(1) [47].

However it is known that both approaches suffer the issue of flat entanglement spectrum. Although the entanglement (Von Neumann) entropy Eq.(2) is consistent with the RT formula, Rényi entropies $S_n(A)$ from both approaches are all identical

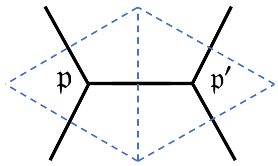

FIG. 2: A simple trivalent tensor network with 2 nodes and 5 links (4 open links). The tensor network is dual to a triangulation with 2 triangles.

to Eq.(2) with trivial n dependence. But the RT formula of Rényi entropy has a nontrivial n dependence since the CFT Rényi entropy does [15]. For instance, in any 2d CFT (CFT_2), the ground state has the universal Rényi entropy [48]

$$S_n(A) = \left(1 + \frac{1}{n}\right) \frac{c}{6} \ln\left(\frac{l_A}{\delta}\right), \tag{3}$$

where c is the central charge, l_A is the length of the region A, and δ is a UV cut-off. It manifests that Rényi entropies of CFT ground state have nontrivial n dependence.

The mismatch of Rényi entropies implies that both tensor network states in [24, 25] fail to approximate the CFT ground state. The reason behind it is not hard to see: In AdS/CFT, the CFT ground state (at strong-coupling) is dual to the bulk semiclassical AdS spacetime geometry. However the tensor networks designed in [24, 25] only consider the geometry of a spatial slice in AdS, without any input about time evolution. On the other hand, in the continuum AdS/CFT context, the correct entanglement spectrum are obtained by considering the spacetime geometry, and taking into account the dynamics given by the Einstein equation [11, 12, 14, 15]. Therefore the issue of entanglement spectrum is equivalent to the issue of dynamical input in tensor networks.

In this work, we resolve the above issue by having dynamical input in random tensor network states. We construct the state $|\Psi\rangle$ which is proposed as an approximation to the CFT ground state. As is anticipated by our proposal, $|\Psi\rangle$ indeed reproduces correctly the RT formula and CFT ground state Rényi entropy S_n with correct n dependence.

In this paper, we focus on 2d CFT and 3d bulk spacetime ($\mathrm{AdS}_3/\mathrm{CFT}_2$) in Euclidean signature. The CFT state $|\Psi\rangle$ is constructed by implementing bulk gravity dynamics to random tensor network states studied in [32]. Random tensor networks constructed in [32] have random tensors at each node $\mathfrak{p}$, and have labels $a_{\mathfrak{p},\mathfrak{p}'}$ on links $(\mathfrak{p},\mathfrak{p}')$. Each $a_{\mathfrak{p},\mathfrak{p}'}$ labels the non-maximal entangled state $|a_{\mathfrak{p},\mathfrak{p}'}\rangle$ on each link. The tensor network is dual to a tiling of 2d spatial slice Σ (FIG.2). The entanglement entropy of $|a_{\mathfrak{p},\mathfrak{p}'}\rangle$ relates to the length L_ℓ of the edge ℓ intersecting $(\mathfrak{p},\mathfrak{p}')$. Thus each random tensor network as boundary CFT state, denoted by $|\vec{a}\rangle$, determines a set of edge lengths L_ℓ in the bulk. When the tiling is a triangulation, edge lengths uniquely determines a discrete geometry on Σ which approximates the continuum. On the other hand, $|\vec{a}\rangle$ form an overcomplete basis in the boundary Hilbert space. So

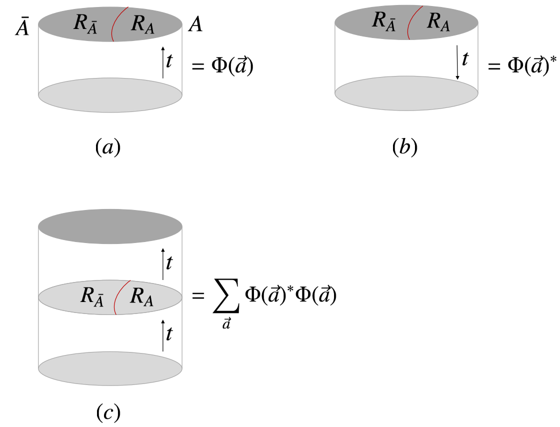

FIG. 3: (Triangulated) 3-manifolds M in (a), $\bar{M}$ in (b), and M_1 in (c).

CFT states are written as

$$|\Psi\rangle = \sum_{\vec{a}} \Phi(\vec{a}) |\vec{a}\rangle. \tag{4}$$

where the coefficients $\Phi(\vec{a})$ can be understood as a wave function of bulk geometry. Eq.(4) is a holographic mapping from the bulk state Φ to the boundary state Ψ.

As mentioned above, CFT ground state is expected dual to a physical state in the bulk which corresponds to semiclassical spacetime geometry. But the labels $\vec{a}$ in $\Phi(\vec{a})$ only relates to the geometry of 2d spatial slice Σ. To let $\Phi(\vec{a})$ encode spacetime geometry, we propose $\Phi(\vec{a})$ to be the Wheeler-deWitt wave function. Namely, $\Phi(\vec{a})$ is a path integral of (Euclidean) Einstein gravity on spacetime M whose boundary contains Σ (FIG.3(a)). The geometry on Σ determined by $\vec{a}$ is the boundary condition of the path integral. Quantum mechanically, $\Phi(\vec{a})$ sums all possible bulk spacetime geometries satisfying the boundary condition. In the semiclassical limit, it localizes at the classical AdS spacetime in 3d. The semiclassical limit relates to the large bond dimension of tensor network.

Since $\vec{a}$ are data of discrete geometry, $\Phi(\vec{a})$ is the discrete version of Wheeler-deWitt wave function: It is a path integral of Regge calculus. Regge calculus is a discretization of Einstein gravity by triangulating spacetime geometries [49]. The discrete spacetime geometry is given by the edge lengths in the triangulation, known as the Regge geometry. $\Phi(\vec{a})$ is a sum over all (Euclidean) Regge geometries on the spacetime M, weighted by the exponentiated Einstein-Regge action [50]. The detailed explanations of $\Phi(\vec{a})$ and random tensor networks are presented in Section II.

The Rényi entropy $\overline{S_n(A)}$ of $|\Psi\rangle$ at arbitrary $n \geq 1$ is computed in Section III. The computation involves averages of the random tensors at nodes $\mathfrak{p}$ in tensor networks [25]. Thanks to the relation between $\Phi(\vec{a})$ and the path integral on M, $\overline{S_n(A)}$ relates to the path integrals of gravity on branch cover 3-manifolds made by $2n$ copies of M. We derives that in the bulk semiclassical limit,

$$\overline{S_n(A)} \simeq \frac{1}{1-n} \left[I_{Bulk}(M_n) - n I_{Bulk}(M_1) \right], \tag{5}$$

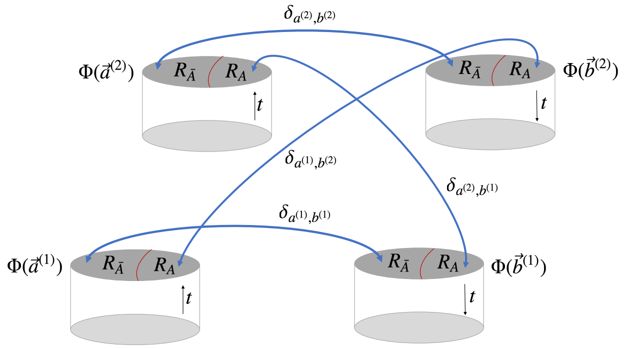

FIG. 4: The (triangulated) manifold M_n ($n = 2$) made by gluing $2n$ copies of M.

where $I_{Bulk}(M_n)$ is the on-shell gravity action evaluated at the bulk solution on the branch cover manifold M_n (FIG.4). The bulk solution has the Z_n replica symmetry. Eq.(5) has been an assumption in AdS/CFT derivations of HEE in e.g. [11, 12, 14, 15]. But it is now derived from $|\Psi\rangle$ and random tensor networks. As a result, we show that $\overline{S_n(A)}$ reproduces the RT formula for holographic Rényi entropy for 2d CFT (Hung-Myers-Smolkin-Yale formula in [14])

$$\overline{S_n(A)} \simeq \left(1+\frac{1}{n}\right)\frac{\mathbf{Ar}_{\min}}{8G_N}. \tag{6}$$

Here $\mathbf{Ar}_{\min}$ is the geodesic length in AdS_3. The above result of $\overline{S_n(A)}$ gives the Rényi entropy Eq.(3) of CFT_2 ground state with correct n dependence.

Section IV analyzes the bound on the fluctuation of Rényi entropy from the random average value $\overline{S_n(A)}$, which shows in the bulk semiclassical limit the fluctuation is generically small.

This work applies discrete geometry method such as Regge calculus to study tensor networks (See [51] for other application of discrete gravity in AdS/CFT). $|\Psi\rangle$ encodes the dynamics of bulk geometries, which is given by the discrete Einstein equation. It is interesting to further understand how the bulk dynamics might relate to the dynamics of boundary CFT, and whether a boundary CFT Hamiltonian might be induced from the bulk dynamics. It is also interesting to compare our proposal of CFT ground state $|\Psi\rangle$ to the existing approach such as multiscale entanglement renormalization ansatz (MERA) [7]. The understanding of these aspects should develop tensor network models to realize the AdS/CFT correspondence at the dynamical level. The research on these aspects is currently undergoing.

II. RANDOM TENSOR NETWORK AND WHEELER-DEWITT WAVE FUNCTION

In this work we consider trivalent random tensor network states. A tensor network is viewed as a discrete 2d spatial slice Σ of 3d bulk spacetime. It is made by a large number of trivalent random tensors $|\mathcal{V}_\mathfrak{p}\rangle \in \mathcal{H}^{\otimes 3} \equiv \mathcal{H}_\mathfrak{p}$ at each tensor network node $\mathfrak{p}$. The Hilbert space $\mathcal{H}$ is of dimension D. We decompose $\mathcal{H}$ into a number of subspaces $\mathcal{H} \simeq \oplus_a V_a$ and denote $\dim(V_a) \equiv d[a]$. Each internal link $(\mathfrak{p},\mathfrak{p}')$ of the tensor network associates with a maximal entangled state in $V_a \otimes V_a$ of certain a,

$$|a_{\mathfrak{p},\mathfrak{p}'}\rangle \equiv \sum_{\mu,\nu} a_{\mu\nu}|\mu\rangle_\mathfrak{p}\otimes|\nu\rangle_{\mathfrak{p}'} = \sum_{\mu\in V_a}\frac{1}{\sqrt{d[a]}}|\mu\rangle_\mathfrak{p}\otimes|\mu\rangle_{\mathfrak{p}'} \tag{7}$$

where $|\mu\rangle_\mathfrak{p}$ is a basis in $\mathcal{H}$. It satisfies $\langle a_{\mathfrak{p},\mathfrak{p}'}|b_{\mathfrak{p},\mathfrak{p}'}\rangle = \delta_{ab}$. A class of random tensor networks $|\vec{a}\rangle$ can be defined by the (partial) inner product between $|\mathcal{V}_\mathfrak{p}\rangle$ at all $\mathfrak{p}$ and $|a_{\mathfrak{p},\mathfrak{p}'}\rangle$ on all internal links

$$|\vec{a}\rangle = \otimes_{\mathfrak{p},\mathfrak{p}'}\langle a_{\mathfrak{p}\mathfrak{p}'}|\otimes_\mathfrak{p}|\mathcal{V}_\mathfrak{p}\rangle. \tag{8}$$

The inner product takes place in $\mathcal{H}$ at each end point $\mathfrak{p}$ or $\mathfrak{p}'$ of each link. $|\vec{a}\rangle$ is a state in the boundary Hilbert space $\mathcal{H}^{N_\partial}$, where N_∂ is the number of open links.

The label $\vec{a}$ relates to the amount of entanglement on each internal link $(\mathfrak{p},\mathfrak{p}')$. The entanglement entropy $S(|a_{\mathfrak{p},\mathfrak{p}'}\rangle)$ of $|a_{\mathfrak{p},\mathfrak{p}'}\rangle$ is $\ln d[a_{\mathfrak{p},\mathfrak{p}'}]$, where $d[a_{\mathfrak{p},\mathfrak{p}'}]$ is effectively the bond dimension on $(\mathfrak{p},\mathfrak{p}')$ in $|\vec{a}\rangle$.

The above class of tensor network states is proposed in [32], in which it is shown that $\{|\vec{a}\rangle\}_{\vec{a}}$ form an overcomplete basis of the boundary Hilbert space. Thus the state in the boundary Hilbert space $|\Psi\rangle$ can be expanded by $\{|\vec{a}\rangle\}_{\vec{a}}$

$$|\Psi\rangle = \sum_{\vec{a}}\Phi(\vec{a})|\vec{a}\rangle. \tag{9}$$

Here we understand the trivalent tensor network to be dual to a triangulation of the spatial slice Σ (FIG.2). Namely, each node $\mathfrak{p}$ located at the center of a triangle $\Delta_\mathfrak{p}$ in the triangulation. Each link $(\mathfrak{p},\mathfrak{p}')$ intersects transversely an internal edge ℓ shared by 2 triangles $\Delta_\mathfrak{p},\Delta_{\mathfrak{p}'}$. Open links in tensor network intersect transversely the edges at the boundary of triangulation.

The label $\vec{a}$ is understood as the discrete geometry in the bulk of Σ [32], in the sense that the edge length L_ℓ of ℓ intersecting $(\mathfrak{p},\mathfrak{p}')$ is proportional to the entanglement entropy $S(|a_{\mathfrak{p},\mathfrak{p}'}\rangle)$ on each link:

$$L_\ell \equiv 4\ell_P \ln d[a_{\mathfrak{p},\mathfrak{p}'}] \tag{10}$$

Here $\ell_P = G_N\hbar$ is the Planck length in 3d. In this proposal, the bulk geometry is understood as emergent from the entanglement in tensor network state. The relation can be obtained from the recent proposal of understanding tensor networks as the effective theory from coarse graining quantum gravity at Planck scale [47], in which one derives that the bond dimension $d[a_{\mathfrak{p},\mathfrak{p}'}]$ of tensor network $|\vec{a}\rangle$ satisfies $d[a_{\mathfrak{p},\mathfrak{p}'}] \simeq e^{L_\ell/4\ell_P}$.

By the relation between $\vec{a}$ and bulk geometry, Eq.(9) is a boundary state by summing over all bulk spatial geometry on Σ, while $\Phi(\vec{a})$ is a wave function of bulk geometry. Eq.(9) defines a holographic mapping from the bulk states of geometry to the boundary states of CFT. We propose the following boundary state $|\Psi\rangle$ whose bulk wave function $\Phi(\vec{a})$ (pre-image of the holographic mapping) is an Wheeler-deWitt wave function in 3d Euclidean gravity. Namely $\Phi(\vec{a})$ is a path integral of gravity on a 3d solid cylinder M, whose boundary includes Σ in addition to the boundary where CFT lives (FIG.3(a)). The geometry $\vec{a}$ is the boundary condition on Σ in the path integral. The path integral may also depend on the boundary conditions at other boundaries of M. But we make those boundary conditions implicit since they play no role in the following analysis.

Since $\vec{a}$ gives a discrete geometry with a set of edge lengths L_ℓ, more precisely, $\Phi(\vec{a})$ is a discrete version of the Wheeler-deWitt wave function. Indeed, we consider a sufficiently refined triangulation of M, and impose discrete metrics on the triangulation. Namely each tetrahedron in the triangulation carries a 3d hyperbolic geometry with constant curvature $-L_{AdS}^{-2}$. Tetrahedron edges are geodesics in the hyperbolic space, and have edge lengths L_ℓ. The set of edge lengths $\{L_\ell\}$ on the triangulation defines a discrete metric of Regge geometry [49, 50, 52]. We define $\Phi(\vec{a})$ to be a path integral of discrete gravity on the triangulation by summing over all L_ℓ in the bulk of M

$$\Phi(\vec{a}) := \sum_{L_\ell} e^{-S_{Regge}(M)} \tag{11}$$

The boundary condition at Σ is $L_{\ell\subset\Sigma} = 4\ell_P \ln d[a_{\mathrm{p,p'}}]$. We use $\sum_{L_\ell}$ instead of integration because L_ℓ are assumed as discrete data, to be consistent with $L_{\ell\subset\Sigma}$. $S_{Regge}(M)$ is the Regge action of Euclidean gravity on the triangulated 3-manifold M evaluated at the discrete metric $\{L_\ell\}$:

$$S_{Regge}(M) = -\frac{1}{8\pi\ell_P}\left[\sum_{\ell\subset\mathrm{bulk}(M)} L_\ell\,\varepsilon_\ell + \sum_{\ell\subset\partial M} L_\ell\,\Theta_\ell - \frac{V(M)}{L_{AdS}^2}\right].$$

ε_ℓ is the bulk deficit angle hinged at the bulk edge ℓ. ε_ℓ is a discretization of the bulk curvature. Each bulk edge ℓ is shared by a number of tetrahedra t. In each t, the dihedral angle between 2 faces joint at ℓ is denoted by $\theta(t,\ell)$. The deficit angle is defined by

$$\varepsilon_\ell = 2\pi - \sum_{t,\,\ell\subset t}\theta(t,\ell), \quad \ell\subset\text{bulk}. \tag{12}$$

Each boundary edge ℓ is shared by 2 boundary triangles. Θ_ℓ is the angle between their outward pointing normals, equivalently

$$\Theta_\ell = \pi - \sum_{t,\,\ell\subset t}\theta(t,\ell), \quad \ell\subset\text{boundary}. \tag{13}$$

Θ_ℓ relates to the boundary extrinsic curvature. The 1st term in S_{Regge} is the discretization of Ricci scalar term of Einstein-Hilbert action, while the 2nd term is the discretization of Gibbons-Hawking boundary term [53]. The last term is the cosmological constant term where $V(M)$ is the total volume of M. All quantities ε_ℓ, Θ_ℓ, and $V(M)$ are determined by edge lengths L_ℓ. The AdS radius L_{AdS} is determined by ℓ_P and the central charge of CFT by $L_{AdS} = \frac{2}{3}c\ell_P$ [54].

In order to be the boundary condition of Regge geometry, $L_{\ell\subset\Sigma} = 4\ell_P \ln d[a_{\mathrm{p,p'}}]$ have to be the edge lengths of hyperbolic triangles, which triangulate Σ. $L_{\ell\subset\Sigma}$ have to be a discrete metric of Σ, which constrains the possible data $\vec{a}$ entering the sum in Eq.(9).

Note that the definition of $\Phi(\vec{a})$ involves the length scale ℓ_P in order to make Regge action dimensionless.

Applying $\Phi(\vec{a})$ in Eq.(11) to the holographic mapping Eq.(9), we obtain a boundary CFT state $|\Psi\rangle$, and we propose the resulting $|\Psi\rangle$ to be the ground state of the boundary CFT, in the bulk semiclassical regime $\ell_P \ll L_\ell$. The motivation of our proposal is the following: As $\ell_P \ll L_\ell$, the bond dimensions are large $\ln d[a] \gg 1$. And the path integral $\Phi(\vec{a})$ localizes at the solution of equation of motion (deriving equation of motion uses the Schläfli identity of hyperbolic tetrahedra $-\delta V(t)/L_{AdS}^2 = \sum_{\ell\subset t} L_\ell \delta\theta(t,\ell)$, see e.g. [52])

$$\varepsilon_\ell = 0, \quad \forall\,\ell\subset\mathrm{bulk}(M). \tag{14}$$

Vanishing ε_ℓ everywhere means that the 3d Regge geometry is a smooth Euclidean AdS_3. So $\Phi(\vec{a})$ is a semiclassical wave function of bulk AdS_3 geometry. The holographic mapping is expected to map the bulk semiclassical state of AdS_3 to the ground state of boundary CFT_2.

In the following discussion, we check our proposal by computing the Rényi entropies S_n of the state $|\Psi\rangle$. We show that $|\Psi\rangle$ indeed reproduces correctly the Rényi entropies of CFT ground state with the correct n dependence, in the regime $\ell_P \ll L_\ell$.

III. RÉNYI ENTROPIES

We compute Rényi entropies S_n of the state $|\Psi\rangle$ by specifying a boundary region $A\subset\partial\Sigma$ which contains a subset of open links. Recall that $|\Psi\rangle$ is made by random tensors at nodes, the n-th Rényi entropy is given by an average over random tensors [25]

$$\overline{S_n(A)} = \frac{1}{1-n}\ln\frac{\overline{\mathrm{tr}(\rho_A^n)}}{\overline{(\mathrm{tr}\rho_A)^n}}. \tag{15}$$

The fluctuation away from the average is discussed in Section IV. ρ_A is the reduced density matrix by tracing out the degrees of freedom located in the complement $\bar{A} = \partial\Sigma\setminus A$. $\mathrm{tr}(\rho_A^n)$ can be conveniently written in terms of the pure density matrix $\rho = |\Psi\rangle\langle\Psi|$

$$\mathrm{tr}(\rho_A^n) = \mathrm{tr}\left[(\rho\otimes\cdots\otimes\rho)C_A^{(n)}\right], \tag{16}$$

where the trace is taken in n copies of boundary Hilbert space. $C_A^{(n)}$ cyclicly permutes the states of region A, leaving the states of $\bar{A}$ invariant:

$$\begin{aligned} &C_A^{(n)}\left(|\mu_\ell^{(1)}\rangle_A|\mu^{(1)}\rangle_{\bar{A}}\otimes\cdots\otimes|\mu^{(n)}\rangle_A|\mu^{(n)}\rangle_{\bar{A}}\right)\\ &= |\mu_\ell^{(2)}\rangle_A|\mu^{(1)}\rangle_{\bar{A}}\otimes\cdots\otimes|\mu^{(n)}\rangle_A|\mu^{(n-1)}\rangle_{\bar{A}}\otimes|\mu^{(1)}\rangle_A|\mu^{(n)}\rangle_{\bar{A}} \end{aligned} \tag{17}$$

where $|\mu\rangle$ forms a basis in the boundary Hilbert space.

Define the pure state density matrix $\rho_P = |E_{\vec{a},\Phi}\rangle\langle E_{\vec{a},\Phi}|$ where $|E_{\vec{a},\Phi}\rangle = \sum_{\vec{a}}\Phi(\vec{a})\otimes_{\mathrm{p,p'}}|a_{\mathrm{pp'}}\rangle$

$$\mathrm{tr}(\rho_A^n) = \mathrm{tr}\left[\left(\rho_P^{\otimes n}\otimes_{\mathrm{p}}|\mathcal{V}_{\mathrm{p}}\rangle\langle\mathcal{V}_{\mathrm{p}}|\right)^{\otimes n}C_A^{(n)}\right]. \tag{18}$$

where the trace is taken in all $\mathcal{H}_{\mathfrak{p}} \equiv \mathcal{H}^{\otimes 3}$ at all nodes.

The random average $\overline{\mathrm{tr}\rho_A^n}$ relates to average $2n$ copies of random tensors $|\mathcal{V}_{\mathfrak{p}}\rangle$ at each node $\mathfrak{p}$. Taking an arbitrary reference state $|0_{\mathfrak{p}}\rangle \in \mathcal{H}_{\mathfrak{p}}$, the random tensor $|\mathcal{V}_{\mathfrak{p}}\rangle = U_{\mathfrak{p}}|0_{\mathfrak{p}}\rangle$ with $U_{\mathfrak{p}}$ unitary transformation on $\mathcal{H}^{\otimes 3} \equiv \mathcal{H}_{\mathfrak{p}}$. The Haar random average is given by [25, 55]:

$$\begin{aligned}\overline{(|\mathcal{V}_{\mathfrak{p}}\rangle\langle\mathcal{V}_{\mathfrak{p}}|)^{\otimes n}} &= \int \mathrm{d}U_{\mathfrak{p}} \left(U_{\mathfrak{p}}|0_{\mathfrak{p}}\rangle\langle 0_{\mathfrak{p}}|U_{\mathfrak{p}}^\dagger\right)^{\otimes n} \\ &= \frac{1}{C_{n,\mathfrak{p}}} \sum_{g_{\mathfrak{p}}\in\mathrm{Sym}_n} g_{\mathfrak{p}} \in \mathcal{H}_{\mathfrak{p}}^{\otimes n} \otimes \mathcal{H}_{\mathfrak{p}}^{*\otimes n} \qquad (19)\end{aligned}$$

where $\mathrm{d}U_{\mathfrak{p}}$ is the Haar measure on the group of all unitary transformations. $\sum_{g_{\mathfrak{p}}\in\mathrm{Sym}_n}$ sums over all permutations $g_{\mathfrak{p}}$ acting on $\mathcal{H}_{\mathfrak{p}}^{\otimes n}$. The overall constant $C_{n,\mathfrak{p}} = \sum_{g_{\mathfrak{p}}\in\mathrm{Sym}_n} \mathrm{tr} g_{\mathfrak{p}} = (\dim\mathcal{H}_{\mathfrak{p}} + n - 1)!/(\dim\mathcal{H}_{\mathfrak{p}} - 1)!$.

Inserting this result in $\mathrm{tr}\rho_A^n$, the average $\overline{\mathrm{tr}\rho_A^n}$ becomes a sum over all permutations $\{g_{\mathfrak{p}}\}$ at all polyhedra $\mathfrak{p}$, where each term associates to a choice of $g_{\mathfrak{p}}$ at each $\mathfrak{p}$. It is straight-forward to compute that (see Appendix A for details) for large bond dimension $D \gg 1$, the sum over $\{g_{\mathfrak{p}}\}$ is dominated by the contribution from $\{g_{\mathfrak{p}}\}$ satisfying the following boundary condition

$$\begin{aligned} g_{\mathfrak{p}}\left(\{\mu_\ell^{(1)}\}_{\bar{A}} \cdots \{\mu_\ell^{(n)}\}_{\bar{A}}\right) &= \left(\{\mu_\ell^{(1)}\}_{\bar{A}} \cdots \{\mu_\ell^{(n)}\}_{\bar{A}}\right) \\ g_{\mathfrak{p}}\left(\{\mu_\ell^{(2)}\}_{A} \cdots \{\mu_\ell^{(n)}\}_{A}\{\mu_\ell^{(1)}\}_{A}\right) &= \left(\{\mu_\ell^{(1)}\}_{A} \cdots \{\mu_\ell^{(n)}\}_{A}\right) \qquad (20)\end{aligned}$$

i.e. for polyhedra $\mathfrak{p}$ connecting to $\partial\Sigma$, $g_{\mathfrak{p}} = I$ if $\mathfrak{p}$ is adjacent to $\bar{A}$, while $g_{\mathfrak{p}} = (C^{(n)})^{-1}$ if $\mathfrak{p}$ is adjacent to A. Each $\{g_{\mathfrak{p}}\}$ corresponds to the following contribution (N_∂ is the number of boundary open links)

$$\begin{aligned}&(D^n)^{N_\partial} \sum_{\{\vec{b}^{(i)}\};\{\vec{a}^{(i)}\};\{\vec{\mu}^{(i)}\}} \Phi^*(\vec{b}^{(1)}) \cdots \Phi^*(\vec{b}^{(n)})\, \Phi(\vec{a}^{(1)}) \cdots \Phi(\vec{a}^{(n)}) \\ &\prod_{\mathfrak{p},\mathfrak{p}'} b^{(1)*}_{g_{\mathfrak{p}}(\mu_{\mathfrak{p}}^{(1)}), g_{\mathfrak{p}'}(\mu_{\mathfrak{p}'}^{(1)})} \cdots b^{(n)*}_{g_{\mathfrak{p}}(\mu_{\mathfrak{p}}^{(n)}), g_{\mathfrak{p}'}(\mu_{\mathfrak{p}'}^{(n)})}\, a^{(1)}_{\mu_{\mathfrak{p}}^{(1)},\mu_{\mathfrak{p}'}^{(1)}} \cdots a^{(n)}_{\mu_{\mathfrak{p}}^{(n)},\mu_{\mathfrak{p}'}^{(n)}} \qquad (21)\end{aligned}$$

where $a^{(i)}_{\mu_{\mathfrak{p}}^{(i)},\mu_{\mathfrak{p}'}^{(i)}}, b^{(i)}_{\nu_{\mathfrak{p}}^{(i)},\nu_{\mathfrak{p}'}^{(i)}}$ come from the ith copy of ρ_P in Eq.(18).

Given $\{g_{\mathfrak{p}}\}$ satisfying the boundary condition Eq.(20), $\{g_{\mathfrak{p}}\}$ contains different domains on Σ with different permutations. We denote by R_g the closed region in which $\mathfrak{p} \in R_g$ are of constant $g_{\mathfrak{p}} = g$. $R_g \cap R_{g'} \equiv \mathcal{S}_{g,g'}$ denotes the domain wall shared by $R_g, R_{g'}$ with two different permutations $g \neq g'$.

Locally at each link $(\mathfrak{p},\mathfrak{p}')$, the result of summing over $\mu_{\mathfrak{p}}^{(i)}, \mu_{\mathfrak{p}'}^{(i)}$ depends on whether $g_{\mathfrak{p}}$ and $g_{\mathfrak{p}'}$ are the same or not, i.e. whether the link $(\mathfrak{p},\mathfrak{p}')$ intersect with any domain wall. When $g_{\mathfrak{p}} = g_{\mathfrak{p}'} = g$, $(\mathfrak{p},\mathfrak{p}')$ located inside a single domain, using Eq.(7)

$$\sum_{\{\mu_{\mathfrak{p}}^{(i)},\mu_{\mathfrak{p}'}^{(i)}\}} \prod_{i=1}^n b^{(i)*}_{g(\mu_{\mathfrak{p}}^{(i)}), g(\mu_{\mathfrak{p}'}^{(i)})}\, a^{(i)}_{\mu_{\mathfrak{p}}^{(i)},\mu_{\mathfrak{p}'}^{(i)}} = \prod_{i=1}^n \delta_{a^{g(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}} \qquad (22)$$

which identifies $b^{(i)}$ and $a^{g(i)}$ in $\Phi(\vec{a}^{(i)}), \Phi(\vec{b}^{(i)})^*$.

For $g_{\mathfrak{p}} \neq g_{\mathfrak{p}'}$, $(\mathfrak{p},\mathfrak{p}')$ cross the domain wall $\mathcal{S}_{g_{\mathfrak{p}},g_{\mathfrak{p}'}}$ we consider the permutation $g_{\mathfrak{p}}^{-1}g_{\mathfrak{p}'}$, in which the set of cycles is denoted by $C(g_{\mathfrak{p}}^{-1}g_{\mathfrak{p}'})$. For each cycle $c \in C(g_{\mathfrak{p}}^{-1}g_{\mathfrak{p}'})$, the cycle length (the number of involved elements $i \in c$) is denoted by n_c, satisfying $\sum_c n_c = n$.

$$\begin{aligned}&\sum_{\{\mu_{\mathfrak{p}}^{(i)},\mu_{\mathfrak{p}'}^{(i)}\}} \prod_{i=1}^n b^{(i)*}_{g_{\mathfrak{p}}(\mu_{\mathfrak{p}}^{(i)}), g_{\mathfrak{p}'}(\mu_{\mathfrak{p}'}^{(i)})}\, a^{(i)}_{\mu_{\mathfrak{p}}^{(i)},\mu_{\mathfrak{p}'}^{(i)}} \\ &= \prod_{c\in C(g_{\mathfrak{p}}^{-1}g_{\mathfrak{p}'})} d[a(c)]^{1-n_c} \prod_{i\in c} \delta_{a^{g_{\mathfrak{p}}(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}}\, \delta_{a^{g_{\mathfrak{p}'}(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}}. \qquad (23)\end{aligned}$$

All $a^{g_{\mathfrak{p}}(i)}, a^{g_{\mathfrak{p}'}(i)}, b^{(i)}$ within a cycle c are identified to be $a(c)$.

In particular, if $g_{\mathfrak{p}} = I$, $g_{\mathfrak{p}'} = (C^{(n)})^{-1}$,

$$\begin{aligned}&\sum_{\{\mu_{\mathfrak{p}}^{(i)},\mu_{\mathfrak{p}'}^{(i)}\}} b^{(1)*}_{\mu_{\mathfrak{p}}^{(1)},\mu_{\mathfrak{p}'}^{(n)}} b^{(2)*}_{\mu_{\mathfrak{p}}^{(2)},\mu_{\mathfrak{p}'}^{(1)}} \cdots b^{(n)*}_{\mu_{\mathfrak{p}}^{(n)},\mu_{\mathfrak{p}'}^{(1)}}\, a^{(1)}_{\mu_{\mathfrak{p}}^{(1)},\mu_{\mathfrak{p}'}^{(1)}} a^{(2)}_{\mu_{\mathfrak{p}}^{(2)},\mu_{\mathfrak{p}'}^{(2)}} \cdots a^{(n)}_{\mu_{\mathfrak{p}}^{(n)},\mu_{\mathfrak{p}'}^{(n)}} \\ &= d[a]^{1-n} \prod_{i=1}^n \delta_{a^{(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}}\, \delta_{a^{(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i+1)}_{\mathfrak{p},\mathfrak{p}'}} \qquad (24)\end{aligned}$$

which identify all $a^{(i)}, b^{(i)}$ to be a.

Inserting the above results, we obtain $\overline{\mathrm{tr}(\rho_A^n)}$ as a sum over all possible $\{g_{\mathfrak{p}}\}$ as $D \gg 1$,

$$\begin{aligned}&\overline{\mathrm{tr}\rho_A^n} \simeq \prod_{\mathfrak{p}} \frac{1}{C_{n,\mathfrak{p}}} (D^n)^{N_\partial} \qquad (25) \\ &\sum_{\{g_{\mathfrak{p}}\}} \sum_{\{\vec{a}^{(i)}\},\{\vec{b}^{(i)}\}} \prod_{i=1}^n \Phi^*(\vec{b}^{(i)})\Phi(\vec{a}^{(i)}) \prod_{(\mathfrak{p},\mathfrak{p}')\not\subset\mathcal{S}} \prod_i \delta_{a^{g_{\mathfrak{p}}(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}} \\ &\prod_{\mathcal{S}_{g,g'}} \prod_{c\in C(g^{-1}g')} \prod_{(\mathfrak{p},\mathfrak{p}')\subset\mathcal{S}_{g,g'}} e^{(1-n_c)\ln d\left[a_{\mathfrak{p},\mathfrak{p}'}(c)\right]} \prod_{i\in c} \delta_{a^{g_{\mathfrak{p}}(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}}\, \delta_{a^{g_{\mathfrak{p}'}(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}}\end{aligned}$$

In the above formula, $(\mathfrak{p},\mathfrak{p}') \subset \mathcal{S}_{g,g'}$ means that the link $(\mathfrak{p},\mathfrak{p}')$ intersects the domain wall $\mathcal{S}_{g,g'}$.

Thanks to the Wheeler-deWitt wave function $\Phi(\vec{a})$ in Eq.(11) as a path integral on a triangulated manifold M, we are able to make an interpretation of Eq.(25) as a sum of path integrals on different manifolds. Recall that $\vec{a}$ is the boundary condition of the path integral on M. The product of $2n$ copies of $\Phi(\vec{a}^{(i)})$ and $\Phi^*(\vec{b}^{(i)})$ is a path integral on the product of $2n$ copies of M and $\bar{M}$ (FIG.3(b)), with identical triangulations. δs in Eq.(25) identifying boundary conditions $a^{(i)}, b^{(j)}$ effectively glue the path integrals on copies of M and $\bar{M}$. In other words, $2n$ copies of M and $\bar{M}$ are glued in certain manner. The path integral is defined on the resulting manifold.

For $(\mathfrak{p},\mathfrak{p}') \not\subset \mathcal{S}$, i.e. $\mathfrak{p},\mathfrak{p}'$ are inside a single domain R_g, $\delta_{a^{g(i)}_{\mathfrak{p},\mathfrak{p}'}, b^{(i)}_{\mathfrak{p},\mathfrak{p}'}}$ glues the i-th copy of $\bar{M}$ with the $g(i)$-th copy of M. This pattern of gluing happens in the domain R_g. However in a different domain $R_{g'}$, the gluing pattern is different: the i-th copy of $\bar{M}$ is glued with the $g'(i)$-th copy of M. Therefore the gluing in both domains results in branch cuts, where the domain wall $\mathcal{S}_{g,g'}$ gives 1d branch curves containing all branch points. Taking all domains with different permutations into account, each $\{g_{\mathfrak{p}}\}$ determines a manifold $M_{\{g_{\mathfrak{p}}\}}$ made by gluing n copies of M and n copies of $\bar{M}$. $M_{\{g_{\mathfrak{p}}\}}$ has a number of branch cuts. FIG.4 illustrates a simple situation with $n = 2$, where there are 2 domains of the identity I and cyclic $C^{(2)}$. The domain wall is a branch curve with $\mathbb{Z}_2$ symmetry. This situation can be easily generalized to a general domain wall

$\mathcal{S}_{g,g'}$ and a cycle c in $g^{-1}g'$. Indeed, each domain wall $\mathcal{S}_{g,g'}$ becomes $\chi(g^{-1}g')$ branching curves $\mathcal{S}_{g,g'}(c)$, where $\chi(g^{-1}g')$ is the number of cycles in $g^{-1}g'$. Each branch curve $\mathcal{S}_{g,g'}(c)$ associates to a cycle $c \in C(g^{-1}g')$, and has a local $\mathbb{Z}_{n_c}$ symmetry.

At each branch curve $\mathcal{S}_{g,g'}(c)$ where $2n_c$ copies of M and $\bar{M}$ meet, Regge actions in $\Phi^*(\vec{b}^{(i)}), \Phi(\vec{a}^{(i)})$ contribute boundary terms $L_\ell\Theta_\ell$ to each $\ell \subset \mathcal{S}_{g,g'}(c)$. In addition, thanks the exponential in Eq.(25) which contributes $(1-n_c)\ln d[a_{\mathrm{p},\mathrm{p}'}(c)] = (1-n_c)\frac{L_\ell}{4\ell_P}$ by Eq.(10), the total contribution to each $\mathcal{S}_{g,g'}(c)$ precisely makes a new bulk term of Regge action:

$$\begin{aligned}&\frac{1}{4\ell_P}(1-n_c)\sum_{\ell\subset\mathcal{S}}L_\ell + \frac{1}{8\pi\ell_P}\sum_{\ell\subset\mathcal{S}}L_\ell\left(2\pi n_c - \sum_{t,\,\ell\subset t}\theta(t,\ell)\right)\\ =\;& \frac{1}{8\pi\ell_P}\sum_{\ell\subset\mathcal{S}}L_\ell\left(2\pi - \sum_{t,\,\ell\subset t}\theta(t,\ell)\right) = \frac{1}{8\pi\ell_P}\sum_{\ell\subset\mathcal{S}}L_\ell\varepsilon_\ell \qquad (26)\end{aligned}$$

On the other hand, it is easy to see that when M glues to $\bar{M}$ inside a domain R_g, a pair of boundary terms from $S_{Regge}(M)$ and $S_{Regge}(\bar{M})$ again makes a bulk term of Regge action on the glued manifold [53].

As a result, $\overline{\mathrm{tr}(\rho_A^n)}$ is written as a sum of discrete path integrals of Regge actions on different manifolds $M_{\{g_\mathrm{p}\}}$:

$$\overline{\mathrm{tr}(\rho_A^n)} \simeq \prod_{\mathrm{p}}\frac{1}{C_{n,\mathrm{p}}}(D^n)^{N_\partial}\sum_{\{g_\mathrm{p}\}}\sum_{\{L_\ell\}}e^{-S_{Regge}\left(M_{\{g_\mathrm{p}\}}\right)}. \qquad (27)$$

It allows us to translate the Rényi entropy of boundary CFT to the bulk geometry.

As is mentioned above, we consider the regime $\ell_P \ll L_\ell$. On each $M_{\{g_\mathrm{p}\}}$, the dominant contribution comes from the solution of equation of motion

$$\varepsilon_\ell = 0, \quad \forall\, \ell \subset \mathrm{bulk}(M_{\{g_\mathrm{p}\}}) \qquad (28)$$

It implies that as the leading contribution, the geometry on $M_{\{g_\mathrm{p}\}}$ is smooth AdS$_3$ everywhere. The on-shell action gives

$$\sum_{\{L_\ell\}}e^{-S_{Regge}\left(M_{\{g_\mathrm{p}\}}\right)} \sim e^{-\frac{1}{8\pi\ell_P}\frac{V\left(M_{\{g_\mathrm{p}\}}\right)}{L_{AdS}^2}+\text{boundary terms}} \qquad (29)$$

Focus on a given $M_{\{g_\mathrm{p}\}}$, locally at each $\mathcal{S}_{g,g'}(c)$ of a given cycle c, the geometry has a local $\mathbb{Z}_{n_c}$ symmetry at both continuum level and discrete level, because we use the same triangulation on all copies of M and $\bar{M}$. We cut a local neighborhood N_{n_c} at $\mathcal{S}_{g,g'}(c)$ from $M_{\{g_\mathrm{p}\}}$, and make a $\mathbb{Z}_{n_c}$ quotient. The orbifold is denoted by $\hat{N}_{n_c} = N_{n_c}/\mathbb{Z}_{n_c}$. The geometry on $\hat{N}_{n_c}$ has a conical singularity at $\mathcal{S}_{g,g'}(c)$ with deficit angle $2\pi\left(1-\frac{1}{n_c}\right)$. In the language of [15], the geometry we derive is back-reacted by a cosmic brane with tension $T_{n_c} = \frac{n_c-1}{4n_c\ell_P}$, located at $\mathcal{S}_{g,g'}(c)$. We may analytic continue n_c by considering arbitrary conical singularity or brane tension.

The geometry of branch curve $\mathcal{S}_{g,g'}(c)$ is determined by the equation of motion as in [12]. Both on-shell geometries on N_{n_c} and $\hat{N}_{n_c}$ are AdS$_3$, except the conical singularity of $\hat{N}_{n_c}$. $\hat{N}_{n_c}$ is a fundamental domain in N_{n_c} of $\mathbb{Z}_n$. $\hat{N}_{n_c}$ may be obtained by cutting N_{n_c} into n_c identical pieces, pick up one piece, followed by identifying its 2 cut boundaries. $\hat{N}_{n_c}$ is AdS$_3$ away from the singularity. So the glued boundaries can be chosen to be identical hyperbolic surfaces intersecting at the singularity. The singularity $\mathcal{S}_{g,g'}(c)$ as the intersection has to be a geodesic (hyperbola) in the hyperbolic plane. The length $L_{\mathcal{S}_{g,g'}(c)}(n_c)$ of $\mathcal{S}_{g,g'}(c)$ explicitly depends on n_c. Since the triangulation of M has been fixed, we only consider $\mathcal{S}_{g,g'}(c)$ made by the edges in the triangulation. Otherwise the equation of motion cannot be satisfied and the domain wall $\mathcal{S}_{g,g'}(c)$ doesn't give leading order contribution.

Consider the volume of $\hat{N}_{n_c}$. We analytic continue n_c and compute the derivative. By Schläfli identity of hyperbolic tetrahedra and keeping $\varepsilon_\ell = 0$ fixed in the bulk

$$\frac{-1}{L_{AdS}^2}\partial_{n_c}V\left(\hat{N}_{n_c}\right) = \sum_{\ell\subset\mathcal{S}_{g,g'}(c)}L_\ell\partial_{n_c}\left(\frac{2\pi}{n_c}\right) = -\frac{2\pi}{n_c^2}L_{\mathcal{S}_{g,g'}(c)}(n_c)$$

Integrating the above relation gives

$$-\frac{V\left(\hat{N}_{n_c}\right)}{L_{AdS}^2} = -\frac{V(N_1)}{L_{AdS}^2} - \int_1^{n_c}\frac{2\pi}{q^2}L_{\mathcal{S}_{g,g'}(c)}(q)\,\mathrm{d}q \qquad (30)$$

where N_1 has no singularity at $\mathcal{S}_{g,g'}(c)$.

Because the geometry is smooth AdS$_3$ on $\hat{N}_{n_c}$ away from $\mathcal{S}_{g,g'}(c)$, to compute $L_{\mathcal{S}_{g,g'}(c)}(q)$, we use the metric on $\hat{N}_q$ in the hyperbolic foliation [13, 14]:

$$\mathrm{d}s^2 = \left(\frac{r^2}{L_{AdS}^2} - \frac{1}{q^2}\right)L_{AdS}^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{L_{AdS}^2}-\frac{1}{q^2}} + r^2\mathrm{d}u^2. \qquad (31)$$

The periodicity of τ is $\tau \sim \tau + 2\pi$. r satisfies $r \geq L_{AdS}/q$. $\int \mathrm{d}u$ is the geodesics length in the hyperbolic plane with unit curvature. $\mathcal{S}_{g,g'}(c)$ is located at the origin $r = L_{AdS}/q$. Define $\frac{1}{2qL_{AdS}}\xi^2 = r - L_{AdS}/q$ and consider the limit $\xi \to 0$

$$\mathrm{d}s^2 \sim \frac{\xi^2}{q^2}\mathrm{d}\tau^2 + \mathrm{d}\xi^2 + \left(\frac{\xi^2}{2qL_{AdS}} + \frac{L_{AdS}}{q}\right)^2\mathrm{d}u^2, \qquad (32)$$

which manifests the conical singularity at $\xi \to 0$. The length of $\mathcal{S}_{g,g'}(c)$ is given by

$$L_{\mathcal{S}_{g,g'}(c)}(q) = \frac{L_{AdS}}{q}\int_{\mathcal{S}_{g,g'}(c)}\mathrm{d}u \equiv \frac{L_{AdS}}{q}l_{\mathcal{S}_{g,g'}}, \qquad (33)$$

where $l_{\mathcal{S}_{g,g'}}$ is the geodesic length of $\mathcal{S}_{g,g'}$ evaluated in the hyperbolic plane with unit curvature. $l_{\mathcal{S}_{g,g'}}$ is independent of c. As a result,

$$-\int_1^{n_c}\frac{2\pi}{q^2}L_{\mathcal{S}_{g,g'}(c)}(q)\,\mathrm{d}q = \frac{1-n_c^2}{n_c^2}\pi L_{AdS}\,l_{\mathcal{S}_{g,g'}} \qquad (34)$$

The volume of N_c: $V(N_{n_c}) = n_c V\left(\hat{N}_{n_c}\right)$, i.e. n_c times Eq.(34). When we glue back N_{n_c} in $M_{\{g_\mathrm{p}\}}$. The first term in Eq.(34) gives $n_c V(N_1)$, and effectively replaces N_{n_c} by n_c copies of N_1, which resolves the branch curve $\mathcal{S}_{g,g'}(c)$ in $M_{\{g_\mathrm{p}\}}$.

When all branch curves are resolved, $M_{\{g_\mathfrak{p}\}}$ reduces to n copies M_1. Therefore when we sum all domain walls and all cycles,

$$\frac{-V\left(M_{\{g_\mathfrak{p}\}}\right)}{8\pi\ell_P L_{AdS}^2} = \frac{-nV(M_1)}{8\pi\ell_P L_{AdS}^2} + \sum_{\mathcal{S}_{g,g'}} \sum_{c\in C(g^{-1}g')} \frac{1-n_c^2}{n_c}\frac{L_{AdS}}{8\ell_P} l_{\mathcal{S}_{g,g'}} \quad (35)$$

It is shown in Appendix B that the maximum of Eq.(35) happens at $\{g_\mathfrak{p}\}$ with only a single domain wall $\mathcal{S}$ separating I in $R_{\bar{A}}$ and $(C^{(n)})^{-1}$ in R_A, where R_A (or $R_{\bar{A}}$) is the region bounded by the boundary region A (or $\bar{A}$) and the domain wall (FIG.3). We denote the corresponding $M_{\{g_\mathfrak{p}\}}$ by M_n

$$\frac{-1}{8\pi\ell_P}\frac{V(M_n)}{L_{AdS}^2} = \frac{-n}{8\pi\ell_P}\frac{V(M_1)}{L_{AdS}^2} + \frac{1-n^2}{n}\frac{L_{AdS}}{8\ell_P} l_{\mathcal{S}} \quad (36)$$

The contribution of any other $M_{\{g_\mathfrak{p}\}}$ is much less than Eq.(36), with the gap of order $L_\ell/\ell_P = 4\ln d[a] \gg 1$.

As a result, the dominant contribution of $\overline{\mathrm{tr}(\rho_A^n)}$ in Eq.(27) is given by

$$\overline{\mathrm{tr}(\rho_A^n)} \simeq \prod_\mathfrak{p} \frac{1}{C_{n,\mathfrak{p}}} (D^n)^{N_\partial} e^{-\frac{1}{8\pi\ell_P}\frac{V(M_n)}{L_{AdS}^2}+\text{boundary terms}} \quad (37)$$

Let's move to the denominator $\overline{\mathrm{tr}(\rho_A)^n}$ in Eq.(15), which can be computed in a very similar manner, since

$$\mathrm{tr}(\rho_A)^n = \mathrm{tr}\left[\left(\rho_P^{\otimes n}\otimes_\mathfrak{p} |\mathcal{V}_\mathfrak{p}\rangle\langle\mathcal{V}_\mathfrak{p}|\right)^{\otimes n}\right]. \quad (38)$$

which different from Eq.(18) by removing $C_A^{(n)}$ in the trace. We still use the Haar random average Eq.(19), and write $\overline{\mathrm{tr}(\rho_A)^n}$ as a sum over all permutations $\{g_\mathfrak{p}\}$ at all nodes. However because $C_A^{(n)}$ is absent, as $D \gg 1$ the dominant configurations of $\{g_\mathfrak{p}\}$ satisfy the boundary condition that $g_\mathfrak{p} = I$ at the entire boundary. Thus suppose $\{g_\mathfrak{p}\}$ has different domains with different $g_\mathfrak{p}$, domain walls are detached from the boundary, and contain closed curves.

Using the same argument as the above, we can write $\overline{\mathrm{tr}(\rho_A)^n}$ as a sum of path integral of Regge action on different $M_{\{g_\mathfrak{p}\}}$, similar to Eq.(27). domain walls become the branch curves in $M_{\{g_\mathfrak{p}\}}$, which contain closed curves. However, since the intersection of two hyperbolic surfaces cannot give closed branch curves, $M_{\{g_\mathfrak{p}\}}$ with closed branch curves doesn't admit AdS_3 geometry. Thus the equation of motion doesn't have any solution, except $g_\mathfrak{p} = I$ identically without any domain wall. As a result $\overline{\mathrm{tr}(\rho_A)^n}$ is dominant at the configuration that $g_\mathfrak{p} = I$ everywhere

$$\overline{\mathrm{tr}(\rho_A^n)} \simeq \prod_\mathfrak{p} \frac{1}{C_{n,\mathfrak{p}}} (D^n)^{N_\partial} e^{-\frac{n}{8\pi\ell_P}\frac{V(M_1)}{L_{AdS}^2}+\text{boundary terms}}. \quad (39)$$

The boundary terms are identical to the ones appearing in Eq.(37).

We find that the average Rényi entropy is given by

$$\overline{S_n(A)} \simeq \frac{1}{1-n}\left[\ln Z(n)_\infty - n\ln Z(1)_\infty\right] \quad (40)$$

where up to a term $\ln\left(\prod_\mathfrak{p} (D^n)^{N_\partial}/C_{n,\mathfrak{p}}\right)$,

$$\ln Z(n)_\infty \equiv -\frac{1}{8\pi\ell_P}\frac{V(M_n)}{L_{AdS}^2} + \text{boundary terms} \quad (41)$$

is the on-shell action of Einstein gravity on 3-manifold M_n. The relation Eq.(40) has been an assumption in the existing derivation of RT formula from AdS/CFT [11, 12, 15]. But it is now derived from the state Eq.(9) using random tensor networks.

Using Eq.(15), we obtain the RT formula of Rényi entropy for CFT_2, which has the nontrivial n dependence.

$$\overline{S_n(A)} \simeq \left(1+\frac{1}{n}\right)\frac{L_{AdS}}{8\ell_P} l_{\mathcal{S}} \quad (42)$$

where $L_{AdS} l_{\mathcal{S}}$ corresponds to $\mathbf{Ar}_{min}$ the geodesic length in AdS_3 in Eq.(1). The usual RT formula is recovered as $n \to 1$. The above result reproduces the Renyi entropy computed by Hung-Myers-Smolkin-Yale in [14] using the AdS/CFT assumptions. To see it is indeed the right Rényi entropy of the boundary CFT_2, recall the central charge of CFT relates to L_{AdS} and ℓ_P by $c = \frac{3L_{AdS}}{2\ell_P}$, and $l_{\mathcal{S}}$ relates to length l_A of boundary interval A by $l_{\mathcal{S}} \simeq 2\ln(l_A/\delta)$ in Poincaré patch, where δ is a UV cut-off. It gives

$$\overline{S_n(A)} \simeq \left(1+\frac{1}{n}\right)\frac{c}{6}\ln\left(\frac{l_A}{\delta}\right) \quad (43)$$

which matches precisely the Rényi entropy Eq.(3) of CFT_2 with correct n dependence [48].

IV. BOUND ON FLUCTUATION

In this section we examine the fluctuation of the Rényi entropy $S_n(A)$ from the above average value $\overline{S_n(A)}$, to qualify how well is the approximation. We show that in the regime $\ell_P \ll L_\ell$ the fluctuation is generically small. The method used in the following is similar to [25].

We denotes by $Z(n) = \mathrm{tr}(\rho_A^n)$ and $Z(n)_\infty$ the average value of $Z(n)$ as $\ell_P \ll L_\ell$ (same as in Eq.(41)). We consider the following fluctuation of $Z(n)$:

$$\begin{aligned}\overline{\left(\frac{Z(n)}{Z(n)_\infty}-1\right)^2} &= \left(\frac{\overline{Z(n)^2}}{Z(n)_\infty^2}-1\right) - 2\left(\frac{\overline{Z(n)}}{Z(n)_\infty}-1\right)\\ &\leq \left(\frac{\overline{Z(n)^2}}{Z(n)_\infty^2}-1\right)\end{aligned} \quad (44)$$

$\overline{Z(n)} \geq Z(n)_\infty$ because in the approximation we made as $\ell_P \ll L_\ell$, the neglected terms in the sums are all non-negative.

$\overline{Z(n)^2}$ is computed in a similar way as the above, using the random average formula Eq.(19), changing n by $2n$. It leads to that the dominant contribution of $\overline{Z(n)^2}$ is again given by a sum over permutations $\{g_\mathfrak{p}\}$ at all $\mathfrak{p}$, whose boundary condition is $g_\mathfrak{p} = I$ in $\bar{A}$ and $g_\mathfrak{p} = (1\cdots n)(n+1\cdots 2n)$ in A. Hence $\overline{Z(n)^2}$ is also written as a sum of path integrals on different

$M_{\{g_{\rm p}\}}$. The situation of a single domain wall separating I in $R_{\bar{A}}$ and $(1\cdots n)(n+1\cdots 2n)$ in R_A gives again the dominant contribution. The 3-manifold $M_{\{g_{\rm p}\}}$ in this case is simply 2 copies of M_n. As a result,

$$\frac{\overline{Z(n)^2}}{Z(n)^2_\infty}=\prod_{\rm p}\frac{C_{2n,\rm p}}{C^2_{n,\rm p}}\left[1+O\left(\frac{\ell_P}{L_\ell}\right)\right],\quad \frac{C_{2n,\rm p}}{C^2_{n,\rm p}}\leq 1 \tag{45}$$

which implies the following bound

$$\overline{\left(\frac{Z(n)}{Z(n)_\infty}-1\right)^2}\leq O\left(\frac{\ell_P}{L_\ell}\right) \tag{46}$$

Bounding the fluctuation of $Z(n)$ by $\varepsilon/4$ has the following probability by Markov inequality,:

$$\text{Prob}\left(\left|\frac{Z(n)}{Z(n)_\infty}-1\right|\geq\frac{\varepsilon}{4}\right)\leq\frac{\overline{\left(\frac{Z(n)}{Z(n)_\infty}-1\right)^2}}{\left(\frac{\varepsilon}{4}\right)^2}\leq O\left(\frac{\ell_P}{\varepsilon^2 L_\ell}\right). \tag{47}$$

Similar conclusion can be drawn for $Z(1)^n$. Bounds on the fluctuations of $Z(n), Z(1)^n$ implies the bound on the fluctuation of $S_n(A)$. The probability of violating the following bound is of $O\left(\ell_P/\varepsilon^2 L_\ell\right)$

$$\begin{aligned}\left|S_n(A)-\overline{S_n(A)}\right| &\leq \frac{1}{n-1}\left(\left|\ln\frac{Z(n)}{Z(n)_\infty}\right|+\left|\ln\frac{Z(1)^n}{Z(1)^n_\infty}\right|\right)\\ &\leq \varepsilon\end{aligned} \tag{48}$$

where we have used that $|\ln(1\pm\varepsilon/4)|\leq\varepsilon/2$ for small ε. When $\ell_P/L_\ell\ll\varepsilon^2$, the above bound of fluctuation is satisfied with a high probability $1-O\left(\ell_P/\varepsilon^2 L_\ell\right)$.

Acknowledgments

MH acknowledges Ling-Yan Hung and Yidun Wan at Fudan University in Shanghai, Yong-Shi Wu at University of Utah, Bei Zeng at University of Waterloo, and Jie Zhou at Perimeter Institute for Theoretical Physics for the hospitality during his visits, stimulating discussions and collaborations. MH acknowledges the useful discussions with Xiao-liang Qi and Zhao Yang, and especially thanks Ling-Yan Hung for explaining the result in [14]. MH also acknowledges support from the US National Science Foundation through grant PHY-1602867, and the Start-up Grant at Florida Atlantic University, USA.

Appendix A: Compute $\overline{\text{tr}(\rho_A^n)}$

Insert the random average Eq.(19) into $\overline{\text{tr}(\rho_A^n)}$ in Eq.(18). Each choice of permutations $\{g_{\rm p}\}$ corresponds to the following contribution in $\overline{\text{tr}(\rho_A^n)}$:

$$\begin{aligned}&\langle\mu_\ell^{(1)}|_A\langle\mu_\ell^{(1)}|_{\bar{A}}\langle E_{\vec{a},\Phi}|\otimes\cdots\otimes\langle\mu_\ell^{(n)}|_A\langle\mu_\ell^{(n)}|_{\bar{A}}\langle E_{\vec{a},\Phi}|\\ &\otimes_{\rm p}g_{\rm p}\,|E_{\vec{a},\Phi}\rangle|\mu_\ell^{(2)}\rangle_A|\mu_\ell^{(1)}\rangle_{\bar{A}}\otimes\cdots\otimes|E_{\vec{a},\Phi}\rangle|\mu_\ell^{(1)}\rangle_A|\mu_\ell^{(n)}\rangle_{\bar{A}}.\end{aligned}$$

μ_ℓ labels a basis in $\mathcal{H}$ at an boundary open link dual to a boundary triangle edge ℓ. The sum over all $\mu_\ell^{(i)}$ and the tensor product over all boundary ℓ has been omitted in the above formula.

Firstly we compute the operator $\otimes_{\rm p}g_{\rm p}$ acting on the right. Using the expression of $|E_{\vec{a},\Phi}\rangle$,

$$\begin{aligned}&\prod_{\rm p}g_{\rm p}\,|E_{\vec{a},\Phi}\rangle|\mu_\ell^{(2)}\rangle_A|\mu_\ell^{(1)}\rangle_{\bar{A}}\otimes\cdots\otimes|E_{\vec{a},\Phi}\rangle|\mu_\ell^{(1)}\rangle_A|\mu_\ell^{(n)}\rangle_{\bar{A}}\\ =&\sum_{\{\vec{a}^{(i)}\},\{\vec{\mu}^{(i)}\}}\Phi(\vec{a}^{(1)})\cdots\Phi(\vec{a}^{(n)})\prod_{\rm p,p'}a^{(1)}_{\mu^{(1)}_{\rm p},\mu^{(1)}_{\rm p'}}\cdots a^{(n)}_{\mu^{(n)}_{\rm p},\mu^{(n)}_{\rm p'}}\\ &\otimes_{\rm p}g_{\rm p}\left(\left|\{\mu_{\rm p}^{(1)}\},\{\mu_\ell^{(2)}\}_A,\{\mu_\ell^{(1)}\}_{\bar{A}}\right\rangle\otimes\cdots\otimes\left|\{\mu_{\rm p}^{(n)}\},\{\mu_\ell^{(1)}\}_A,\{\mu_\ell^{(n)}\}_{\bar{A}}\right\rangle\right)\end{aligned}$$

Taking the inner product gives

$$\begin{aligned}&\sum_{\{\vec{b}^{(i)}\};\{\vec{\nu}^{(i)}\}}\sum_{\{\vec{a}^{(i)}\};\{\vec{\mu}^{(n)}\}}\sum_{\{\mu_\ell^{(i)}\}_A;\{\mu_\ell^{(i)}\}_{\bar{A}}}\Phi^*(\vec{b}^{(1)})\cdots\Phi^*(\vec{b}^{(n)})\,\Phi(\vec{a}^{(1)})\cdots\Phi(\vec{a}^{(n)})\\ &\prod_{\rm p,p'}b^{(1)*}_{\nu^{(1)}_{\rm p},\nu^{(1)}_{\rm p'}}\cdots b^{(n)*}_{\nu^{(n)}_{\rm p},\nu^{(n)}_{\rm p'}}\prod_{\rm p,p'}a^{(1)}_{\mu^{(1)}_{\rm p},\mu^{(1)}_{\rm p'}}\cdots a^{(n)}_{\mu^{(n)}_{\rm p},\mu^{(n)}_{\rm p'}}\prod_{\rm p}\delta_{\{\nu_{\rm p}^{(i)}\},\ g_{\rm p}\{\mu_{\rm p}^{(i)}\}}\\ &\delta_{\left(\{\mu_\ell^{(1)}\}_A\cdots\{\mu_\ell^{(n)}\}_A\right),\ g_{\rm p}\left(\{\mu_\ell^{(2)}\}_A\cdots\{\mu_\ell^{(n)}\}_A\{\mu_\ell^{(1)}\}_A\right)}\ \delta_{\left(\{\mu_\ell^{(1)}\}_{\bar{A}}\cdots\{\mu_\ell^{(n)}\}_{\bar{A}}\right),\ g_{\rm p}\left(\{\mu_\ell^{(1)}\}_{\bar{A}}\cdots\{\mu_\ell^{(n)}\}_{\bar{A}}\right)}.\end{aligned}$$

The last two δs associates $g_{\rm p}$s close to the boundary regions A and $\bar{A}$ respectively. To maximize the sum $\sum_{\{\mu_\ell^{(i)}\}_A;\{\mu_\ell^{(i)}\}_{\bar{A}}}$, $g_{\rm p}$ close to A has to be a cyclic permutation, and $g_{\rm p}$ close to $\bar{A}$ has to be an identity. So we obtain the boundary condition in Eq.(20). Eq.(21) is obtained by performing the sum over $\{\nu_{\rm p}^{(i)}\}$.

Appendix B: Domain Walls in Sym$_n$ Spin Model

In this section, we prove that the configuration $\{g_{\rm p}\}$ with a single domain wall indeed gives the leading contribution to $\sum_{\{g_{\rm p}\}}$. Let's consider a more generic case shown in FIG.5(a), where more than one domain-walls are created in the bulk of Σ. We are going to show that this configuration always contribute less than a single domain-wall.

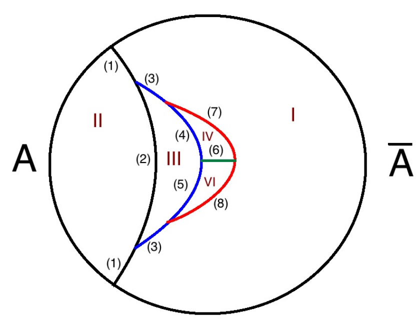

FIG. 5: shows the space Σ with boundary $\partial\Sigma$ divided into regions A and $\bar{A}$. Σ contains the domain-walls $(1),(2),\cdots,(8)$, which divide the bulk of Σ into regions I, II, $\cdots$, VI. Each bulk region associates a permutation $g_{I,II,\cdots,VI}$, with $g_I=I$ and $g_{II}=(C^{(n)})^{-1}$.

Given the multi-domain-wall configuration, each domain-wall carries the contribution proportional to

$$l_{S_{g,g'}} \sum_{c\in C(g^{-1}g')} \frac{1-n_c^2}{n_c} \tag{B1}$$

in Eq.(35) where l_S is the geodesic length on the hyperbolic plane with unit curvature. Given trivalent intersection of domain-walls, which separate three domains with permutations g_1, g_2, g_3 (FIG.6), we have the following "triangle inequality" (see Appendix C for a proof)

$$\sum_{c\in C(g_1^{-1}g_3)} \frac{1-n_c^2}{n_c} \geq \sum_{c\in C(g_1^{-1}g_2)} \frac{1-n_c^2}{n_c} + \sum_{c\in C(g_2^{-1}g_3)} \frac{1-n_c^2}{n_c}. \tag{B2}$$

It implies each trivalent intersection gives the following contribution

$$l_{S_{13}} \sum_{c\in C(g_1^{-1}g_3)} \frac{1-n_c^2}{n_c} + l_{S_{12}} \sum_{c\in C(g_1^{-1}g_2)} \frac{1-n_c^2}{n_c} + l_{S_{23}} \sum_{c\in C(g_2^{-1}g_3)} \frac{1-n_c^2}{n_c}$$
$$\leq \sum_{c\in C(g_1^{-1}g_3)} \frac{1-n_c^2}{n_c} \left[l_{S_{13}} + \min\left(l_{S_{12}}, l_{S_{23}}\right)\right]$$

which is less than a single domain wall contribution.

For any intersection with e.g. 4 domain walls, one can always shift the end point of one domain wall away from the intersection and obtain a smaller l_S (greater contribution to Eq.(35)). It reduces the 4-valent intersection back to trivalent situation, which implies the contribution after the above shift is still smaller than the single domain wall configuration. The same argument applies to the intersection with larger number of domain walls.

Therefore we find that the contribution of the multi-domain-wall configuration is less or equal to the single domain-wall configuration

$$l_{S_{g,g'}} \sum_{c\in C(g^{-1}g')} \frac{1-n_c^2}{n_c} \leq \frac{1-n^2}{n} l_S. \tag{B3}$$

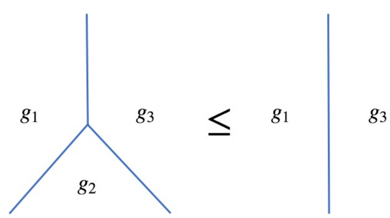

FIG. 6: A trivalent intersection of domain walls has less contribution than a single domain wall.

Appendix C: Proof of triangle inequality (B2)

For every permutation $g \in \mathrm{Sym}_n$ (in the following discussion, we assume that $n \geq 2$), we can decompose g as

$$g = \prod_{i=1}^{k} c_i,$$

where $c_i \in \mathrm{Sym}_n$ are disjoint cycles such that $\sum_{i=1}^{k} n_{c_i} = n$. Denote

$$C(g) = \{c_1, \dots, c_k\}$$

as the set of disjoint cycles whose product is g.

Let $d : \mathrm{Sym}_n \to \mathbb{R}$ be a function which satisfies that there exists a function $f \in C^2[1, +\infty)$ such that

(i) $f''(x) \leq 0$ for $x \geq 1$.

(ii) $\left(\frac{f(x)}{x}\right)' \geq 0$ for $x \geq 1$.

(iii) For each permutation $g \in \mathrm{Sym}_n$, we have

$$d(g) = \sum_{c\in C(g)} f(n_c),$$

We say d is a **norm** on Sym_n, and f is the **generator** of d.

Lemma 1. *Let f be a generator of a norm d on Sym_n, then $f(1) = 0$.*

Proof. Let $g = (1)(2\dots n)$, then $d(g) = f(1) + f(n-1) = f(1) + d(g)$, hence $f(1) = 0$. □

Lemma 2. $f'(x) \geq 0$.

Proof. Let $g(x) = f(x)/x$. We have $f'(x) = [x \cdot g(x)]' = x \cdot g'(x) + g(x) \geq g(x) = \int_1^x g'(x) \geq 0$. □

Lemma 3. *Let f be a generator of a norm on Sym_n. For every $x_1, \dots x_k \geq 1$, we have $\sum_{i=1}^{k} f(x_i) \leq f(\sum_{i=1}^{k} x_i)$.*

Proof. We have

$$\sum_{i=1}^{k} f(x_i) = \sum_{i=1}^{k} x_i \frac{f(x_i)}{x_i} \leq \sum_{i=1}^{k} x_i \frac{f(\sum_{i=1}^{k} x_i)}{\sum_{i=1}^{k} x_i} = f(\sum_{i=1}^{k} x_i).$$

□

Lemma 4. *Let f be a generator of a norm on Sym_n. For every $x_1, \dots x_k \geq 1$, we have $\sum_{i=1}^{k} f(x_i) \geq f(\sum_{i=1}^{k} x_i - k + 1)$.*

Proof. When $k = 2$, we have

$$f(x_1) + f(x_2) = \int_1^{x_1} f'(x)dx + \int_1^{x_2} f'(x)dx$$
$$\geq \int_1^{x_1} f'(x)dx + \int_{x_1}^{x_1+x_2-1} f'(x)dx = f(x_1 + x_2 - 1).$$

Suppose the argument holds when $k = k'$. When $k = k' + 1$, we have

$$\sum_{i=1}^{k'+1} f(x_i) \geq f(\sum_{i=1}^{k'} x_i - k' + 1) + f(x_{k'+1}) \geq f(\sum_{i=1}^{k'+1} x_i - k').$$

□

Theorem 1. *Let $d : Sym_n \to \mathbb{R}$ be a norm on Sym_n whose generator is f, $g \in Sym_n$ be a permutation, $c \in Sym_n$ be a cycle. Then we have $d(cg) \leq d(c) + d(g)$.*

Proof. Let $A \subseteq C(g)$ be the set of all cycles in $C(g)$ that are disjoint with c. Then we have $A \subseteq C(cg)$. Let $B_1 = C(g) \setminus A$, $B_2 = C(cg) \setminus A$. We have

$$\begin{aligned} & d(c) + d(g) - d(cg) \\ = & \; f(n_c) + \sum_{r\in B_1} f(n_r) - \sum_{r\in B_2} f(n_r) \end{aligned}$$

Let $N = \sum_{r\in B_1} n_c = \sum_{r\in B_2} n_c$. By Lemma 3, we have

$$\sum_{r\in B_2} f(n_r) \le f(N).$$

By lemma 2,4 and the fact that $|B_1| \le n_c$, we have

$$\sum_{r\in B_1} f(n_r) \ge f(N - |B_1| + 1) \ge f(N - n_c + 1).$$

Therefore

$$d(c) + d(g) - d(cg) \ge f(n_c) + f(N - n_c + 1) - f(N) \ge 0.$$

□

Theorem 2. *Let $d : Sym_n \to \mathbb{R}$ be a norm on Sym_n whose generator is f. For $g_1, g_2 \in S_n$, we have $d(g_1g_2) \le d(g_1) + d(g_2)$.*

Proof. Let $C(g_1) = \{c_1, \ldots, c_k\}$. We have

$$\begin{aligned} d(g_1g_2) &= d((\prod_{i=1}^{k} c_i)g_2) \le d(c_1) + d((\prod_{i=2}^{k} c_i)g_2) \\ &\le \ldots \le \sum_{i=1}^{k} d(c_i) + d(g_2) = d(g_1) + d(g_2). \end{aligned}$$

□

Corollary 1. *The function*

$$d : Sym_n \to \mathbb{R}, d(g) = \sum_{c\in C(g)} \frac{1 - n_c^2}{n_c}$$

is a norm on Sym_n.

Proof. It is sufficient to show that $f(x) = \frac{x^2-1}{x}$ is a generator. We have

$$\left(\frac{f(x)}{x}\right)' = \frac{2}{x^3} \ge 0$$

when $x \ge 1$, and

$$f''(x) = -\frac{2}{x^3} \le 0.$$

□

[1] J. C. Bridgeman and C. T. Chubb (2016), 1603.03039.
[2] G. Evenbly and G. Vidal, Physical Review Letters **115**, 180405 (2015), 1412.0732.
[3] A. J. Ferris and D. Poulin, Phys. Rev. Lett. **113**, 030501 (2014).
[4] E. Miles Stoudenmire and D. J. Schwab, ArXiv e-prints (2016), 1605.05775.
[5] A. Novikov, D. Podoprikhin, A. Osokin, and D. P. Vetrov, CoRR **abs/1509.06569** (2015), URL `http://arxiv.org/abs/1509.06569`.
[6] R. Orus, Eur. Phys. J. **B87**, 280 (2014), 1407.6552.
[7] B. Swingle, Phys. Rev. **D86**, 065007 (2012), 0905.1317.
[8] M. Van Raamsdonk, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015* (2017), pp. 297–351, 1609.00026, URL `https://inspirehep.net/record/1484863/files/arXiv:1609.00026.pdf`.
[9] E. P. Verlinde (2016), 1611.02269.
[10] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006), hep-th/0603001.
[11] A. Lewkowycz and J. Maldacena, JHEP **08**, 090 (2013), 1304.4926.
[12] X. Dong, A. Lewkowycz, and M. Rangamani, JHEP **11**, 028 (2016), 1607.07506.
[13] H. Casini, M. Huerta, and R. C. Myers, JHEP **05**, 036 (2011), 1102.0440.
[14] L.-Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, JHEP **12**, 047 (2011), 1110.1084.
[15] X. Dong, Nature Commun. **7**, 12472 (2016), 1601.06788.
[16] M. Rangamani and T. Takayanagi (2016), 1609.01287.
[17] J. de Boer and J. I. Jottar, JHEP **04**, 089 (2014), 1306.4347.
[18] B. Chen and J.-q. Wu, JHEP **09**, 015 (2016), 1605.06753.
[19] X. Dong, Phys. Rev. Lett. **116**, 251602 (2016), 1602.08493.
[20] W. Song, Q. Wen, and J. Xu, JHEP **02**, 067 (2017), 1610.00727.
[21] M. Ammon, A. Castro, and N. Iqbal, JHEP **10**, 110 (2013), 1306.4338.
[22] Y. Ling, P. Liu, C. Niu, J.-P. Wu, and Z.-Y. Xian, JHEP **04**, 114 (2016), 1502.03661.
[23] Q. Hu and G. Vidal (2017), 1703.04798.
[24] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, JHEP **06**, 149 (2015), 1503.06237.
[25] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang, JHEP **11**, 009 (2016), 1601.01694.
[26] M. Miyaji, T. Takayanagi, and K. Watanabe, Phys. Rev. **D95**, 066004 (2017), 1609.04645.
[27] A. Bhattacharyya, Z.-S. Gao, L.-Y. Hung, and S.-N. Liu, JHEP

08, 086 (2016), 1606.00621.
[28] B. Czech, P. H. Nguyen, and S. Swaminathan, JHEP **03**, 090 (2017), 1612.05698.
[29] Z.-X. Luo, E. Lake, and Y.-S. Wu, ArXiv e-prints (2016), 1611.01140.
[30] K. Li, M. Han, G. Long, Y. Wan, D. Lu, B. Zeng, and R. Laflamme (2017), 1705.00365.
[31] X.-L. Qi (2013), 1309.6282.
[32] X.-L. Qi, Z. Yang, and Y.-Z. You (2017), 1703.06533.
[33] A. Bhattacharyya, L.-Y. Hung, Y. Lei, and W. Li (2017), 1703.05445.
[34] A. May (2016), 1611.06220.
[35] Y. Li, M. Han, M. Grassl, and B. Zeng (2016), 1612.04504.
[36] J. Cotler, P. Hayden, G. Salton, B. Swingle, and M. Walter (2017), 1704.05839.
[37] G. Chirco, D. Oriti, and M. Zhang (2017), 1701.01383.
[38] F. Pastawski and J. Preskill (2016), 1612.00017.
[39] A. Peach and S. F. Ross, Class. Quant. Grav. **34**, 105011 (2017), 1702.05984.
[40] D. A. Roberts and B. Yoshida, JHEP **04**, 121 (2017), 1610.04903.
[41] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, JHEP **02**, 004 (2016), 1511.04021.
[42] L. Smolin (2016), 1608.02932.
[43] M. Han, W. Huang, and Y. Ma, Int.J.Mod.Phys. **D16**, 1397 (2007), gr-qc/0509064.
[44] A. Ashtekar and J. Lewandowski, Class.Quant.Grav. **21**, R53 (2004), gr-qc/0404018.
[45] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, 2007).
[46] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2014), ISBN 9781107069626, URL `https://books.google.com/books?id=4VjeBAAAQBAJ`.
[47] M. Han and L.-Y. Hung, Phys. Rev. **D95**, 024011 (2017), 1610.02134.
[48] P. Calabrese and J. L. Cardy, J. Stat. Mech. **0406**, P06002 (2004), hep-th/0405152.
[49] T. Regge, Nuovo Cim. **19**, 558 (1961).
[50] R. Friedberg and T. Lee, Nucl.Phys. **B242**, 145 (1984).
[51] S. S. Gubser, M. Heydeman, C. Jepsen, M. Marcolli, S. Parikh, I. Saberi, B. Stoica, and B. Trundy (2016), 1612.09580.
[52] B. Bahr and B. Dittrich, Phys.Rev. **D80**, 124030 (2009), 0907.4323.
[53] J. Hartle and R. Sorkin, Gen.Rel.Grav. **13**, 541 (1981).
[54] M. Henningson and K. Skenderis, JHEP **07**, 023 (1998), hep-th/9806087.
[55] A. W. Harrow (2013), 1308.6595.