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On rainbow-k-connectivity of random graphs ☆

Jing He, Hongyu Liang *

Institute for Interdisciplinary Information Sciences, FIT Building 4-609, Tsinghua University, Beijing 100084, China

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ABSTRACT

A path in an edge-colored graph is called a *rainbow path* if the edges on it have distinct colors. For $k \geqslant 1$, the *rainbow-k-connectivity* of a graph G, denoted by $rc_k(G)$, is the minimum number of colors required to color the edges of G in such a way that every two distinct vertices are connected by at least k internally vertex-disjoint rainbow paths. In this paper, we study rainbow-k-connectivity in the setting of random graphs. We show that for every fixed integer $d \geqslant 2$ and every $k \leqslant O(\log n)$, $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leqslant d$. This substantially generalizes a result in [Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, Electron. J. Comb. 15 (2008)], stating that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_1(G(n,p)) \leqslant 2$. As a by-product, we obtain a polynomial-time algorithm that makes $G(n,p)$ rainbow-k-connected using at most one more than the optimal number of colors with probability $1 - o(1)$, for all $k \leqslant O(\log n)$ and $p = n^{-\epsilon(1\pm o(1))}$ for any constant $\epsilon \in [0, 1)$.

1. Introduction

All graphs considered in this paper are finite, simple, undirected and contain at least 2 vertices. We follow the notation and terminology of [3]. The following notion of *rainbow-k-connectivity* was proposed by Chartrand et al. [8,9] as a strengthening of the canonical connectivity concept in graphs. Given an edge-colored graph G, a path in G is called a *rainbow path* if its edges have distinct colors. For an integer $k \geqslant 1$, an edge-colored graph is called *rainbow-k-connected* if any two different vertices of G are connected by at least k internally vertex-disjoint rainbow paths. The *rainbow-k-connectivity* of G, denoted by $rc_k(G)$, is the minimum number of colors required to color the edges of G to make it rainbow-k-connected. Note that such coloring does not exist if G is not k-vertex-connected, in which case we simply let $rc_k(G) = \infty$. When $k = 1$ it is alternatively called *rainbow-connectivity* or *rainbow connection number* in literature, and is conventionally written as $rc(G)$ with the subscript k dropped.

Besides its theoretical interest as being a natural combinatorial concept, rainbow connectivity also finds applications in networking and secure message transmitting [6,11,15]. The following motivation is given in [6]: Suppose we want to route messages in a cellular network such that each link on the route between two vertices is assigned with a distinct channel. Then the minimum number of used channels is exactly the rainbow-connectivity of the underlying graph.

Some easy observations regarding rainbow-k-connectivity include that $rc_k(G) = 1$ if and only if $k = 1$ and G is a clique, that $rc(G) \leqslant n - 1$ for all connected G, and that $rc(G) = n - 1$ if and only if G is a tree, where n is the number of vertices in G. Chartrand et al. [8] determined the rainbow-connectivity of several special classes of graphs, including complete multipartite graphs. In [9] they investigated rainbow-k-connectivity in complete graphs and regular complete bipartite graphs. The extremal graph-theoretic aspect of rainbow-connectivity was studied by Caro et al. [5], who proved that $rc(G) = O_\delta(n \log\delta/\delta)$ with δ being the minimum degree of G. This tradeoff was later improved to

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* Corresponding author.
E-mail addresses: he-j08@mails.tsinghua.edu.cn (J. He), lianghy08@mails.tsinghua.edu.cn (H. Liang).

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$rc(G) < 20n/\delta$ by Krivelevich and Yuster [13], and was recently shown to be $rc(G) \leqslant 3n/(\delta+1)+3$ by Chandran et al. [7] which is essentially tight. Chakraborty et al. [6] studied the computational complexity perspective of this notion, proving among other results that given a graph G deciding whether $rc(G)=2$ is NP-complete.

Another important setting that has been extensively explored for studying various graph concepts is the Erdős–Rényi random graph model $G(n,p)$ [10], in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability p independently from other pairs. We say an event $\mathcal{E}$ happens *almost surely* if the probability that it happens approaches 1 as $n \to \infty$, i.e., $\mathbf{Pr}[\mathcal{E}] = 1 - o_n(1)$. We will always assume that n is the variable that tends to infinity, and thus omit the subscript n from the asymptotic notations. For a graph property P, a function $p(n)$ is called a *threshold function* of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ almost surely satisfies P; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ almost surely does not satisfy P.

Furthermore, $p(n)$ is called a *sharp threshold function* of P if there exist two positive constants c and C such that:

- for every $r(n) \geqslant C \cdot p(n)$, $G(n, r(n))$ almost surely satisfies P; and
- for every $r'(n) \leqslant c \cdot p(n)$, $G(n, r'(n))$ almost surely does not satisfy P.

Clearly a sharp threshold function of a graph property is also a threshold function of it; yet the converse may not hold, e.g., the property of containing a triangle [2].

It is known that every non-trivial monotone graph property possesses a threshold function [4,12]. Obviously for every k, d, the property $rc_k(G) \leqslant d$ is monotone, and thus has a threshold. Caro et al. [5] proved that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_1(G(n,p)) \leqslant 2$. In this paper, we significantly extend their result by establishing sharp thresholds for the property $rc_k(G(n,p)) \leqslant d$ for all constants d and logarithmically increasing k. Our main theorem is as follows.

Theorem 1. *Let $d \geqslant 2$ be a fixed integer and $k = k(n) \leqslant O(\log n)$. Then $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leqslant d$.*

We also investigate rainbow-k-connectivity from the algorithmic point of view. The NP-hardness of determining $rc(G)$ is shown by Chakraborty et al. [6]. We show that the problem (even the search version) becomes easy in random graphs, by designing an algorithm for coloring random graphs to make it rainbow-k-connected with near-optimal number of colors.

Theorem 2. *For any constant $\epsilon \in [0,1)$, $p = n^{-\epsilon(1\pm o(1))}$ and $k \leqslant O(\log n)$, there is a randomized polynomial-time algorithm that, with probability $1-o(1)$, makes $G(n,p)$ rainbow-k-connected using at most one more than the optimal number of colors, where the probability is taken over both the randomness of $G(n,p)$ and that of the algorithm.*

Our result is quite strong, since almost all natural edge probability functions p encountered in various scenarios satisfy $p = n^{-\epsilon(1\pm o(1))}$ for some $\epsilon > 0$. Note that $G(n, n^{-\epsilon})$ is almost surely disconnected when $\epsilon > 1$ [10], which makes the problem become trivial. We therefore ignore these cases.

In Section 2 we present the proof of Theorem 1, and in Section 3 we show the correctness of Theorem 2.

2. Threshold of rainbow-*k*-connectivity

This section is devoted to proving Theorem 1. Throughout the paper "ln" denotes the natural logarithm, and "log" denotes the logarithm to the base 2. Hereafter we assume $d \geqslant 2$ is a fixed integer, $c_0 \geqslant 1$ a positive constant, and $k = k(n) \leqslant c_0 \log n$ for all sufficiently large n. To establish a sharp threshold function for a graph property the proof should be two-fold. We first show the easy direction.

Theorem 3. *$rc_k(G(n, (\ln n)^{1/d}/n^{(d-1)/d})) \geqslant d+1$ almost surely holds.*

We need the following fact proved by Bollobás [1].

Lemma 1. *(See restatement of part of Theorem 6 in [1].) Let c be a positive constant and $d \geqslant 2$ a fixed integer. Let $p' = (\ln(n^2/c))^{1/d}/n^{(d-1)/d}$. Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\big[G\big(n,p'\big) \text{ has diameter at most } d\big] = e^{-c/2}.$$

Proof of Theorem 3. Fix an arbitrary $\epsilon > 0$ and choose a constant $c > 0$ so that $e^{-c/2} < \epsilon/2$. Let $p' = (\ln(n^2/c))^{1/d}/n^{(d-1)/d}$ and $p = (\ln n)^{1/d}/n^{(d-1)/d}$. Clearly $p \leqslant p'$ for all $n > c$.

By Lemma 1 and the definition of limits, there exists an $N_1 > 0$ such that for all $n > N_1$, $\mathbf{Pr}[G(n,p')$ has diameter at most $d] < e^{-c/2} + \epsilon/2 < \epsilon$, by our choice of c. Thus, for every $n > \max\{c, N_1\}$,

$$\begin{aligned}&\mathbf{Pr}\big[G(n,p) \text{ has diameter at most } d\big]\\ &\quad\leqslant \mathbf{Pr}\big[G\big(n,p'\big) \text{ has diameter at most } d\big] < \epsilon.\end{aligned}$$

Due to the arbitrariness of ϵ, this implies that the probability of $G(n,p)$ having diameter at most d is $o(1)$. This completes the proof of Theorem 3, since the rainbow-k-connectivity of a graph is at least as large as its diameter. □

We are left with the other direction stated below. Fix $C = 2^{20} \cdot c_0$.

Theorem 4. *$rc_k(G(n, C(\log n)^{1/d}/n^{(d-1)/d})) \leqslant d$ almost surely holds.*

The key component of our proof of Theorem 4 is the following theorem.

Theorem 5. *With probability at least $1 - n^{-\Omega(1)}$, every two different vertices of $G(n, C(\log n)^{1/d}/n^{(d-1)/d})$ are connected by at least $2^{10d}c_0 \log n$ internally vertex-disjoint paths of length exactly d.*

Before demonstrating Theorem 5, we show how Theorem 4 follows from it.

Proof of Theorem 4. Let G be an instance of $G(n, C(\log n)^{1/d}/n^{(d-1)/d})$ for which the condition in Theorem 5 holds; that is, every two different vertices of G have at least $c_1 \log n$ internally vertex-disjoint paths of length d connecting them, where $c_1 := 2^{10d}c_0$. To establish Theorem 4 it suffices to prove that $rc_k(G) \leqslant d$ for every such G, since by Theorem 5 the condition holds with probability at least $1 - n^{-\Omega(1)} = 1 - o(1)$.

Let $S = \{1, 2, \dots, d\}$ be a set of d distinct colors. We randomly color the edges of G with colors from S. Fix two distinct vertices u and v. Let P be a path of length d connecting u and v. The probability that P becomes a rainbow path under the random coloring is

$$q := d!/d^d \geqslant (d/e)^d/d^d \geqslant 4^{-d},$$

by Stirling's formula. Since there are at least $c_1 \log n$ such paths and they are all edge-disjoint, we can upper-bound the probability that at most $k-1$ of them become rainbow paths by

$$\begin{aligned}
&\binom{c_1 \log n}{k-1}(1-q)^{c_1 \log n-(k-1)} \\
&\quad\leqslant \binom{c_1 \log n}{c_0 \log n}\left(1-4^{-d}\right)^{(c_1-c_0)\log n} \\
&\quad\leqslant 2^{c_1 \log n \cdot H(c_0/c_1)} \cdot 2^{-4^{-d}(c_1-c_0)\log n} \\
&\quad= n^{-(4^{-d}(c_1-c_0)-c_1 \cdot H(c_0/c_1))},
\end{aligned}$$

where the second inequality follows from the fact that

$$\binom{m}{\alpha m} \leqslant 2^{m \cdot H(\alpha)}$$

for all constants $\alpha \in (0, 1)$ and sufficiently large m, H being the binary entropy function defined as

$$H(\epsilon) = \epsilon \log(1/\epsilon) + (1-\epsilon)\log\bigl(1/(1-\epsilon)\bigr),$$

and that

$$1 - x \leqslant e^{-x} \leqslant 2^{-x}, \quad \text{for all } x \geqslant 0.$$

It is easy to verify that $\log x \leqslant \sqrt{x}$ whenever $x \geqslant 100$. Also, since $1 + x \leqslant e^x \leqslant 2^{2x}$, we have $\log(1+x) \leqslant 2x$ for all $x > -1$. Recalling that $c_1 = 2^{10d}c_0 > 200c_0$, we get

$$\begin{aligned}
H(c_0/c_1) &= (c_0/c_1)\log(c_1/c_0) \\
&\qquad + (1-c_0/c_1)\log\bigl(1 + c_0/(c_1-c_0)\bigr) \\
&\leqslant (c_0/c_1)\sqrt{c_1/c_0} + (1-c_0/c_1) \cdot 2c_0/(c_1-c_0) \\
&= \sqrt{c_0/c_1} + 2c_0/c_1 \leqslant 3\sqrt{c_0/c_1}.
\end{aligned}$$

We thus have

$$\begin{aligned}
&4^{-d}(c_1-c_0) - c_1 \cdot H(c_0/c_1) \\
&\quad\geqslant 4^{-d}(c_1-c_0) - 3\sqrt{c_1 c_0} \\
&\quad= 4^{-d}c_1\left(1-2^{-10d}\right) - 3\sqrt{2^{-10d} \cdot c_1^2} \\
&\quad\geqslant 2^{-2d-1}c_1 - 2^{-5d+2}c_1 \\
&\quad\geqslant c_1 \cdot 2^{-2d-2} \\
&\quad= c_0 \cdot 2^{10d} \cdot 2^{-2d-2} > 100,
\end{aligned}$$

since $c_0 \geqslant 1$ and $d \geqslant 2$. Therefore, the probability that there exist at most $k-1$ rainbow paths between u and v is at most

$$n^{-(4^{-d}(c_1-c_0)-c_1 \cdot H(c_0/c_1))} < n^{-100}.$$

By the Union Bound, with probability at least

$$1 - \binom{n}{2}n^{-100} \geqslant 1 - n^{-90},$$

every two distinct vertices of G have at least k internally vertex-disjoint rainbow paths connecting them. In particular, there exists a d-coloring of the edges of G under which G becomes k-rainbow-connected, implying that $rc_k(G) \leqslant d$. This concludes the proof of Theorem 4. □

We now prove Theorem 5.

Proof of Theorem 5. Let $p = C(\log n)^{1/d}/n^{(d-1)/d}$ where $C = 2^{20}c_0$. Let V be the set of all vertices in $G(n, p)$. For every $S \subseteq V$ and $u \in S$, let $\mathcal{A}(S, u)$ be the event that u is adjacent to at least $pn/10$ $(= Cn^{1/d}(\log n)^{1/d}/10)$ distinct vertices in $V \setminus S$. The following lemma is needed for our proof.

Lemma 2. *For every S, u such that $u \in S$ and $|S| \leqslant d \cdot (pn/10)^{d-1}$,*

$$\mathbf{Pr}\bigl[\mathcal{A}(S, u)\bigr] \geqslant 1 - 2^{-\Omega(n^{1/d})}.$$

Proof. Fix $S \subseteq V$ with $|S| \leqslant d \cdot (pn/10)^{d-1}$, and $u \in S$. We have

$$\begin{aligned}
|V \setminus S| &\geqslant n - d \cdot (pn/10)^{d-1} \\
&= n - d \cdot (C/10)^{d-1}n^{(d-1)/d}(\log n)^{(d-1)/d} \geqslant n/2,
\end{aligned}$$

for all sufficiently large n. Take T to be any subset of $V \setminus S$ of cardinality $n/2$. Let X be the random variable counting the number of neighbors of u inside T. It is obvious that X can be expressed as the sum of $n/2$ independent random variables, each of which taking 1 with probability p and 0 with probability $1-p$. Thus $\mathbf{E}[X] = pn/2$. By the Chernoff–Hoeffding Bound (see e.g. Theorem 4.2 of [14]), we have

$$\begin{aligned}
&\mathbf{Pr}\bigl[X < (1-4/5)pn/2\bigr] \\
&\quad\leqslant \exp\bigl(-(1/2)(4/5)^2(pn/2)\bigr) = 2^{-\Omega(n^{1/d})},
\end{aligned}$$

which gives precisely what we want. □

We now continue the proof of Theorem 5. Fix $u, v \in V$, $u \neq v$. Let $S_0 = \{u\}$. A *t-ary tree* with a designated root is a tree whose non-leaf vertices all have exactly t children. Consider the following process of "choosing" a $(pn/10)$-ary tree of depth $d-1$ rooted at u:

1. Let $i \leftarrow 1$ and $S_i \leftarrow \emptyset$.
2. For every vertex $w \in S_{i-1}$ (in an arbitrary order), choose $pn/10$ distinct neighbors of w from the set $V \setminus (\{v\} \cup \bigcup_{j=0}^{i} S_j)$, and add them to S_i. (Note that S_i is updated every time after the processing of a vertex w, so that one vertex cannot be chosen and added to S_i more than once. This ensures that at the end of this step, $|S_i| = (pn/10)^i$.)
3. Let $i \leftarrow i+1$. If $i \leqslant d-1$ then go to Step 2, otherwise stop.

Of course the process may fail during Step 2, since with nonzero probability w will have no neighbor in $V \setminus (\{v\} \cup \bigcup_{j=0}^{i} S_j)$. (In fact, with nonzero probability the graph becomes empty.) However, noting that at any time during the process,

$$\left|\{v\} \cup \bigcup_{j=0}^{i} S_j\right| \leqslant 1 + \sum_{j=0}^{d-1} (pn/10)^j \leqslant d \cdot (pn/10)^{d-1},$$

for all sufficiently large n,

we can deduce from Lemma 2 that every execution of Step 2 fails with probability at most $2^{-\Omega(n^{1/d})}$. Since Step 2 can be conducted for at most $d \cdot (pn/10)^{d-1}$ times, we obtain that, with probability at least

$$1 - d \cdot (pn/10)^{d-1} \cdot 2^{-\Omega(n^{1/d})} = 1 - 2^{-\Omega(n^{1/d})},$$

the process will successfully terminate. At the end of the process, the sets $S_0, S_1, \dots, S_{d-1}$ naturally induces a $(pn/10)$-ary tree T of depth $d-1$ rooted at u, with S_i being the collection of vertices in the i-th level. The number of leaves in T is exactly $|S_{d-1}| = (pn/10)^{d-1}$.

Now we assume that T has been successfully constructed. Let Y be a random variable denoting the number of neighbors of v inside S_{d-1}. (Recall that $v \notin S_{d-1}$.) It is clear that

$$\mathbf{E}[Y] = p \cdot |S_{d-1}| = p^d n^{d-1}/10^{d-1} = 10 \cdot (C/10)^d \log n.$$

As before, using the Chernoff–Hoeffding Bound, we have

$$\begin{aligned}&\mathbf{Pr}\big[Y < (C/10)^d \log n\big]\\ &\quad\leqslant \exp\big(-(1/2)(9/10)^2 (C/10)^d \cdot 10 \log n\big) \leqslant n^{-10},\end{aligned}$$

for our choice of C.

It is clear that each neighbor v' of v inside S_{d-1} induces a length-d path between u and v (by simply combining the path from u to v' in tree T and the edge (v', v)). The problem is that these paths may not be internally vertex-disjoint. We next address this issue.

For every $w \in S_1$, denote by T_w the subtree of T of depth $d-2$ rooted at w. Call these T_w *vice-trees*. Clearly every vice-tree contains $(pn/10)^{d-2}$ leaves.

The reason for defining such vice-trees is that, by simple observations, any two leaves of T that belong to different vice-trees must correspond to edge-disjoint root-to-leaf paths in T. Thus, to establish a large number of internally vertex-disjoint paths between u and v, it suffices to show that we can find many neighbors of v inside S_{d-1} that belong to distinct vice-trees.

For each vice-tree T_w, let $\mathcal{B}_w$ be the event that v has at least $10d$ neighbors inside the set of leaves of T_w. Noting that T_w has exactly $(pn/10)^{d-2}$ leaves, we have

$$\begin{aligned}\mathbf{Pr}[\mathcal{B}_w] &\leqslant \binom{(pn/10)^{d-2}}{10d} p^{10d}\\ &= \binom{(Cn^{1/d}(\log n)^{1/d}/10)^{d-2}}{10d} \cdot \left(\frac{C(\log n)^{1/d}}{n^{(d-1)/d}}\right)^{10d}\\ &\leqslant \left(\big(Cn^{1/d}(\log n)^{1/d}/10\big)^{d-2} \cdot \frac{C(\log n)^{1/d}}{n^{(d-1)/d}}\right)^{10d}\\ &= C^{10d}(C/10)^{10d(d-2)}(\log n)^{10(d-1)} n^{-10}\\ &\leqslant O\big(n^{-9}\big).\end{aligned}$$

By applying the Union Bound, we obtain

$$\mathbf{Pr}\left[\bigvee_{w \in S_1} \mathcal{B}_w\right] \leqslant (pn/10) \cdot O\big(n^{-9}\big) \leqslant O\big(n^{-7}\big).$$

Combined with previous results, we deduce that with probability at least

$$1 - 2^{-\Omega(n^{1/d})} - n^{-10} - O\big(n^{-7}\big) \geqslant 1 - O\big(n^{-6}\big),$$

the following three events simultaneously happen:

1. The tree T is successfully constructed.
2. v has at least $(C/10)^d \log n$ neighbors that are leaves of T.
3. Every vice-tree T_w contains at most $10d$ leaves that are neighbors of v.

When all these three events happen, we can choose $((C/10)^d/(10d)) \log n$ neighbors of v, every two of which come from different vice-trees. This immediately leads to $((C/10)^d/(10d)) \log n$ length-d internally vertex-disjoint paths between u and v, where, by our choice of $C = 2^{20} c_0$,

$$\big((C/10)^d/(10d)\big) \log n \geqslant 2^{10d} c_0 \log n.$$

Using the Union Bound again gives us that, with probability at least

$$1 - \binom{n}{2} \cdot O\big(n^{-6}\big) = 1 - n^{-\Omega(1)},$$

every two distinct vertices have at least $2^{10d} c_0 \log n$ internally vertex-disjoint paths of length d connecting them. The proof of Theorem 5 is thus completed. □

3. Rainbow-coloring random graphs

In this section we prove Theorem 2.

Proof of Theorem 2. First note that for every G with at least 2 vertices, $rc_k(G) = 1$ if and only if $k = 1$ and G is a clique, which can be easily checked. Thus, in the following we assume w.l.o.g. that $rc_k(G(n, p)) \geqslant 2$.

It is easy to see that there exists a (unique) integer $d \geqslant 2$ such that $(d-2)/(d-1) \leqslant \epsilon < (d-1)/d$. We have $p = \omega((\log n)^{1/d}/n^{(d-1)/d})$, which, by Theorem 4, implies that $rc_k(G(n,p)) \leqslant d$ almost surely holds. Moreover, a scrutiny into the proof of Theorem 5 tells us that for such p, a random coloring of $G(n,p)$ using d colors will make it rainbow-k-connected with probability $1 - n^{-\Omega(1)}$. Thus, it suffices for us to show that with probability $1 - o(1)$, $rc_k(G(n,p)) \geqslant d-1$. This is trivial for $d \leqslant 3$, since we have assumed that $rc_k(G(n,p)) \geqslant 2$. When $d \geqslant 4$, we have $p = o((\log n)^{1/(d-2)}/n^{(d-3)/(d-2)})$. Due to Theorem 3, $G(n,p)$ with such p almost surely satisfies $rc_k(G(n,p)) \geqslant d-1$.

Hence, we have shown that with probability $1 - o(1)$, a random coloring with d colors will make $G(n,p)$ rainbow-k-connected and the number of colors used is at most one more than the optimum, where the probability is taken over both the randomness of $G(n,p)$ and that of the algorithm. This completes the whole proof. □

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