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Lower Bounds for the Determinantal Complexity of Explicit Low Degree Polynomials

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Abstract The determinantal complexity of a polynomial $f(x_1, x_2, \ldots, x_n)$ is the minimum m such that $f = \det_m(L(x_1, x_2, \ldots, x_n))$, where $L(x_1, x_2, \ldots, x_n)$ is a matrix whose entries are affine forms in the x_is over some field $\mathbb{F}$.

Asymptotically tight lower bounds are proven for the determinantal complexity of the elementary symmetric polynomial S_n^d of degree d in n variables, $2d$-fold iterated matrix multiplication of the form $\langle u|X^1X^2 \ldots X^{2d}|v\rangle$, and the symmetric power sum polynomial $\sum_{i=1}^n x_i^d$, for any fixed $d > 1$.

A restriction of determinantal computation is considered in which the underlying affine map $\lambda x.L(x)$ must satisfy a rank lowerability property: L mapping to $m \times m$ matrices is said to be r-lowerable, if there exists an $a \in \mathbb{F}^n$ such that $\text{rank}(L(a)) \leq m - r$. In this model strongly nonlinear and exponential lower bounds are proved for several polynomial families. For example, for S_n^{2d} it is proved that the determinantal complexity using r-lowerable maps is $\Omega(n^{d/(2d-r)})$, for constants d and r with $2 \leq d + 1 \leq r < 2d$. For $r = 2d - 1$ and $d = \lfloor n^{1/5-\epsilon} \rfloor$, a lower bound is given for S_n^{2d} of magnitude $n^{\Omega(\epsilon n^{1/5-\epsilon})}$, for any $\epsilon \in (0, 1/5)$.

Keywords Computational complexity · Arithmetic circuits · Determinant versus permanent · Elementary symmetric polynomial

1 Introduction

The main open problem in algebraic complexity theory is the resolution of Valiant's Hypothesis, which states that the complexity classes VP and VNP are distinct. The

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question is attractive to study as a stepping stone towards the P versus NP conundrum, as in this area algebraic tools are more readily available. Over the field of complex numbers VP $\neq$ VNP is known to be implied by NP $\not\subseteq$ P/*poly*, provided the *Generalized Riemann Hypothesis* is true [4]. Currently, we do not know of a reverse implication.

The complexity class VNP is characterized by the permanent polynomial $per_n = \sum_\sigma \prod_{i\in[n]} x_{i\sigma(i)}$, over fields of characteristic other than two [15], cf. [3]. Here the summation is over all permutations σ of $[n]$. For comparison, we have the determinant polynomial defined by $\det_n = \sum_\sigma \mathrm{sgn}(\sigma)\prod_{i\in[n]} x_{i\sigma(i)}$, where $\mathrm{sgn}(\sigma)$ denotes the *signature* of the permutation σ. Resolution of Valiant's Hypothesis comes down to proving lower bounds for the *determinantal complexity* of the permanent. In general, the determinantal complexity of a polynomial $f(x)$, denoted by $\mathrm{dc}(f)$, is the minimum m such that $f = \det_m(L(x))$, where $L(x)$ is a matrix whose entries are affine forms in x. Proving $\mathrm{dc}(per_n) = n^{\omega(\log n)}$ is known to be sufficient for separating VP and VNP. This is a plausible condition, knowing that computing the permanent of a 0, 1-matrix over the integers is #P-complete [16]. By Toda's Theorem [13] this implies the permanent is PH-hard. On the other hand, integer determinants can be computed with NC_2-circuits [2].

Currently the best known lower bound on determinantal complexity of the permanent is $\mathrm{dc}(per_n) \geq n^2/2$ [7] (cf. [5]). Their lower bound technique is a dimension argument employing second order partial derivatives. Using partial derivatives of order at most two limits the technique to proving lower bounds that are *linear* in the number of input variables. In this paper the question is investigated whether any stronger results can be obtained by considering higher than second order partial derivatives. Higher order partial derivatives have been instrumental in previous landmark work in algebraic complexity theory, e.g. [8–12].

A fundamental observation is that determinantal complexity minorizes arithmetic formula size up to constant factors, i.e. $\mathrm{dc}(f) \leq 2L(f) + 2$, where $L(f)$ denotes formula size [15], cf. [6]. As a matter of fact, $\mathrm{dc}(f) = O(B(f))$, where $B(f)$ denotes algebraic branching program size of f (see [8]). In the latter model, currently no nonlinear lower bounds are known beyond the trivial bound $B(f) = \Omega(\deg(f))$ and the geometric degree bounds of Baur-Strassen [1]. The later bounds are of level $\Omega(n \log \deg(f))$, and are established for general arithmetic circuits. For explicit f of "reasonable" degree, proving strongly nonlinear lower bounds for $B(f)$, and even more so for $\mathrm{dc}(f)$, is a major open problem.

In this paper the first aim is to investigate under what additional restrictions to the determinantal model we can achieve above goal for $\mathrm{dc}(f)$. It is shown the Mignon-Ressayre technique can be generalized to higher order partial derivatives. This will be applied to a version of determinantal complexity, in which the affine map $\lambda x.L(x)$ is restricted to be so-called *r-lowerable*. This condition stipulates that for L mapping to $m \times m$ matrices, there exists a point a, such that $\mathrm{rank}\, L(a) \leq m - r$.

For low degree polynomials this will yield strongly nonlinear and exponential lower bounds. For example, for the elementary symmetric polynomial of degree d in n variables, defined by $S_n^d(X) = \sum_{I\subset[n],|I|=d} \prod_{i\in I} x_i$, it will be shown that the determinantal complexity of S_n^{2d} when restricted to using r-lowerable maps is $\Omega(n^{d/(2d-r)})$, for any constants d and r with $2 < d + 1 \leq r < 2d$. In the most restrictive setting, for $r = 2d - 1$, an exponential lower bound of level $n^{\Omega(\epsilon n^{1/5-\epsilon})}$ is observed for S_n^{2d}, for $d = \lfloor n^{1/5-\epsilon} \rfloor$ and any constant $\epsilon \in (0, 1/5)$. We stress that for both of these examples, currently we do not know of a finite upper bound for the restricted determinantal complexity.

The second aim of this paper is to consider unrestricted determinantal complexity of several important polynomial families. For example, it will be shown that $\mathrm{dc}(S_{2n}^{2d-1}) \geq n$ and $\mathrm{dc}(S_{2n+2}^{2d}) \geq n/2$, for $d > 1$, over fields of characteristic zero. For constant d, this determines the determinantal complexity of S_n^d up to a constant factor, since $L(S_n^d) = O(nd^3 \log d)$ [12].

1.1 Organization

The rest of this paper is divided as follows. Section 2 contains preliminaries. Section 3 briefly revisits the Mignon-Ressayre technique. In Sect. 4 lower bounds are proven for determinantal complexity under lowerability restrictions. In Sect. 5 lower bounds are proven for the unrestricted determinantal complexity of elementary symmetric polynomials and several other important polynomial families. Finally, some open problems are raised in Sect. 6.

2 Preliminaries

For integer n, $[n]$ denotes $\{1, 2, \ldots, n\}$. Let F be a field. Let $X = \{x_1, \ldots, x_N\}$ and $Y = \{y_1, \ldots, y_M\}$ be sets of variables. Let polynomials $f \in F[X]$ and $g \in F[Y]$ be given. For a vector $r = (r_1, r_2, \ldots, r_M)^T \in F[X]^M$ denote by $g(r)$ the polynomial obtained by substitution of y_i by r_i in g, for all $1 \leq i \leq M$. For a matrix G whose entries are elements in $F[Y]$, let $G(r)$ denote the matrix that has ijth entry $G(r)_{ij} = G_{ij}(r)$. We generalize the framework of [7] by considering arbitrary order partial derivatives. Their results are obtained by setting $k = 1$ in the following suite of results (Lemma 1, Proposition 1, and Lemma 2):

2.1 Partial Derivatives Matrix

Let $k \geq 1$ be an integer. Define the *partial derivatives matrix* of a polynomial f of order $2k$, denoted by $T^{2k} f$, to be an $N^k \times N^k$ matrix of *formal partial derivatives*, with rows and columns indexed by k-tuples $v, w \in X^k$, respectively, where

$$\left(T^{2k} f\right)_{v,w} = \frac{\partial^{2k} f}{\partial v \partial w},$$

where ∂v is shorthand for $\partial v_1 \partial v_2 \ldots \partial v_k$. We use $(i_1, i_2, \ldots, i_k)$ as a shorthand for the index of a row or column given by the k-tuple $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$. In expressions, taking derivatives is given precedence over substitution, e.g. $T^{2k} g\ (L(x))$ means $(T^{2k} g)(L(x))$.

Lemma 1 *Let $k \geq 1$ be an integer. Let C be an $M \times N$ matrix with entries from F. Let $c = (c_1, c_2, \ldots, c_M)^T \in F^M$. Let $x = (x_1, x_2, \ldots, x_N)^T$ be a vector of variables. Suppose for a M-vector $L(x) = Cx + c$ of affine forms that $f = g(L(x))$. Then*

$$T^{2k} f = \left(C^T\right)^{\otimes k} \cdot \left(T^{2k} g\ (L(x))\right) \cdot C^{\otimes k},$$

and hence for any $a \in F^n$,

$$\operatorname{rank} T^{2k} f(a) \le \operatorname{rank} T^{2k} g\big(L(a)\big).$$

Proof Lemma 1 is proved by induction on k. The basis $k = 1$ is given by Lemma 3.1 in [7], and is obtained using the chain rule as follows: $\frac{\partial f}{\partial x_i} = \sum_{k=1}^{M} \frac{\partial g}{\partial y_k}(L(x)) \cdot \frac{\partial (L(x))_k}{\partial x_i} = \sum_{k=1}^{M} \frac{\partial g}{\partial y_k}(L(x)) \cdot C_{ki}$. Hence $(T^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_{k=1}^{M} \sum_{l=1}^{M} \frac{\partial^2 g}{\partial y_k \partial y_l} \times (L(x)) \cdot \frac{\partial (L(x))_l}{\partial x_j} \cdot C_{ki} = \sum_{k=1}^{M} \sum_{l=1}^{M} \frac{\partial^2 g}{\partial y_k \partial y_l}(L(x)) \cdot C_{lj} C_{ki} = \sum_{k=1}^{M} \sum_{l=1}^{M} \times (T^2 g\ (L(x)))_{kl} \cdot C_{lj} C_{ki} = (C^T \cdot T^2 g\ (L(x)) \cdot C)_{ij}$.

Now assume the statement of the lemma is true for a $k \ge 1$. Let $(v, i)(w, j) \in [N]^{k+1}$. We have that

$$\begin{aligned}
\big(T^{2k+2} f\big)_{(v,i)(w,j)} &= \frac{\partial^2 (T^{2k} f)_{vw}}{\partial x_i \partial x_j} \\
&= \frac{\partial^2 [(C^T)^{\otimes k} \cdot (T^{2k} g\ (L(x))) \cdot C^{\otimes k}]_{vw}}{\partial x_i \partial x_j} \\
&= \sum_{s,t \in [N]^k} \big[(C^T)^{\otimes k}\big]_{vs} \cdot \frac{\partial^2 [T^{2k} g(L(x))]_{st}}{\partial x_i \partial x_j} \cdot \big[C^{\otimes k}\big]_{tw}. \qquad (1)
\end{aligned}$$

Let $h_{st} = [T^{2k} g\ (L(x))]_{st}$. We have $h_{st} = [T^{2k} g]_{st}(L(x))$. By the base case

$$\begin{aligned}
\frac{\partial^2 h_{st}}{\partial x_i \partial x_j} &= \big[C^T \cdot \big[T^2 \big[T^{2k} g\big]_{st}\big]\ (L(x)) \cdot C\big]_{ij} \\
&= \sum_{k,l \in [N]} C^T_{ik} \cdot \big[\big[T^2 \big[T^{2k} g\big]_{st}\big]\big(L(x)\big)\big]_{kl} \cdot C_{lj} \\
&= \sum_{k,l \in [N]} C^T_{ik} \cdot \big[T^{2k+2} g\ (L(x))\big]_{(s,k)(t,l)} \cdot C_{lj}.
\end{aligned}$$

Substituting this into (1), we get

$$\begin{aligned}
&\big(T^{2k+2} f\big)_{(v,i)(w,j)} \\
&\quad = \sum_{s,t \in [N]^k} \sum_{k,l \in [N]} \big[(C^T)^{\otimes k}\big]_{vs} C^T_{ik} \cdot \big[T^{2k+2} g\ \big(L(x)\big)\big]_{(s,k)(t,l)} \cdot C_{lj} \big[C^{\otimes k}\big]_{tw} \\
&\quad\quad \times \sum_{s,t \in [N]^k} \sum_{k,l \in [N]} \big[(C^T)^{\otimes (k+1)}\big]_{(v,i)(s,k)} \cdot \big[T^{2k+2} g\ \big(L(x)\big)\big]_{(s,k)(t,l)} \\
&\quad\quad \cdot \big[C^{\otimes (k+1)}\big]_{(t,l)(w,j)}.
\end{aligned}$$

We conclude that $T^{2k+2} f = (C^T)^{\otimes (k+1)} \cdot T^{2k+2} g\ (L(x)) \cdot C^{\otimes (k+1)}$. □

2.2 Rank Deficiency of the Determinant

Proposition 1 *Let $k \geq 1$ be an integer. Let A and C be invertible $m \times m$ matrices. Then for any $m \times m$ matrix B,*

$$\operatorname{rank} T^{2k} \det_m(B) = \operatorname{rank} T^{2k} \det_m(ABC).$$

Proof Let X be a $m \times m$ matrix with variables with ij-th entry equal to x_{ij}. The linear map $L : X \mapsto AXC$ is invertible. So if we let $f(X) = \det_m(L(X))$, we have by Lemma 1 that $\operatorname{rank} T^{2k} f(B) = \operatorname{rank} T^{2k} \det_m(L(B))$. However, $f(X) = \det_m(AXC) = \mu \det_m(X)$, where $\mu = \det(A)\det(C) \neq 0$. Hence $T^{2k} f = \mu \cdot T^{2k} \det_m$. We conclude that $\operatorname{rank} T^{2k} \det_m(B) = \operatorname{rank} T^{2k} f(B) = \operatorname{rank} T^{2k} \times \det_m(L(B))$. □

Lemma 2 *Let $2k \geq r \geq 1$ be integers. Suppose B is an $m \times m$ matrix of rank at most $m - r$. Then* $\operatorname{rank} T^{2k} \det_m(B) \leq ((2k)!/(2k-r)!)^2 m^{2k-r}$.

Proof By Proposition 1, we can assume wlog. that B equals $\operatorname{diag}(0, 0, \ldots, 0, 1, 1, \ldots, 1)$, where there are r zeroes on the diagonal. Let $H = T^{2k} \det_m$. For $i, j, s, t \in [m]^k$, on row (i, j) and column (s, t) of H we have

$$\frac{\partial^{2k} \det_m}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \ldots \partial x_{i_k j_k} \cdot \partial x_{s_1 t_1} \partial x_{s_2 t_2} \ldots \partial x_{s_k t_k}}.$$

For this entry to be nonzero when evaluated at B, we must have that both $[r] \subseteq \{i_1, i_2, \ldots, i_k, s_1, s_2, \ldots, s_k\}$ and $[r] \subseteq \{j_1, j_2, \ldots, j_k, t_1, t_2, \ldots, t_k\}$. There are $((2k)!/(2k-r)!)^2$ ways of fixing particular indices to be $1, 2, \ldots, r$. Once fixed we are left with $2k - r$ variables that index rows and $2k - r$ variables that index columns. These take values in the range $[m]$. Hence each choice of fixing values gives rise to a submatrix of rank at most m^{2k-r}. Observe this implies we can write H as a sum of $((2k)!/(2k-r)!)^2$ matrices, each with rank bounded by m^{2k-r}. Hence $\operatorname{rank} H \leq ((2k)!/(2k-r)!)^2 m^{2k-r}$. □

Note in [7] it is proved that $\operatorname{rank} T^2 \det_m (B) \leq 2m$, for singular B. The following lemma and its proof are similar to Proposition 3.3 in [11].

Lemma 3 *Let f be homogeneous of degree $2d$ in variables $X = \{x_1, x_2, \ldots, x_n\}$, and let g be homogeneous of degree $2e$ in variables $Y = \{y_1, y_2, \ldots, y_m\}$. Then* $\operatorname{rank} T^{2(d+e)} fg \geq \operatorname{rank} T^{2d} f \cdot \operatorname{rank} T^{2e} g$.

Proof Let $H = T^{2(d+e)} fg$. The rows and columns of H are indexed by $(d+e)$-sequences v and w of variables from $X \cup Y$. $H_{v,w}$ equals the coefficient of the monomial that is the product of the entries of (v, w) in fg. We will only consider a minor H' of H, where v (and similarly w) consists of d X-variables and e Y-variables. Let $s = n^d$ and $t = n^e$. Enumerate all d-sequences in X as $a_1, a_2, \ldots, a_s$. Enumerate all e-sequences in Y as $b_1, b_2, \ldots, b_t$. Consider the minor with rows and columns $(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_t)$, $(a_2, b_1), (a_2, b_2), \ldots, (a_2, b_t), \ldots, (a_s, b_1), (a_s, b_2),$

$\ldots, (a_s, b_t)$. Observe that the entry $H'_{(a_i,b_j),(a_p,b_q)} = (T^{2d} f)_{(a_i,a_p)}(T^{2e} g)_{(b_j,b_q)}$. Hence $H' = (T^{2d} f) \otimes (T^{2e} g)$, and the lemma follows. □

3 The Mignon-Ressayre Bound

Mignon and Ressayre's lower bound proof for the permanent uses the special case where $k = 1$ in the framework of Sect. 2. The proof strategy can be outlined as follows. Suppose $f(x) = \det_m(L(x))$, for some affine map $\lambda x.L(x)$. The objective is to find a point $a \in F^n$ that simultaneously *minimizes* $\operatorname{rank} T^{2k} \det_m(L(a))$ and *maximizes* $\operatorname{rank} T^{2k} f(a)$. For points a such that $f(a) = 0$, $L(a)$ must be singular. Mignon and Ressayre prove that this[1] implies $\operatorname{rank} T^2 \det_m (L(a)) \leq 2m$. By Lemma 1, for any a, $\operatorname{rank} T^2 f(a) \leq \operatorname{rank} T^2 \det_m(L(a))$. For $f = per_n$ it is possible to find a with $per(a) = 0$ and $\operatorname{rank} T^2 per_n(a)$ at the maximum value n^2, implying $\mathrm{dc}(per_n) \geq n^2/2$.

4 *r*-Determinantal Complexity

Results in this section hold for any choice of the underlying field F.

Definition 1 Call a map $L : F^n \to M_m(F)$ *r-lowerable* if there exists $a \in F^n$ such that $\operatorname{rank} L(a) \leq m - r$. We define the *r-determinantal complexity* of a polynomial f, denoted by $\mathrm{dc}_r(f)$, to be the minimum m for which $f(x) = \det_m(L(x))$, where L is an r-lowerable affine map.

For any $r \geq 0$, $\mathrm{dc}_r(f) \leq \mathrm{dc}_{r+1}(f)$. Note that $\mathrm{dc}(f) = \mathrm{dc}_0(f)$, and if $f^{-1}(0) \neq \emptyset$, then $\mathrm{dc}(f) = \mathrm{dc}_1(f)$.

It should be remarked that the affine maps obtained by the constructions that show universality of the determinant of [6, 15] will not be 2-lowerable, since these constructions create an upper triangular minor in $L(x)$ of size $m - 1$. As a matter of fact, universality fails in the r-determinantal model for *inhomogeneous* polynomials, for $r > 2$. To give an example, let us consider computing a polynomial f of degree four using a 3-lowerable map, where f is of the form:

$$f = x_1 x_2 (x_3 x_4) + x_5 x_6 (x_3 x_4 - 1) + g(x_7, x_8, \ldots, x_n).$$

Assume that $f(x) = \det_m(L(x))$, where for some a, $\operatorname{rank} L(a) \leq m - 3$. We have that all $(m-2) \times (m-2)$ minors of $L(a)$ are singular. Hence all 2nd order partial derivatives of $\det_m$ vanish at $L(a)$. Since the second order partial derivatives of $f(x)$ are in the linear span of those of $\det_m$ (see e.g. the proof of Lemma 1), we have that all second order partial derivatives of $f(x)$ vanish at a. For the given f, $\partial^2 f/\partial x_1 \partial x_2 = x_3 x_4$ and $\partial^2 f/\partial x_5 \partial x_6 = x_3 x_4 - 1$, which clearly cannot simultaneously vanish.

[1] Note that Lemma 2 would already yields $\operatorname{rank} T^2 \det_m(L(a)) \leq 4m$.

Whether universality fails for *homogeneous* polynomials in the r-determinantal model is an open problem, for $1 < r < \deg(f)$. We do however have the following general lower bound theorem:

Theorem 1 *For any polynomial f of degree $2d$ and integer r such that $1 \le r < 2d$,* $\mathrm{dc}_r(f) \ge (\frac{(2d-r)!}{(2d)!})^{2/(2d-r)} (\mathrm{rank}\, T^{2d} f)^{1/(2d-r)}$.

Proof Suppose we can write $f(X) = \det_m(L(X))$, where L is r-lowerable. Since f is of degree $2d$, for any $a \in F^n$, $\mathrm{rank}\, T^{2d} f(a) = \mathrm{rank}\, T^{2d} f$. Fix arbitrary a such that $L(a)$ has rank at most $m - r$. By Lemma's 1 and 2, $\mathrm{rank}\, T^{2d} f(a) \le \mathrm{rank}\, T^{2d}\, \mathrm{Det}_m(L(a)) \le ((2d)!/(2d-r)!)^2 m^{2d-r}$. □

4.1 Applications

Theorem 2 $\mathrm{dc}_r(S_n^{2d}) \ge (\frac{(2d-r)!}{(2d)!})^{2/(2d-r)} \binom{n}{d}^{1/(2d-r)}$, *for* $1 \le r < 2d < n$.

Proof Consider any $\binom{n}{d} \times \binom{n}{d}$ minor H of $T^{2d} S_n^{2d}$, where rows and columns are indexed by all d-subsets $I = \{i_1, i_2, \dots, i_d\}$ and $J = \{j_1, j_2, \dots, j_d\}$ of $[n]$, respectively. Then

$$H_{I,J} = \frac{\partial^{2d} S_n^{2d}}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_d} \partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_d}} = \begin{cases} 1 & \text{if } I \cap J = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

In other words H is the *communication matrix* of set disjointness, which is known to have rank $\binom{n}{d}$ (see page 175, [14]). The result now follows by applying Theorem 1. □

Corollary 1 $\mathrm{dc}_r(S_n^{2d}) = \Omega(n^{d/(2d-r)})$, *for* $1 < d < r < 2d = O(1)$.

In the most restrictive setting one has that $\mathrm{dc}_{2d-1}(S_n^{2d}) = \Omega(n^d)$, for constant $d > 1$. As a matter of fact, Theorem 2 yields an exponential lower bound in this case for non-constant d:

Corollary 2 $\mathrm{dc}_{2d-1}(S_n^{2d}) = n^{\Omega(\epsilon n^{1/5-\epsilon})}$, *for any fixed* $0 < \epsilon < 1/5$, *where* $d = \lfloor n^{1/5-\epsilon} \rfloor$.

The above is the most restrictive case in which computability of S_n^{2d} is not yet ruled out by Theorem 1. Careful inspection of the proof of Theorem 1 does show that S_n^{2d} cannot be computed using a $2d$-lowerable map if $\binom{n}{d} > (2d!)^2$, e.g. for all large enough n, if d is constant.

Using Lemma 3, we obtain lower bounds for products of symmetric polynomials on disjoint variables sets.

Theorem 3 *For constants $d > 1$, $p \ge 1$, and r with $dp < r < 2dp$, let $X^1, X^2, \dots, X^p$ be disjoint sets of variables of size n each. Then*

$$\mathrm{dc}_r\Big(\prod_{i \in [p]} S_n^{2d}(X^i)\Big) = \Omega\big(n^{2dp/(2dp-r)}\big).$$

Proof By Lemma 3, $\operatorname{rank} T^{2dp} \prod_{i\in[p]} S_n^{2d}(X^i) \geq \prod_{i\in[p]} \operatorname{rank} T^{2d} S_n^{2d}(X^i) \geq \binom{n}{d}^p$. The latter inequality follows from the proof of Theorem 2. Applying Theorem 1 yields the result. □

Next, we consider iterated matrix product. Define $IMM_{n,d}$ by summing entries of an iterated matrix multiplication:

$$IMM_{n,d} = \sum_{i,j\in[n]} \left(\prod_{r\in[d]} X^r \right)_{ij},$$

where $X^1, X^2, \dots, X^d$ are $n \times n$ matrices with disjoint sets of variables. Since $IMM_{n,d}$ has algebraic branching programs of size dn, we have that $\operatorname{dc}(IMM_{n,d}) = O(nd)$, which is *sublinear* in the number of variables n^2d.

Lemma 4 *For* $d \geq 1$, $\operatorname{rank} T^{2d} IMM_{n,2d} \geq n^d$.

Proof We prove the following claim, by induction on d:

Claim 1 *Let* $(g_1, g_2, \dots, g_n) = (1, 1, \dots, 1)X^1X^2 \dots X^{2d-1}X^{2d}$. *Then for each* i, $\operatorname{rank} T^{2d} g_i \geq n^d$.

The case $d = 1$ follows directly by inspection. Now suppose, $(g_1, g_2, \dots, g_n) = (1, 1, \dots, 1)X^1X^2 \dots X^{2d-1}X^{2d}$. Let $G_i = T^{2d} g_i$. Let $(h_1, h_2, \dots, h_n) = (g_1, g_2, \dots, g_n)X^{2d+1}X^{2d+2}$. Consider the minor H of $T^{2d+2} \sum_i h_i$ where rows contain a variable from X^{2d+1}, but not from X^{2d+2}, and vice-versa for columns. Provided we order at the top level rows according to variables from X^{2d+1} and columns according to variables from X^{2d+2}, this minor will be of the following form:

$$H = \begin{pmatrix}
G_1 & \dots & G_1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
 & 0 & & G_1 & \dots & G_1 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & \dots & G_1 & \dots & G_1 \\
G_2 & \dots & G_2 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
 & 0 & & G_2 & \dots & G_2 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & \dots & G_2 & \dots & G_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
G_n & \dots & G_n & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
 & 0 & & G_n & \dots & G_n & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & \dots & G_n & \dots & G_n
\end{pmatrix}.$$

For any i, $T^{2d+2}h_i$ is obtained by setting to zero in the above columns which at the top level are indexed by variables X_{kj}^{2d+2} with $j \neq i$. Hence for each i, $\operatorname{rank} T^{2d+2}h_i \geq n \cdot \max_{k\in[n]} \operatorname{rank} G_k \geq n^{d+1}$. This proves the claim.

Observe that in the above the individual $T^{2d+2}h_i$ do not interfere, in the sense that $\operatorname{rank} H \geq \operatorname{rank} T^{2d}h_i$, for each i. Hence $\operatorname{rank} T^{2d}\sum_i g_i \geq n^d$. □

Combining Theorem 1 and Lemma 4 yields the following results:

Theorem 4 $\mathrm{dc}_r(IMM_{n,2d}) \geq (\frac{(2d-r)!}{(2d)!})^{2/(2d-r)} n^{d/(2d-r)}$, *for* $1 \leq r < 2d$.

Corollary 3 $\mathrm{dc}_r(IMM_{n,2d}) = \Omega(n^{d/2d-r})$, *for* $1 \leq d < r < 2d = O(1)$.

Corollary 4 $\mathrm{dc}_{2d-1}(IMM_{n,2d}) = n^{\Omega(\epsilon n^{1/4-\epsilon})}$, *for any fixed* $0 < \epsilon < 1/4$, *where* $d = \lfloor n^{1/4-\epsilon} \rfloor$.

5 Unrestricted Determinantal Complexity

Results in this section hold for fields F of characteristic zero.

5.1 Elementary Symmetric Polynomials

Note using Theorem 2 with $d = 1$ and $r = 1$ yields that $\mathrm{dc}(S_n^2) \geq n/4$. For $n \geq 1$, we define $2n$-vector $p_n = (1, -1, 1, -1, \dots, 1, -1)$. Easily verified by induction on n, is that the univariate polynomial $((t-1)(t+1))^n$ equals $\sum_{r=0}^n t^{2r}(-1)^{n-r}\binom{n}{r}$. Since $S_{2n}^d(p_n)$ is the coefficient of t^{2n-d} we obtain the following:

Proposition 2 *For any* $1 \leq d \leq n$, $S_{2n}^{2d-1}(p_n) = 0$.

Proposition 3 *For any* $0 \leq d \leq n$, $S_{2n}^{2d}(p_n) = (-1)^d\binom{n}{d}$.

Lemma 5 $\operatorname{rank} T^2 S_{2n}^{2d-1}(p_n) = 2n$, *for* $2 \leq d \leq n$.

Proof Let $H = T^2 S_{2n}^{2d-1}$. Then $H_{ij} = S_{2n-2}^{2d-3}(X/x_i, x_j)$, if $i \neq j$, and zero otherwise. By symmetry and Proposition 2, if i and j have distinct parity, $H_{ij}(p_n) = S_{2n-2}^{2d-3}(p_{n-1}) = 0$. Let H^o and H^e be the $n \times n$ minors of H formed by all odd, respectively even, rows and columns. H acts independently on the odd and even indices of a vector, so $\operatorname{rank} H = \operatorname{rank} H^o + \operatorname{rank} H^e$. H^o has zeroes on its diagonal, and by symmetry off-diagonal entries will be the same value $S_{2n-2}^{2d-3}(-1, -1; p_{n-2})$. Provided this is a nonzero value, H^o will be non-singular. A straightforward calculation shows this value is $(-1)^{d-2}\binom{n-2}{d-2}$. Similarly $\operatorname{rank} H^e = n$. □

Following the proof strategy as outlined in Sect. 3, and using the bound $\operatorname{rank} T^2 \det_m(B) \leq 2m$, for singular B of [7], we obtain the following corollary:

Corollary 5 $\mathrm{dc}(S^{2d-1}_{2n+1}) \geq \mathrm{dc}(S^{2d-1}_{2n}) \geq n$, *for* $2 \leq d \leq n$.

For even degree, a simple calculation shows that $S^{2d}_{2n+2}(a, b, p_n) = 0$ whenever $2ab = (n+1)/d$.

Lemma 6 *For* $2 \leq d \leq n$, *there exists constant* μ, *such that for* $q_{n+2} = (\mu(n+1), \frac{1}{2d\mu}, p_n)$, $\mathrm{rank}\, T^2 S^{2d}_{2n+2}(q_{n+2}) \geq n$.

Proof Let $H = T^2 S^{2d}_{2n+2}$. Consider the $n \times n$ minor H' of H given by rows i and columns j, where $i, j \geq 3$ and i, j are both odd. The diagonal of H' has all entries zero. By symmetry, $H'_{i,j}(q_{n+2})$ has the same value $S^{2d-2}_{2n}(\mu(n+1), \frac{1}{2d\mu}, -1, -1, p_{n-2})$, for all for $i \neq j$. A simple calculation shows that one can easily pick μ large enough so this value does not vanish, in which case H' has rank n. □

Corollary 6 $\mathrm{dc}(S^{2d}_{2n+3}) \geq \mathrm{dc}(S^{2d}_{2n+2}) \geq n/2$, *for* $2 \leq d \leq n$.

Shpilka and Wigderson give depth 6 arithmetic formulas of size $O(nd^3 \log d)$ for S^d_n [12]. By the universality of the determinant [15] (cf. [6]) we have $\mathrm{dc}(S^d_n) = O(nd^3 \log d)$. Hence the following theorem holds:

Theorem 5 *For any constant* $d > 1$, $\mathrm{dc}(S^d_n) = \Theta(n)$.

In other words, for constant d, $\mathrm{dc}(S^d_n) = \Theta(L(S^d_n))$. For non-constant d, it is an open problem whether any improved upper bound can be given for $\mathrm{dc}(S^d_n)$ beyond applying the universality of the determinant to best-known formulas. We note this can easily be done for the *permanental complexity* of S^d_n. Also permanental complexity minorizes formula size, i.e. for any polynomial f, $\mathrm{pc}(f) = O(L(f))$. However, for any d, $\mathrm{pc}(S^d_n) \leq n$, over fields of characteristic zero. Namely, the permanent of a matrix A whose first d rows equal $(x_1, x_2, \ldots, x_n)$, and with other entries equal to one, is $d!(n-d)!S^d_n$.

Note one could try to obtain a polynomial h^d_n that has the same support as S^d_n by taking $h^d_n = \det_n(A \circ C)$, where $\circ$ denotes the Hadamard product, and C is some matrix of constants $C = ({}_{ij})_{i,j\in[n]}$. In this case for $I \subset [n]$ of size d, the monomial $\prod_{i\in I} x_i$ appears with coefficient $\pm\det(C_{[d],I})\det(C_{\{d+1,d+2,\ldots,n\},[n]/I})$, where $C_{I,J}$ denotes the submatrix with rows I and columns J. It is easily possible to choose C so that all these coefficients are nonzero. However, requiring all coefficients to equal 1 will not be feasible in general. We immediately know this from our previous investigation, since the mapping defined this way is d-lowerable. We know by Theorem 1, one cannot compute S^{2d}_n using $2d$-lowerable maps, if $\binom{n}{d} > (2d!)^2$.

5.2 Iterated Matrix Product

Theorem 6 *For any* $d \geq 1$, $\mathrm{dc}(IMM_{n,2d}) \geq n/2$.

Proof Consider the minor of $T^2 IMM_{n,2d}$ corresponding to rows with variables $X_{11}^{2d-1}, \dots, X_{1n}^{2d-1}$ and columns with variables of X^{2d}. With the appropriate ordering of variables from X^{2d}, this minor looks like:

$$\begin{pmatrix} g_{11} & \dots & g_{1n} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ & 0 & & g_{21} & \dots & g_{2n} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & g_{n1} & \dots & g_{nn} \end{pmatrix},$$

where $g_{ij} = \partial^2 IMM_{n,2d}/\partial X_{1i}^{2d-1} \partial X_{ij}^{2d}$. Note g_{ij} is the same for all $i, j \in [n]$. It suffices to find point a, such that $IMM_{n,2d}(a) = 0$ and $g_{ij}(a) \neq 0$. Since g_{ij} does not contain any variables from X^{2d-1} and X^{2d}, this can easily be achieved by setting to zero all variables in X^{2d-1} and X^{2d}, and setting all other variables to 1. □

By the remarks before Lemma 4, we have that $\mathrm{dc}(IMM_{n,2d}) = \Theta(n)$, for constant d. More generally, by means of reduction, we have for row vector $\langle u|$ and column vector $|v\rangle$ of variables the following corollary:

Corollary 7 *For constant* d, $\mathrm{dc}(\langle u|X^1 X^2 \dots X^{2d}|v\rangle) = \Theta(n)$.

5.3 Symmetric Power Sum Polynomials

The symmetric power sum polynomial of degree d in n variables is defined by $P_n^d = \sum_{i\in[n]} x_i^d$. P_n^d has arithmetic formula size $L(P_n^d) = O(nd)$. The partial derivatives matrix $T^2 P_n^d$ is given by the diagonal matrix $d(d-2)\mathrm{diag}(x_1^{d-2}, x_2^{d-2}, \dots, x_n^{d-2})$. By the strategy described in Sect. 3, it suffices to find a zero a of P_n^d with all entries $a_i \neq 0$ to obtain that for all $d \geq 2$, $\mathrm{dc}(P_n^d) \geq n/2$. For an arbitrary field F of characteristic zero one achieves this by going to an extension field G of F where the dth root of $(1-n)$ exists. Over G, one takes $a = (1, 1, \dots, 1, (1-n)^{1/d})$ to show $\mathrm{dc}_G(P_n^d) \geq n/2$. The lower bound follows since $\mathrm{dc}_F(P_n^d) \geq \mathrm{dc}_G(P_n^d)$.

Corollary 8 *For any constant* $d \geq 2$, $\mathrm{dc}(P_n^d) = \Theta(n)$.

Note the statement of the corollary would be false if we drop the restriction (which holds section-wide) on the characteristic of the underlying field, e.g. $\mathrm{dc}(P_n^2) = 2$ over $GF(2)$.

6 Conclusions

In this paper efforts have been made to overcome the current barrier to proving lower bounds for the determinantal complexity of explicit polynomials, that scale beyond linear in the number of variables. It has been demonstrated that this barrier can be broken through under the mathematically natural restriction of r-lowerability. It will be interesting to see whether one can do this under any weaker assumptions. The inquiry leaves us with some intriguing questions regarding determinantal representation:

Problem 1 *Can S_n^{2d} be computed using an r-lowerable map, with $d < r < 2d$?*

Problem 2 *Can* every *homogeneous polynomial of degree $2d$ be computed using an r-lowerable map for some r such that $d < r < 2d$?*

Problem 3 *For non-constant d depending on n, can we give better upper bounds for* $\mathrm{dc}(S_n^d)$ *than would be obtained by applying Valiant's universality construction of the determinant to best-known formulas for S_n^d?*

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