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On the sensitivity complexity of bipartite graph properties

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ABSTRACT

Sensitivity complexity, introduced by Cook, Dwork, and Reischuk (1982, 1986) in [2,3], is an important complexity measure of Boolean functions. Turán (1984) [7] initiated the study of sensitivity complexity for graph properties. He conjectured that for any non-trivial graph property on n vertices, the sensitivity complexity is at least $n-1$. He proved that it is greater than $n/4$ in his paper. Wegener (1985) [8] verified this conjecture for all monotone graph properties. Recently Sun (2011) [6] improved the lower bound to $\frac{6}{17}n$ for general graph properties. We follow their steps and investigate the sensitivity complexity of bipartite graph properties. In this paper we propose the following conjecture about the sensitivity of bipartite graph properties, which can be considered as the bipartite analogue of Turán's conjecture: for any non-trivial $n \times m$ bipartite graph property f,

$$s(f) \geq \max\left\{\left\lceil \frac{n+1}{m+1} m \right\rceil, \left\lceil \frac{m+1}{n+1} n \right\rceil\right\}.$$

We prove this conjecture for all $n \times 2$ bipartite graph properties. For general $n \times m$ bipartite graph properties, we show a $\max\{\lceil n/2 \rceil, \lceil m/2 \rceil\}$ lower bound. We also prove this conjecture when the bipartite graph property can be written as a composite function.

1. Introduction

Sensitivity complexity is a useful measure of Boolean functions which shows how sensitive the value of a Boolean function is to the changes in the input. It was first introduced by Cook, Dwork and Reischuk in [2,3] (under the name critical complexity). They showed that its logarithm (in base $(5+\sqrt{21})/2 \approx 4.79$) is a lower bound for the time needed by a PRAM to compute function f.

Definition 1. Let $f : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function. The sensitivity $s(f, \mathbf{x})$ of f on input $\mathbf{x}$ is defined as the number of bits on which the function is sensitive, i.e. $s(f, \mathbf{x}) = \left|\{i | f(\mathbf{x}) \neq f(\mathbf{x}^i)\}\right|$, where $\mathbf{x}^i$ is obtained by flipping the i-th bit of $\mathbf{x}$. Define the *sensitivity* of f as $s(f) = \max\left\{s(f, \mathbf{x}) | \mathbf{x} \in \{0, 1\}^n\right\}$ and the 0-*sensitivity* and 1-*sensitivity* of f as $s_0(f) = \max\left\{s(f, \mathbf{x}) | \mathbf{x} \in \{0, 1\}^n, f(\mathbf{x}) = 0\right\}$, $s_1(f) = \max\left\{s(f, \mathbf{x}) | \mathbf{x} \in \{0, 1\}^n, f(\mathbf{x}) = 1\right\}$.

Sensitivity complexity is closely related to the concept of decision tree complexity and other complexity measures of Boolean functions. For an excellent survey of decision tree complexity we refer the readers to [1]. The study of the decision tree complexity of graph properties dates back to 1970s [4,5]. Informally, a graph property is a property preserved under all possible isomorphisms of a graph.

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Definition 2. A graph property is a Boolean function $f : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ such that for every input $\mathbf{x} \in \{0,1\}^{\binom{n}{2}}$,

$$\forall \pi \in S_n, \quad f(x_{1,2}, \ldots, x_{n-1,n}) = f(x_{\pi(1),\pi(2)}, \ldots, x_{\pi(n-1),\pi(n)}),$$

where the input $\mathbf{x}$ represents the adjacency matrix of an undirected graph G with n vertices.

Turán [7] initiated the study of sensitivity of graph properties. He proved that the sensitivity is at least $n/4$ for graph properties and made the following conjecture in his paper:

Conjecture 1 (*Turán [7]*). *For any non-trivial graph property on n vertices, the sensitivity complexity is at least* $n-1$.

Here $(n-1)$ is tight since the "contained an isolated vertex" property reaches this bound. Wegener [8] verified this conjecture for all monotone graph properties. Recently Sun [6] improved the bound from $\frac{n}{4}$ to $\frac{6}{17}n$ for general graph properties. There is still a large gap there.

In this paper we focus our attention on bipartite graph properties. Suppose that $G = (U, V, E)$ is a bipartite graph where $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$.

Definition 3. A *bipartite graph property* on G is a Boolean function $f : \{0,1\}^{nm} \to \{0,1\}$ such that for every input $\mathbf{x} \in \{0,1\}^{nm}$,

$$\forall \pi_1 \in S_n, \forall \pi_2 \in S_m, \quad f(x_{1,1}, \ldots, x_{n,m}) = f\left(x_{\pi_1(1),\pi_2(1)}, \ldots, x_{\pi_1(n),\pi_2(m)}\right),$$

where input $\mathbf{x} \in \{0,1\}^{nm}$ represents the adjacency matrix of G, i.e. $x_{i,j} = 1$ iff $\{u_i, v_j\} \in E$. Besides, we define $\mathcal{B}_{n,m}$ as the class of all the bipartite graph properties.

We propose the following conjecture about the sensitivity complexity of bipartite graph properties, it can be considered as the bipartite analogue of Turán's conjecture [7].

Conjecture 2. *For any non-trivial bipartite graph property* $f \in \mathcal{B}_{n,m}$*, the sensitivity complexity*

$$s(f) \geq \max\left\{\left\lceil \frac{n+1}{m+1} m \right\rceil, \left\lceil \frac{m+1}{n+1} n \right\rceil\right\},$$

particularly for any non-trivial $f \in \mathcal{B}_{n,n}$, $s(f) \geq n$.

Remark 1. The lower bound $\max\{\lceil \frac{n+1}{m+1} m \rceil, \lceil \frac{m+1}{n+1} n \rceil\}$ is tight, see Section 4 for functions which achieve this bound.

In this paper we prove this conjecture when $\min\{m, n\} = 2$. Actually we prove a stronger result which gives a tradeoff between $s_0(\cdot)$ and $s_1(\cdot)$ in this case.

Theorem 1. *For any non-trivial bipartite graph property* $f \in \mathcal{B}_{n,2}$,

$$\max\{s_0(f) + 2s_1(f), 2s_0(f) + s_1(f)\} \geq 2n + 2,$$

particularly, $s(f) \geq \lceil 2(n+1)/3 \rceil$.

For general bipartite graph properties in $\mathcal{B}_{n,m}$, we cannot even prove that the bound is monotone on m and n. We can only prove the following weaker lower bound.

Theorem 2. *For any non-trivial bipartite graph property* $f \in \mathcal{B}_{n,m}$, $s(f) \geq \max\{n, m\}/2$.

We also prove the conjecture for an important special case when the bipartite graph property is a composite function. Informally, composite functions consider vertices in V and subgraphs on $U \times \{v\}$ separately, which take count on the subgraphs of size $n \times 1$ that keep some properties to decide the value.

Definition 4. A Boolean function $f : \{0,1\}^{nm} \to \{0,1\}$ is a composition of Boolean functions $g : \{0,1\}^m \to \{0,1\}$ and $h : \{0,1\}^n \to \{0,1\}$, if for every input $\mathbf{x} = (x_{1,1}, \ldots, x_{n,m}) \in \{0,1\}^{nm}$,

$$f(\mathbf{x}) = g(h(x_{1,1}, \ldots, x_{n,1}), h(x_{1,2}, \ldots, x_{n,2}), \ldots, h(x_{1,m}, \ldots, x_{n,m})),$$

written as $f = g \circ h$.

Theorem 3. *For any non-trivial* $f \in \mathcal{B}_{n,m}$, *if* $f = g \circ h$ *is composed of two symmetric Boolean functions* $g : \{0,1\}^m \to \{0,1\}$ *and* $h : \{0,1\}^n \to \{0,1\}$, *then we have*

$$s(f) \geq \max\left\{\left\lceil \frac{n+1}{m+1} m \right\rceil, \left\lceil \frac{m+1}{n+1} n \right\rceil\right\}.$$

The rest of the paper is organized as follows: in Section 2 we prove Theorem 2 first. We prove Theorem 1 in Section 3. We consider the composite function case in Section 4 and discuss the feasible region of $(s_0(f), s_1(f))$ in Section 5. We conclude the paper in Section 6 with a general conjecture.

2. Lower bound for general bipartite graph properties

We prove Theorem 2 in this section. W.l.o.g. we assume that $f(E_{n,m}) = 0$, where $E_{n,m}$ denotes the empty bipartite graph. For any bipartite graph $G = (U, V, E)$ where U, V are the sets of vertices and E is the set of edges. We use the notation $U_{\geq 1}$ to denote the set of non-isolated vertices in U. We need the following lemma in the proof.

Lemma 1. *For any non-trivial $f \in \mathcal{B}_{n,m}$, if $s(f) < \frac{n}{2}$, then for any graph $G \neq E_{n,m}$, there exists a graph G' such that $f(G) = f(G')$ and $|U_{\geq 1}(G')| < |U_{\geq 1}(G)|$.*

Once Lemma 1 is proved, then Theorem 2 followed immediately: Since f is non-trivial, there exists a graph G such that $f(G) = 1$. We choose G to be the graph which minimizes the size of $U_{\geq 1}(G)$. Then from the lemma above, there is another graph G' such that $f(G') = f(G) = 1$ and $|U_{\geq 1}(G')| < |U_{\geq 1}(G)|$, which is a contradiction. By symmetry we can also rule out the case $s(f) < \frac{m}{2}$.

The proof of Lemma 1 is based on the following fact: If we find at least $s(f) + 1$ adjacent[1] graphs of any specific graph G, there will be at least one graph G' among them such that $f(G) = f(G')$. Since $s(f) < \frac{n}{2}$, we just need to find $\frac{n}{2}$ such graphs in each step.

Proof. For any non-empty bipartite graph $G = (U, V, E)$, we divide the vertices in U into two different sets:

- For each $u_i \in U$, we say u_i is *regulated* iff $\Gamma(u_i) = \{v_1, v_2, \ldots, v_{\deg(u_i)}\}$, where $\Gamma(u_i)$ denotes the set of vertices that are adjacent to u_i in V. In other words, u_i is regulated when it is adjacent to the first k vertices in V, where k is the degree of u_i. We use the notation U_{reg} to denote the set of regulated vertices.
- Otherwise, we say that u_i is unregulated. We use the notation U_{unr} to denote the set of unregulated vertices.

Notice that isolated vertices in U are also regulated, therefore $U_{unr} \subseteq U_{\geq 1}$. We can relabel all the vertices in U and V, such that $u_1 \in U_{\geq 1}$ and u_1 is regulated. Then we get $|U_{unr}| < |U_{\geq 1}|$.

The proof is rather similar to a proof based on induction. Starting with graph G, each time we either add an edge to it or delete an edge from it while keeping the function value unchanged, such that one of the following thing would happen:

1. $d(U_{unr})$ is decreased. Here $d(U_{unr})$ denotes the total degree of all vertices in U_{unr}, i.e.

$$d(U_{unr}) = \sum_{u \in U_{unr}} \deg(u). \tag{1}$$

2. $H(U_{reg})$ is decreased. Here $H(U_{reg})$ is the *potential energy function* of U_{reg}, which is defined as follows:

$$H(U_{reg}) = \sum_{u \in U_{reg}} (|\Delta(U_{reg}) - 1 - \deg(u)| + \Delta(U_{reg})). \tag{2}$$

Here we use the notation $\Delta(U_{reg})$ to denote the maximum degree of vertices in U_{reg}, i.e.

$$\Delta(U_{reg}) = \max\{\deg(u) | u \in U_{reg}\}.$$

To achieve the goal above, we can either delete an edge $e = (u, v)$ with $u \in U_{unr}$, or we can add an edge $e = (u, v)$ with $u \in U_{reg}$ and $\deg(u) < \Delta(U_{reg}) - 1$ such that u remains regulated after the insertion of e. Additionally, for any vertex $u \in U_{reg}$ with maximum degree, we can delete an edge associated with u while keeping u regulated after the deletion of the edge.

There are $d(U_{unr})$ ways to delete an edge to achieve the first goal (to decrease $d(U_{unr})$). For the second goal, for each $u \in U_{reg}$ that $d(u) \neq \Delta(U_{reg}) - 1$, we can always add/delete an edge to decrease $H(U_{reg})$. In the special case when there is exactly one vertex u in U_{reg} with maximum degree, deletion of the edge will also decrease $H(U_{reg})$. Therefore, the total number of ways to add/delete an edge is equal to

$$D = d(U_{unr}) + |U_{reg}| - \sum_{u \in U_{reg}} I\{\deg(u) = \Delta(U_{reg}) - 1\}. \tag{3}$$

Here $I\{\deg(u) = \Delta(U_{reg}) - 1\}$ is the indicator function.

Notice that $d(U_{unr}) + |U_{reg}| \geq |U_{unr}| + |U_{reg}| = n$, therefore unless there are more than $\frac{n}{2}$ vertices u in U_{reg} satisfying $\deg(u) = \Delta(U_{reg}) - 1$, we can always find enough neighbours so that at least one among them would keep the function value unchanged.

We keep doing the step above until we cannot find enough ways to add or delete edges, which means that $D \leq s(f) < \frac{n}{2}$. At this time, we have:

$$\sum_{u \in U_{reg}} I\{\deg(u) = \Delta(U_{reg}) - 1\} > \frac{n}{2}. \tag{4}$$

[1] We call two graphs are adjacent if they differ on one edge.

By relabelling the vertices in U, we assume w.l.o.g. that u_1 takes the maximum degree among all vertices in U_{reg}, i.e. $\deg(u_1) = \Delta(U_{reg})$. Assume $\deg(u_1) = k$. Since u_1 is regulated, we have $\Gamma(u_1) = \{v_1, v_2, \dots, v_k\}$. We claim that if we delete the edge $e = (u_1, v_k)$ from the current graph G, the function value $f(G)$ would not change.

In fact, consider the graph $G' = G - (u_1, v_k)$, there are at least $\frac{n}{2} + 1$ vertices in U_{reg} with the same degree $k - 1 (= \Delta(U_{reg}) - 1)$. Since they are all regulated, their adjacency sets would be identical ($\{v_1, \dots, v_{k-1}\}$). Therefore, the graph $G' = G - (u_1, v_k)$ has at least $\frac{n}{2} + 1$ ways to add an edge to get an isomorphic graph of G. Thus $f(G') = f(G)$, i.e. there still exists a way to delete an edge from G while keeping $f(G)$ unchanged. The deletion of such edge will also decrease $H(U_{reg})$.

There is only one exception that we cannot continue this procedure: all vertices in U_{reg} become isolated. If this happens, we already achieved the requirements described in Lemma 1 (arrive at a graph G' with $f(G') = f(G)$ and smaller $U_{\geq 1}(*)$).

Finally we need to prove that the above procedure cannot drop into a cycle. Indeed, in each step either $d(U_{unr})$ or $H(U_{reg})$ is decreased, and $d(U_{unr})$ and $H(U_{reg})$ are both upper bounded by $O(mn)$. $H(U_{reg})$ might increase in some cases, but this can only happen for finite times. Actually, $H(U_{reg})$ may increase when some vertex u in U_{unr} becomes regulated after some step. Notice that whenever some vertex u becomes regulated, it cannot change back to unregulated again. It is clear that $H(U_{reg})$ will only increase for finite times. □

3. Lower bound for functions in $\mathcal{B}_{n,2}$

We give the proof of Theorem 1 in this section. The proof is by contradiction. We claim that for any function $f \in \mathcal{B}_{n,2}$, if $s_0(f) + 2s_1(f) < 2n+2$ and $s_1(f) + 2s_0(f) < 2n+2$, then f must be a constant function. We will use the following notations in the proof.

For an input $\mathbf{x} = (x_{1,1}, \dots, x_{n,2}) \in \{0,1\}^{2n}$, let $N_{00}(\mathbf{x}) = |\{i \in [n] | x_{i,1} = 0, x_{i,2} = 0\}|$, $N_{10}(\mathbf{x}) = |\{i | x_{i,1} = 1, x_{i,2} = 0\}|$, $N_{01}(\mathbf{x}) = |\{i | x_{i,1} = 0, x_{i,2} = 1\}|$, and $N_{11}(\mathbf{x}) = |\{i | x_{i,1} = 1, x_{i,2} = 1\}|$. If $f \in \mathcal{B}_{n,2}$, then $f(x_{1,1}, \dots, x_{n,2})$ only depends on these four numbers $N_{00}(\mathbf{x})$, $N_{10}(\mathbf{x})$, $N_{01}(\mathbf{x})$, and $N_{11}(\mathbf{x})$. We let

$$F(N_{00}(\mathbf{x}), N_{10}(\mathbf{x}), N_{01}(\mathbf{x}), N_{11}(\mathbf{x})) = f(x_{1,1}, \dots, x_{n,2}),$$

and

$$S((N_{00}(\mathbf{x}), N_{10}(\mathbf{x}), N_{01}(\mathbf{x}), N_{11}(\mathbf{x}))) = s(f, \mathbf{x}).$$

A configuration C is a collection of isomorphic bipartite graphs. We use a tuple (a, b, c, d) to represent it:

$$\mathbf{x} \in C \quad \text{iff} \quad (N_{00}(\mathbf{x}), N_{10}(\mathbf{x}), N_{01}(\mathbf{x}), N_{11}(\mathbf{x})) = (a, b, c, d) \textbf{ or } (a, c, b, d).$$

Obviously for any *legal* configuration (a, b, c, d), we have $a + b + c + d = n$. Notice that (a, b, c, d) and (a, c, b, d) represent the same configuration.

For two configurations C_1 and C_2, for each $\mathbf{x} \in C_1$, let $w(C_1, C_2)$ denote the number of edges (i, j) such that $\mathbf{x}^{(i,j)}$ is in C_2, i.e.

$$w(C_1, C_2) = \left|\{(i, j) | \mathbf{x} \in C_1,\ \mathbf{x}^{(i,j)} \in C_2\}\right|,$$

where $\mathbf{x}^{(i,j)}$ represents the vector by flipping $x_{i,j}$ in $\mathbf{x}$. $w(\cdot)$ is closely related to the sensitivity complexity:

$$S(C) = \sum_{F(C') \neq F(C)} w(C, C').$$

For some particular configurations C_1 and C_2, the values of $w(C_1, C_2)$ can be computed as follow:

C_1	C_2	$w(C_1, C_2)$
(a, b, c, d)	$(a-1, b+1, c, d)$	a
(a, b, c, d)	$(a-1, b, c+1, d)$	a
(a, b, c, d)	$(a+1, b-1, c, d)$	b
(a, b, c, d)	$(a+1, b, c-1, d)$	c
(a, b, c, d)	$(a, b-1, c, d+1)$	b
(a, b, c, d)	$(a, b, c-1, d+1)$	c
(a, b, c, d)	$(a, b+1, c, d-1)$	d
(a, b, c, d)	$(a, b, c+1, d-1)$	d

There are some exceptions on the boundary (when $b = c$):

C_1	C_2	$w(C_1, C_2)$
(a, b, b, d)	$(a-1, b, b+1, d)$	$2a$
(a, b, b, d)	$(a+1, b-1, b, d)$	$2b$
(a, b, b, d)	$(a, b, b+1, d-1)$	$2d$
(a, b, b, d)	$(a, b-1, b, d+1)$	$2b$

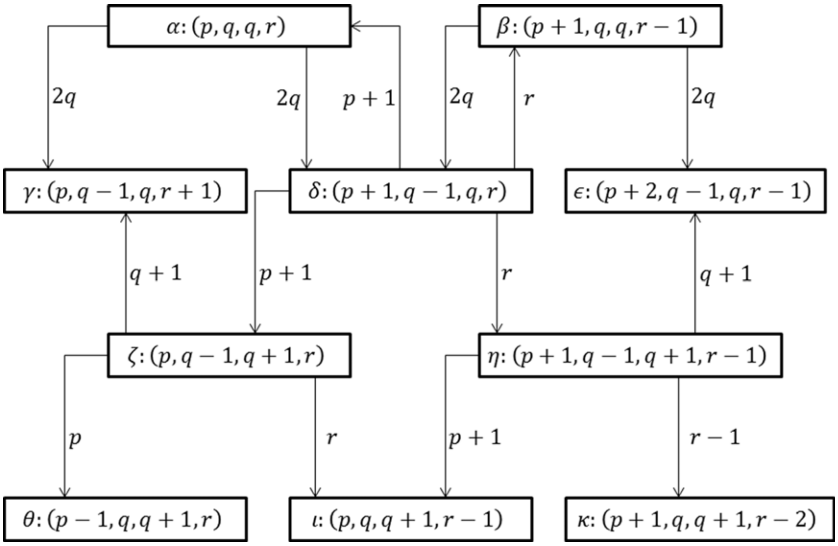

Fig. 1. Sensitivity between configurations, where $n - k = 2q$ and $p + r = k$.

We start the proof with the following lemma.

Lemma 2. *If $s_0(f) + 2s_1(f) < 2n + 2$ and $s_1(f) + 2s_0(f) < 2n + 2$, then $F(p, q_0, q_1, r)$ is constant for all (p, q_0, q_1, r) such that $|q_0 - q_1| \le 1$.*

Proof. We prove the lemma by induction on $p + r$. For the case $p + r = 0$, notice that there is exactly one configuration (p, q_0, q_1, r) satisfying $|q_0 - q_1| \le 1$. For $n = 2t$, it is $(0, t, t, 0)$. For $n = 2t + 1$, it is $(0, t, t + 1, 0)$. Therefore, we can assume w.l.o.g. that $F(0, q_0, q_1, 0) = 0$ when $|q_0 - q_1| \le 1$. Assume for all $p + r \le k$, $|q_0 - q_1| \le 1$, $F(p, q_0, q_1, r) = 0$. We want to prove that $F(p, q_0, q_1, r) = 0$ also holds when $p + r = k + 1$.

We prove it in the following two cases: First we consider the case $n - k = 2q$, which means that n and k has the same parity. Consider the following configurations:

$$\alpha = (p, q, q, r), \quad \beta = (p + 1, q, q, r - 1), \quad \gamma = (p, q - 1, q, r + 1), \quad \delta = (p + 1, q - 1, q, r),$$
$$\epsilon = (p + 2, q - 1, q, r - 1),$$
$$\zeta = (p, q - 1, q + 1, r), \quad \eta = (p + 1, q - 1, q + 1, r - 1), \quad \theta = (p - 1, q, q + 1, r),$$
$$\iota = (p, q, q + 1, r - 1), \quad \kappa = (p + 1, q, q + 1, r - 2)$$

where $p + r = k$. As shown in Fig. 1, it can be verified that

$$w(\alpha, \gamma) = w(\alpha, \delta) = w(\beta, \delta) = w(\beta, \epsilon) = 2q, \; w(\delta, \alpha) = w(\delta, \zeta) = p + 1,$$

and

$$\begin{aligned} w(\delta, \beta) &= w(\delta, \eta) = r, \; w(\zeta, \theta) = p, \; w(\zeta, \iota) = r, \; w(\eta, \iota) = p + 1, \; w(\eta, \kappa) \\ &= r - 1, \; w(\zeta, \gamma) = w(\eta, \epsilon) = q + 1. \end{aligned}$$

By the induction hypothesis, $F(\alpha) = F(\beta) = 0$. All we need to show is that $F(\gamma) = F(\delta) = 0$. We do it for $F(\delta)$ here, the proof of $F(\gamma) = 0$ is similar (if we flip all the bits, configuration (p, q_0, q_1, r) becomes (r, q_1, q_0, p)).

Case 1: $s_0(f) < 2q$. Noticing that $S(\alpha) \le s_0(f) < 2q = w(\alpha, \delta)$, we have $F(\delta) = F(\alpha) = 0$.

Case 2: $2q \le s_0(f) < 4q$. Since $s_0(f) + 2s_1(f) < 2n + 2$ and $s_1(f) + 2s_0(f) < 2n + 2$, we have $s_1(f) < n - q + 1$ and $s_1(f) < 2n - 4q + 2$.

Assume $F(\delta) = 1$. Since $w(\alpha, \gamma) + w(\alpha, \delta) = 4q > s_0(f) \ge S(\alpha)$, we have $F(\gamma) = 0$, and $F(\epsilon) = 0$ for the same reason. Also we have $F(\theta) = F(\iota) = 0$ from the induction hypothesis. Then we claim that $F(\zeta) = 0$, otherwise $S(\zeta) \ge w(\zeta, \gamma) + w(\zeta, \theta) + w(\zeta, \iota) = (q + 1) + p + r = n - q + 1 > s_1(f)$. Similarly we can prove that $F(\eta) = 0$. Finally we have $S(\delta) \ge w(\delta, \alpha) + w(\delta, \beta) + w(\delta, \zeta) + w(\delta, \eta) = (p + 1) + r + (p + 1) + r = 2n - 4q + 2 > s_1(f)$. It is a contradiction. Therefore $F(\delta) = 0$.

Case 3: $s_0(f) \ge 4q$, then from $s_0(f) + 2s_1(f) < 2n + 2$ we get $s_1(f) < n - 2q + 1$. Notice that $w(\delta, \alpha) + w(\delta, \beta) = (p + 1) + r = n - 2q + 1 > s_1(f)$, we have $F(\delta) = 0$.

Now let us consider the case $n - k = 2q + 1$. In Fig. 2, suppose $p + r = k + 1$. Same as previous case, $\lambda, \mu, \xi, \ldots$ are configurations. $\lambda = (p - 1, q, q + 1, r)$, $\mu = (p, q, q + 1, r - 1)$, $\nu = (p, q - 1, q + 1, r)$, $\xi = (p, q, q, r)$,

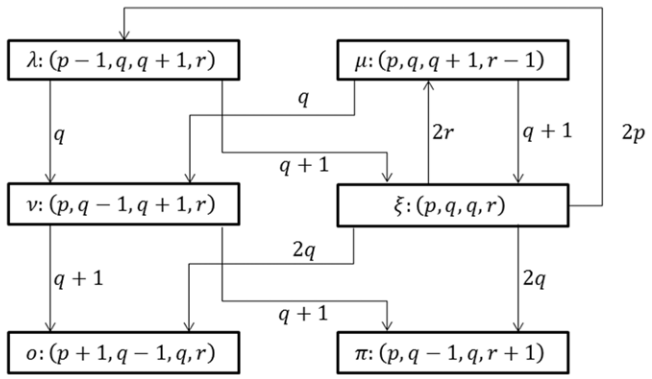

Fig. 2. Sensitivity between configurations, where $n - k = 2q + 1$ and $p + r = k + 1$.

$o = (p + 1, q - 1, q, r),\ \pi = (p, q - 1, q, r + 1)$. The transitions between them are:

$$w(\lambda, \nu) = w(\mu, \nu) = q,\ w(\lambda, \xi) = w(\mu, \xi) = q + 1,$$
$$w(\nu, o) = w(\nu, \pi) = q + 1,\ w(\xi, o) = w(\xi, \pi) = 2q,\ w(\xi, \lambda) = 2p,\ w(\xi, \mu) = 2r.$$

The goal is to prove that $F(\xi) = 0$. By induction hypothesis we have $F(\lambda) = F(\mu) = 0$.

Case 1: $s_0(f) \le q$. Since $S(\lambda) \le s_0(f) < q + 1 = w(\lambda, \xi)$, we have $F(\xi) = F(\lambda) = 0$.

Case 2: $q + 1 \le s_0(f) < 2q + 1$. Since $2s_0(f) + s_1(f) < 2n + 2$, we have $s_1(f) < 2n - 2q$. Assume $F(\xi) = 1$. Since $w(\lambda, \nu) + w(\lambda, \xi) = 2q + 1 > s_0(f)$, we have $F(\nu) = 0$. Notice that $w(\nu, o) + w(\nu, \pi) = 2q + 2 > s_0(f)$, at least one of $F(o)$ and $F(\pi)$ is equal to 0. Then $S(\xi) \ge \min\{w(\xi, o), w(\xi, \pi)\} + w(\xi, \lambda) + w(\xi, \mu) = 2n - 2q > s_1(f)$. It is a contradiction. Therefore $F(\xi) = 0$.

Case 3: $s_0(f) \ge 2q + 1$, then $s_1(f) < 2n - 4q$. Noticing that $w(\xi, \lambda) + w(\xi, \mu) = 2n - 4q > s_1(f)$, we have $F(\xi) = 0$.

In conclusion, all configurations (p, q_0, q_1, r) in $\mathcal{B}_{n,2}$ such that $|q_0 - q_1| \le 1$ must take the same function value. □

Now we can use this lemma to prove Theorem 1. The sketch of the proof is to prove that for every possible configuration C, there exists another configuration C', such that $F(C) = F(C')$, and $C' = (p, q_0, q_1, r)$ where $|q_0 - q_1| \le 1$.

Lemma 3. *If $s(f) < n$, then for any configuration (p, q_0, q_1, r), $q_0 \ne q_1$, there exists another configuration (p', q_0', q_1', r') such that $F(p, q_0, q_1, r) = F(p', q_0', q_1', r')$, and $|q_0' - q_1'| < |q_0 - q_1|$.*

Proof. Assume that there is a configuration C that does not satisfy the lemma above, we assume w.l.o.g. that $C = (p, q_0, q_1, r)$ and $q_0 > q_1$. Notice that $C_1 = (p + 1, q_0 - 1, q_1, r)$, $C_2 = (p, q_0 - 1, q_1, r + 1)$, $C_3 = (p - 1, q_0, q_1 + 1, r)$ and $C_4 = (p, q_0, q_1 + 1, r - 1)$ are all legal configurations, and from the assumption these configurations should have different function value with $F(C)$. Therefore,

$$S(C) \ge w(C, C_1) + w(C, C_2) + w(C, C_3) + w(C, C_4) = q_0 + q_0 + p + r > p + q_0 + q_1 + r = n > s(f) \quad (5)$$

which contradicted to the condition $s(f) < n$. □

By using Lemma 3 iteratively, we show that every configuration C has the same function value with another configuration $C' = (p, q, q, r)$. However, by Lemma 2, all configurations of the form (p, q, q, r) should have the same function value. Therefore the function f must be a constant function.

4. Sensitivity of composite Boolean functions

In this section, we will present an upper bound for our conjecture, particularly we also specialize the upper bound for non-trivial $f \in \mathcal{B}_{n,2}$ to show that Theorem 1 is tight. Besides the proof, we also give a lower bound for those f, which are composed of two symmetric Boolean functions g and h. Notice that according to the symmetry and non-trivialness of f both g and h should be symmetric and non-constant Boolean functions.

Proof of Theorem 3. Suppose $f = g \circ h$, where $g : \{0, 1\}^m \to \{0, 1\}$ and $h : \{0, 1\}^n \to \{0, 1\}$ are both symmetric and non-trivial Boolean functions.

Since g is symmetric, its value only depends on the number of 1-bits in the input $\mathbf{x}$, so there exists a function $g^* : [m] \to \{0, 1\}$, such that $\forall\ \mathbf{x} \in \{0, 1\}^m$, $g(\mathbf{x}) = g^*(|\mathbf{x}|)$, where $|\mathbf{x}| = \left|\{i | x_i = 1\}\right|$ denotes the number of 1-bits in $\mathbf{x}$. Also we have such a function h^* for h.

Since g is not a constant function, there must exist a $0 \le j \le m - 1$, such that $g^*(j) \ne g^*(j + 1)$, w.l.o.g. assume $g^*(j) = 0$ and $g^*(j + 1) = 1$. Similarly we assume $h^*(i) = 0$ and $h^*(i + 1) = 1$, where $0 \le i \le n - 1$. (Otherwise, if such an i does not exist, we can flip all the input bits and substitute h^* by $\overline{h^*}$.) Then we have

$$s_0(g) \ge s_0(g, 1^j 0^{m-j}) = m - j,\ s_1(g) \ge s_1(g, 1^{j+1} 0^{m-j-1}) = j + 1,$$

and

$$s_0(h) \geq s_0(h, 1^i0^{n-i}) = n - i, \; s_1(h) \geq s_1(h, 1^{i+1}0^{n-i-1}) = i + 1.$$

Then we focus on the sensitivity of f. We calculate $s_0(f)$ first. Suppose $\mathbf{x} = \mathbf{x}^{(1)}\mathbf{x}^{(2)} \dots \mathbf{x}^{(m)}$, where each $\mathbf{x}^{(k)}$ is a n-bits string ($k \in [m]$), i.e. $\mathbf{x}^{(k)} = x_1^{(k)}x_2^{(k)} \dots x_n^{(k)}$. Then $f(\mathbf{x}) = g(\mathbf{y})$, where $\mathbf{y} = y_1 \dots y_m$ and $y_k = h(\mathbf{x}^{(k)})$, for $k = 1, 2, \dots, m$.

Now let us consider the following $\mathbf{x} = \mathbf{x}^{(1)}\mathbf{x}^{(2)} \dots \mathbf{x}^{(m)}$, where

$$\mathbf{x}^{(k)} = \begin{cases} 1^{i+1}0^{n-i+1}, & \text{if } 1 \leq k \leq j; \\ 1^i0^{n-i}, & \text{otherwise.} \end{cases}$$

It is easy to see that the corresponding $\mathbf{y} = 1^j0^{m-j}$. Then g is sensitive on $y_{j+1}y_{j+2} \dots y_m$, which are all 0-bits. For $j + 1 \leq k \leq m$, each $y_k = h(\mathbf{x}^{(k)})$ is sensitive on $x_{i+1}^{(k)}x_{i+2}^{(k)} \dots x_n^{(k)}$. Now if we flip any $x_l^{(k)}$ where $i + 1 \leq l \leq n$, then the value of $y_k = h(x^{(k)})$ will flip, hence the value of $g(y)$ will also change. Therefore,

$$s_0(f) = s_0(g \circ h) \geq s_0(g \circ h, \mathbf{x}) = \sum_{k=j+1}^{m} s_0(h, \mathbf{x}^{(k)}) = (m - j)(n - i). \tag{6}$$

Similarly,

$$s_1(f) \geq (j + 1)(i + 1). \tag{7}$$

Now we prove that $\max\{m \cdot s_0(f) + s_1(f), s_0(f) + m \cdot s_1(f)\} \geq mn + m$. W.l.o.g., suppose $(m - j)(n - i) \geq (j + 1)(i + 1)$, then we claim that

$$m \cdot s_0(f) + s_1(f) \geq mn + m. \tag{8}$$

In fact from Eqs. (6) and (7), we have

$$m \cdot s_0(f) + s_1(f) \geq m \cdot (m - j)(n - i) + (j + 1)(i + 1). \tag{9}$$

Notice that

$$\begin{aligned} m \cdot (m - j)(n - i) + (j + 1)(i + 1) - (mn + m) &= m(m - j - 1)(n - i) + m(n - i) + (j + 1)(i + 1) - (mn + m) \\ &= m(m - j - 1)(n - i) + (j + 1)(i + 1) - mi - m \\ &= m(m - j - 1)(n - i) + (j + 1 - m)(i + 1) \\ &= (m - j - 1)[m(n - i) - (i + 1)]. \end{aligned} \tag{10}$$

Since we assumed $(m - j)(n - i) \geq (j + 1)(i + 1)$, it implies that

$$m(n - i) \geq j(n - i) + (j + 1)(i + 1) \geq (j + 1)(i + 1) \geq i + 1. \tag{11}$$

Combining Eq. (10), (11) and the fact that $j \leq m - 1$, we get

$$m \cdot (m - j)(n - i) + (j + 1)(i + 1) \geq mn + m. \tag{12}$$

Eqs. (9) and (12) imply $m \cdot s_0(f) + s_1(f) \geq mn + m$.

Similarly, when $(m - j)(n - i) \leq (j + 1)(i + 1)$ we can show

$$s_0(f) + m \cdot s_1(f) \geq mn + m.$$

Finally by the symmetry of m and n in Eqs. (6) and (7), we have

$$\begin{aligned} \max\{m \cdot s_0(f) + s_1(f), s_0(f) + m \cdot s_1(f)\} &\geq mn + m, \\ \max\{n \cdot s_0(f) + s_1(f), s_0(f) + n \cdot s_1(f)\} &\geq mn + n. \quad \square \end{aligned}$$

Tightness of Conjecture 2

Assume $m \leq n$, then $\max\{\lceil \frac{n+1}{m+1}m \rceil, \lceil \frac{m+1}{n+1}n \rceil\} = \lceil \frac{n+1}{m+1}m \rceil$. Consider the following function $f : \{0, 1\}^{mn} \to \{0, 1\}$,

$$f = T_1^m \circ T^n_{\lceil \frac{n+1}{m+1}m \rceil},$$

where $T_k^l : \{0, 1\}^l \to \{0, 1\}$ is the threshold function, $T_k^l(\mathbf{x}) = 1$ if and only if $|\mathbf{x}| \geq k$. Notice that

$$s_0(T_k^l) = s_0(T_k^l, 1^{k-1}0^{l+1-k}) = l + 1 - k, \; s_1(T_k^l) = s_1(T_k^l, 1^k0^{l-k}) = k.$$

Then we calculate the sensitivity of f. For the outer function T_1^m, 0-sensitivity is non-zero if and only if the input is all zero. If any of these zeros flips, the outcome of f would change. On the other hand, to flip any one of them, we have exactly $s_0(T^n_{\lceil\frac{n+1}{m+1}m\rceil})$ ways, by flipping any of the sensitive bits in the input, to change the interim outcome. Hence

$$\begin{aligned}
s_0(f) &= s_0(T_1^m)\cdot s_0(T^n_{\lceil\frac{n+1}{m+1}m\rceil}) = m\cdot\left(n+1-\left\lceil\frac{n+1}{m+1}m\right\rceil\right)\\
&= m(n+1) - m\left\lceil\frac{n+1}{m+1}m\right\rceil\\
&= \left\lceil\frac{n+1}{m+1}m\right\rceil + m(n+1) - (m+1)\left\lceil\frac{n+1}{m+1}m\right\rceil\\
&\le \left\lceil\frac{n+1}{m+1}m\right\rceil,
\end{aligned}$$

the last "$\le$" is due to $\lceil\frac{n+1}{m+1}m\rceil \ge \frac{n+1}{m+1}m$. Similarly

$$s_1(f) = s_1(T_1^m)\cdot s_1(T^n_{\lceil\frac{n+1}{m+1}m\rceil}) = \left\lceil\frac{n+1}{m+1}m\right\rceil.$$

So $s(f) = \max\{s_0(f), s_1(f)\} = \lceil\frac{n+1}{m+1}m\rceil$. Particularly for $m = 2$, if we use the *configuration* notation described in Section 3, the function is like

$$F(a,b,c,d) = \begin{cases} 0, & \text{if } a \ge \frac{2}{3}(n+1);\\ 1, & \text{otherwise}\end{cases}$$

which gives an upper bound of Theorem 1.

5. The region of feasible $(s_0(f), s_1(f))$

In Section 3 we have proved that for any non-trivial $f \in \mathcal{B}_{n,2}$, the sensitivity of f satisfies that

$$\max\{s_0(f) + 2s_1(f),\ 2s_0(f) + s_1(f)\} \ge 2n + 2. \tag{13}$$

We have also shown some functions with $(s_0(f), s_1(f))$ on the *boundary* (equality holds in Eq. (13)). However, it does not mean that for every pair $(s_0(f), s_1(f))$ with $\max\{s_0(f)+2s_1(f),\ 2s_0(f)+s_1(f)\} \ge 2n+2$, there exists a non-trivial $f \in \mathcal{B}_{n,2}$. Next we show that $(1, m)$ with $m < 2n$ is not feasible for $(s_0(f), s_1(f))$.

Theorem 4. *There does not exist function $f \in \mathcal{B}_{n,2}$ with $s_0(f) = 1$ and $s_1(f) \le 2n - 1$.*

Proof. We use the same terminology as in Section 3. Besides, for any $k \in \mathbb{N}$ and configurations A, B and C, we use notations $A \xrightarrow{k} B$ to indicate $w(A,B) \ge k$ and $A \xrightarrow{k} B + C$ to indicate $w(A,B) + w(A,C) \ge k$ in the following proof. Notice that if $k > s_{f(A)}(f)$, then $A \xrightarrow{k} B$ implies $f(A) = f(B)$. Similarly if $k > s_{f(A)}(f)$ and $f(B) = f(C)$, then $A \xrightarrow{k} B + C$ implies $f(A) = f(B) = f(C)$.

The intuition is to prove that f is a constant function under conditions $s_0(f) = 1$ and $s_1(f) \le 2n-1$. The proof contains three parts. The first two parts prove that either $F(n,0,0,0) = 1$ or $F(0,0,0,n) = 1$ will lead to the result that f becomes constant. The third part proves that under conditions $F(n,0,0,0) = 0$ and $F(0,0,0,n) = 0$, f will also become constant.

On the contrary, assume such f exists.

1. When $F(n,0,0,0) = 1$, since $(n,0,0,0) \xrightarrow{2n} (n-1,0,1,0)$ and $s_1(f) \le 2n-1$, $F(n-1,0,1,0) = 1$. Let (a,b,c,d) be the configuration that $F(a,b,c,d) = 0$ and $b+c+2d$ has the least value. Since $F(n,0,0,0) = 1$ and $F(n-1,0,1,0) = 1$, $b + c + 2d \ge 2$.
   (a) If $d \ge 1$ and $b = c$, then $(a,b,c,d) \xrightarrow{2} (a,b,c+1,d-1)$. So $F(a,b,c+1,d-1) = F(a,b,c,d) = 0$. But $b + (c+1) + 2(d-1) < b + c + 2d$ which indicates $F(a,b,c+1,d-1) = 1$. Thus we get a contradiction.
   (b) If $d \ge 1$ and $b \ne c$, then $(a,b,c,d) \xrightarrow{2} (a,b+1,c,d-1)+(a,b,c+1,d-1)$. So $F(a,b,c+1,d-1) = F(a,b,c,d) = 0$. But $b + (c+1) + 2(d-1) < b + c + 2d$ which indicates $F(a,b,c+1,d-1) = 1$. Thus we get a contradiction.
   (c) If $d = 0$ and $c \ge 2$, then $(a,b,c,d) \xrightarrow{2} (a+1,b,c-1,d)$. So $F(a+1,b,c-1,d) = F(a,b,c,d) = 0$. But $b + (c-1) + 2d < b + c + 2d$ which indicates $F(a+1,b,c-1,d) = 1$. Thus we get a contradiction.
   (d) If $d = 0$ and $c < 2$, since $b + c + 2d \ge 2$, then $b = c = 1$. Then $(a,b,c,d) \xrightarrow{2} (a+1,b-1,c,d)$ and so $F(a+1,b-1,c,d) = F(a,b,c,d) = 0$. But $(b-1)+c+2d < b+c+2d$ which indicates $F(a+1,b-1,c,d) = 1$. Thus we get a contradiction.

   As above, when $F(n,0,0,0) = 1$, f becomes a constant function which contradicts the fact that f is non-trivial.

2. When $F(0, 0, 0, n) = 1$, it is similar to the first case. We can also get a contradiction.
3. The left case is when $F(n, 0, 0, 0) = F(0, 0, 0, n) = 0$. From this two conditions together with the restriction that $s_0(f) = 1$ and $s_1(f) \leq 2n - 1$, we can prove that f is a constant function step by step:

   For each configuration (a, b, c, d):

   (a) When $a \geq 1, c \geq 1, (a + 1, b, c - 1, d) \xrightarrow{2} (a, b, c, d)$.

   (b) When $a \geq 2, b = c = 0, (a, 0, 1, d - 1) \xrightarrow{2} (a - 1, 1, 1, d - 1) \xrightarrow{2} (a - 1, 0, 1, d)$ and $(a, 0, 0, d) \xrightarrow{2n} (a, 0, 1, d - 1) + (a - 1, 0, 1, d)$.

   (c) When $(a, b, c, d) = (1, 0, 0, n - 1), (0, 0, 0, n) \xrightarrow{2} (1, 0, 0, n - 1)$.

   (d) When $a = 0, d \neq 0$, it is similar to $d = 0, a \neq 0$. The configurations with $d = 0, a \neq 0$ can be treated similarly as (a)–(c), since we have $F(n, 0, 0, 0) = 0$ and $F(0, 0, 0, n) = 0$.

   (e) When $a = 0, d = 0, (0, b, c, 0) \xrightarrow{2n}$ some configurations in previous cases.

   We can sort all the configurations (a, b, c, d) using a as the first keyword and d as the second keyword, both in decreasing order. Then these five steps give a way for proving $f(x) = 0$ from the base $F(n, 0, 0, 0) = F(0, 0, 0, n) = 0$. Just follow these five steps and then we can get that f is a constant function which contradicts the fact that f is non-trivial. □

6. Conclusion and discussion

In this paper we proposed analogue conjecture of Turán conjecture about the sensitivity of bipartite graph properties and show examples which achieve the lower bound. We proved our conjecture for $n \times 2$ bipartite graph properties and gave a weaker lower bound for general $m \times n$ bipartite graph properties. We also proved the conjecture for composite Boolean functions. The proofs of Theorems 1 and 3 actually suggest the following stronger conjecture:

Conjecture 2′. *For any non-trivial bipartite graph property $f \in \mathcal{B}_{n,m}$, the 0-sensitivity and 1-sensitivity of f satisfy*

$$\max\{s_0(f) + m \cdot s_1(f), m \cdot s_0(f) + s_1(f)\} \geq m(n + 1).$$

It is also interesting to know which is the feasible region of the pair $(s_0(f), s_1(f)) \in \mathbb{N}^2$.

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