Fans Economy and All-Pay Auctions
with Proportional Allocations[*]

Pingzhong Tang, Yulong Zeng, Song Zuo
Institute for Interdisciplinary Information Sciences
Tsinghua University, Beijing, China
kenshinping@gmail.com,cengyl13@mails.tsinghua.edu.cn,songzuo.z@gmail.com

Abstract

In this paper, we analyze an emerging economic form, called *fans economy*, in which a fan donates money to the host and gets allocated proportional to the amount of his donation (normalized by the overall amount of donation). Fans economy is the major way live streaming apps monetize and includes a number of popular economic forms ranging from crowdfunding to mutual fund.

We propose an auction game, coined *all-pay auctions with proportional allocation* (APAPA), to model the fans economy and analyze the auction from the perspective of revenue. Comparing to the standard all-pay auction, which normally has no pure Nash-Equilibrium in the complete information setting, we solve the pure Nash-Equilibrium of the APAPA in closed form and prove its uniqueness. Motivated by practical concerns, we then analyze the case where APAPA is equipped with a reserve and show that there might be multiple equilibria in this case. We give an efficient algorithm to compute all equilibria in this case. For either case, with or without reserve, we show that APAPA always extracts revenue that 2-approximates the second-highest valuation. Furthermore, we conduct experiments to show how revenue changes with respect to different reserves.

Introduction

Lately, there has been a surge of mobile apps focusing on the so-called *peer-to-peer* live streaming. Prominent examples are Momo (Nasdaq: MOMO), Periscope, Twitch, and Douyu. In essence, such apps can be seen as a social network in which each peer (fan) in the network can watch the live videos shared by another peer (host) he/she follows. A popular host (either commentator or player) on Twitch and Douyu, both of which focus on video games, can gather over a million of fans in a typical game night.

What is interesting about this type of apps is the way they monetize. While all such apps can make money via traditional advertisements, a few such apps take the initiative to design monetization mechanisms as follows: the app allows fans to buy and send gifts to the host and then charges a commission proportional to the price of the gift. In return, the app rewards the fan (either by displaying his or her name to the community (i.e., all fans) or designing a virtual badge, similar to charity auction (Conitzer and Sandholm 2004; Ghosh and Mahdian 2008) for each gift he or she sends. The fan then derives positive utility for the fraction of community attention he or she gets[1]. This new monetization method has turned out to be very successful and has been witnessed by the recent surge in stock price of MOMO (from 8 dollars per share to 24 dollars per share in the past two months since it switched to this monetization mechanism).

The situation above can be modeled as an *all-pay auction with proportional allocation* in which each bidder (fan) submits a bid (the number of gifts he or she buys) to the auctioneer (the app), who allocates one divisible item (community attention) among the bidders proportional to his or her bid.

The simple auction game above turns out to be general enough to cover a number of interesting economic scenarios. For example, a major form of crowdfunding (Alaei, Malekian, and Mostagir 2016) aims to collect money from crowd in order to achieve a revenue target and if succeed, pay back to each agent an amount proportional to overall profit, to which each agent has a private estimation (type). Clearly, such form of crowdfunding can be conveniently modeled by the APAPA game. Similar examples abound, ranging from buying raffle tickets to mutual funds (Piliavin and Charng 1990; Carhart 1997).

In fact, we are not the first to consider the all-pay auction with proportional allocation[2]. A number of papers (Johari and Tsitsiklis 2004; 2009; Nisan et al. 2007, Chapter 21.2) study APAPA from the point of social welfare. They show that, when each bidder is risk-averse in the sense that he or she has a concave utility function towards proportional allo-

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[1]Strictly speaking, the fan also derives positive utility for how happier the host becomes because of his or her gifts. Note that this part of utility he or she gets is also proportional to the number of gifts sent. Therefore, it is also compatible with our APAPA model.

[2]More work on all-pay auctions (Baye, Kovenock, and de Vries 1996; Bertoletti 2006; DiPalantino and Vojnovic 2009; Dechenaux, Kovenock, and Sheremeta 2015). We omit the discussions due to space limit.

cations, the APAPA has a unique pure Nash equilibrium.

Our contribution

In this paper, we investigate APAPA from the revenue perspective of the apps. We make the following contribution:

- We first prove that APAPA has a unique pure Nash equilibrium in *closed form*, thus generalizes Johari's result (which does not give a closed-form solution) in the linear case. Our proof is notably simple for this special case and may be of independent interest.

- We then show that for either case, with or without reserves, the revenue that the APAPA extracts always 2-approximates the second highest valuation[3].

- We notice that, in the case where APAPA has a reserve, there might be multiple equilibria of the auction game. We provide several *partial characterizations* of the sets of equilibria which then enable an *efficient algorithm* to find all equilibria of the auction game.

Settings

In an APAPA, there is a single seller, who has a divisible item for sale, and n bidders, who have linear valuations for receiving a fraction $x_i \in [0, 1]$ of the item, i.e., $v_i x_i, i \in [n]$, where $v_i \in \mathbb{R}_+$. The item is allocated to a bidder with a fraction equal to its bid over the sum of bids from all bidders, i.e.,

$$x_i(b_1, \ldots, b_n) = \frac{b_i}{\sum_{i' \in [n]} b_{i'}}, \tag{AL}$$

where $b_i \in \mathbb{R}_+$ is the bid of bidder i. Note that the allocation rule (AL) is defined only when $\sum_{i \in [n]} b_i > 0$. If all the bidders bid 0, we define the allocation to be $x_i \equiv 0$ for all $i \in [n]$. Throughout this paper, we only consider the case where at least one bidder has non-zero bid.

Bidder i's payment is always b_i. The utility of bidder i is

$$\begin{aligned} u_i(b_i, b_{-i}) &= v_i \cdot x_i(b_i, b_{-i}) - b_i \\ &= \frac{b_i}{\sum_{i' \in [n]} b_{i'}} v_i - b_i, \end{aligned} \tag{UT}$$

where b_{-i} denotes the bids of all bidders other than i. The seller's revenue is simply the sum of the bids, denoted by $S = \sum_{i \in [n]} b_i$.

Without loss of generality, we assume that $v_1 \geq \cdots \geq v_n \geq 0$. A bidding profile $(b_1, \ldots, b_n)$ is a pure Nash equilibrium (PNE), if

$$\forall i \in [n], b_i' \in \mathbb{R}_+, \ u_i(b_i, b_{-i}) \geq u_i(b_i', b_{-i}).$$

Therefore in any pure Nash equilibrium, bidder i's utility $u_i(b_i, b_{-i}) \geq u_i(0, b_{-i}) = 0$. In other words, any bidder's utility is guaranteed to be non-negative as long as he/she best responds to other bidders' strategies.

Note that for bidders with 0 valuation, their bids must be 0 in any equilibrium, otherwise their utilities will be negative. So in the remainder of this paper, we remove all bidders with 0 valuation and assume that $v_1 \geq \cdots \geq v_n > 0$.

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[3]It is worth pointing out that the second highest valuation is a standard benchmark in competitive analysis of complete information settings, cf. (Feige et al. 2005; Alaei, Malekian, and Srinivasan 2009).

Closed form pure Nash equilibrium

Our first step is to characterize the pure Nash equilibrium of APAPA by the following lemma, where we prove the uniqueness of the pure Nash equilibrium and provide a closed form solution. We remark that Johari (Nisan et al. 2007, Ch 21) has an alternative proof of the existence and uniqueness of the pure Nash equilibrium in a more general model (known as *proportional allocation mechanism*) but without closed form solution.

Lemma 1. *Any APAPA has a unique pure Nash equilibrium. Moreover, it can be expressed in closed form as follows,*

$$S^* = \max_{i \in [n]} \frac{i-1}{\sum_{i' \leq i} \frac{1}{v_{i'}}}, \tag{1}$$

$$b_i^* = \max \left\{ 0, S^* - \frac{S^{*2}}{v_i} \right\}, \ \forall i \in [n]. \tag{2}$$

To characterize the pure Nash equilibrium of APAPA, we first figure out what the bidder's best response is. The following two observations state that in an APAPA, the best response of bidder i against any other bidders' bids b_{-i} is $b_i = \max\{S - S^2/v_i, 0\}$.

Observation 1. *Given b_{-i}, bidder i that bids $b_i > 0$ has positive (negative) utility if and only if $v_i > S$ ($v_i < S$).*

Observation 2. *Given b_{-i}, if it is feasible for bidder i to bid $b_i = S - S^2/v_i$, then it is the unique best response of bidder i.*

Here a bid is feasible if it is non-negative (or no less than the reserve in the next section). Observation 2 follows from the first-order and second-order conditions:

$$\frac{\mathrm{d}u_i}{\mathrm{d}b_i} = \frac{(S-b_i)v_i}{S^2} - 1 = 0, \quad \frac{\mathrm{d}^2 u_i}{\mathrm{d}b_i^2} = -\frac{2(S-b_i)v_i}{S^3} < 0.$$

Definition 1. *For each $i \in [n]$, define*

$$S_i = \frac{i-1}{\sum_{i' \leq i} \frac{1}{v_{i'}}}.$$

Lemma 2. *For any $k \in [n]$, the following three statements are equivalent.*

1) $v_{k+1} \geq S_k$
2) $v_{k+1} \geq S_{k+1}$
3) $S_{k+1} \geq S_k$

The result also holds if we simultaneously change the $\geq$ signs above to $\leq$.

Proof. 1) $\iff$ 2):

$$v_{k+1} \geq S_{k+1} \iff v_{k+1} \geq \frac{k}{\sum_{i=1}^{k+1} \frac{1}{v_i}}$$

$$\iff 1 + v_{k+1} \sum_{i=1}^{k} \frac{1}{v_i} \geq k \iff v_{k+1} \geq S_k$$

3) $\iff$ 1):

$$S_{k+1} \geq S_k \iff (k-1) \sum_{i=1}^{k+1} \frac{1}{v_i} \leq k \sum_{i=1}^{k} \frac{1}{v_i}$$

$$\iff \sum_{i=1}^{k} \frac{1}{v_i} \geq \frac{k-1}{v_{k+1}} \iff v_{k+1} \geq S_k$$

For the case of $\leq$, the proof is similar. $\qquad\square$

As we know the best responses of the bidders, we are ready to prove Lemma 1 — to demonstrate the close form of the pure Nash equilibrium and prove its uniqueness.

Throughout this and the next section, we define k as the smallest number in $[n]$ such that $S_k = \max_{i \in [n]} S_i$.

Proof of Lemma 1. Note that $S^* = S_k$.
Closed form First, for all $i > k$, since $S_k \geq S_{k+1}$, by Lemma 2 we have $v_i \leq v_{k+1} \leq S_k$. So $b_i = \max\{0, S^* - S^{*2}/v_i\} = 0$. Similarly, for all $i \leq k$, since $S_k > S_{k-1}$, from Lemma 2 we get $v_k > S_k$, so $v_i > S_k$ and thus $b_i = S^* - S^{*2}/v_i > 0$.

We conclude that $b_i > 0 \iff v_i > S_k \iff i \leq k$. By definition, the revenue

$$S = \sum_{i=1}^k b_i = kS_k - S_k^2\big(\sum_{i=1}^k v_i^{-1}\big) = S_k.$$

Now we remain to show that b_i is the unique best response for each bidder i when all other bidders bid b_{-i}.

- For all $i > k$, $b_i = 0$ is the best response for bidder i: when bidder i bids any $b_i' > 0$, then $v_i < S_k + b_i'$, by Observation 1 bidder i has negative utility.
- For all $i \leq k$, then $b_i > 0$. By Observation 2 $b_i = S - S^2/v_i$ is the unique best response of bidder i.

So bidding profile $(b_1, \ldots, b_n)$ is a pure Nash equilibrium.

Uniqueness To prove the uniqueness, we assume by contradiction that there exists another equilibrium with m participants $T \subseteq [n]$, where each of them bids more than 0 in the equilibrium. Note that for any bidder i bids 0 (hence $i \notin T$) $v_i \leq S$ must hold. Otherwise he/she can bid $b_i' > 0$ such that $b_i' + S < v_i$ and receives a positive utility, contradiction. Also note that for any bidder $i \in T$, $v_i > S$ must hold by Observation 1. Thus $T = [m]$.

Then we show that m must equal to k to reach the desired contradiction.

For each $i \in T = [m]$, by Observation 2, we have $b_i = S - S^2/v_i$. By solving these equations we get $S = S_m$. Since $v_m > S_m$, by Lemma 2, we have $S_{m-1} < S_m$, and hence $v_{m-1} \geq S_m > S_{m-1}$. Similarly, we have $S_1 < \cdots < S_m$.

On the other hand, for each $i \notin T$, since $v_{i'} \leq S = S_m$ for all $m+1 \leq i' \leq i$, we have

$$S_i = \frac{i-1}{v_1^{-1} + \cdots + v_i^{-1}} = \frac{i-1}{(m-1)/S_m + v_{m+1}^{-1} + \cdots + v_i^{-1}}$$
$$\leq \frac{i-1}{(m-1)/S_m + (i-m)S_m^{-1}} = S_m.$$

Hence $S_1 < \cdots < S_m \leq S_{m+1}, \ldots, S_n$, therefore $m = k$, contradiction. $\square$

The result shows that, from bidder 1 to bidder n, the bidder participates the game if and only if his participation increases revenue.

Revenue Guarantee for APAPA

From the auctioneer's perspective, a natural question is: How much revenue can an APAPA guarantee at the (unique) pure Nash equilibrium? In this paper, we study this question based on the analysis of pure Nash equilibria in the complete information setting (cf. (Milchtaich 1996; Fang,

Tang, and Zuo 2016)), in contrast with the rich literatures on Bayesian settings (Wang and Tang 2014; 2015; Tang, Wang, and Zhang 2016; Mirrokni et al. 2016a; 2016b; Tang and Zeng 2016).

To quantify such revenue guarantee (if exists), we need a reasonable definition of benchmark for APAPA to compare with. For the complete information setting, it is standard from the digital good literature that v_2, the second highest value, is used as the benchmark for revenue.

In this section, we prove our first main result that the revenue of any APAPA 2-approximates the benchmark v_2.

Theorem 1. *When $n \geq 2$, the equilibrium revenue of any APAPA is at least $v_2/2$.*

Moreover, the bound is tight, namely, there exists an instance for $n \geq 2$, such that the revenue of the APAPA equals to $v_2/2$ at the unique pure Nash equilibrium.

Proof of Theorem 1. Note that $S_1 = 0$, so k is at least 2. Hence the revenue S is at least $v_2/2$:

$$S = S_k \geq S_2 = \frac{v_1 v_2}{v_1 + v_2} \geq v_2/2.$$

For instances where $v_1 = v_2$ and $v_3, v_4, \ldots, v_n < v_2/2$, the revenue is,

$$S = \frac{v_1 v_2}{v_1 + v_2} = v_2/2.$$

In fact, the lower bound is reached only on these instances. $\square$

APAPA with Reserves

A common way to boost revenue is to set a reserve price in an auction (Hartline and Roughgarden 2009; Ostrovsky and Schwarz 2011; Tang and Sandholm 2012). In the APAPA model, the reserves can be easily implemented by deleting any bid less than the reserve. In the fans economy scenario, the reserve corresponds the price of the cheapest gift.

In this section, we show that an appropriately set reserve can increase the revenue, but may lead to multiple equilibria or no equilibrium at all. We also give an algorithmic characterization of the set of all equilibria.

Theorem 2. *For APAPA with reserve, given $v_i, i \in [n]$, all equilibria, if any, can be computed in polynomial time.*

Throughout this section, we refer r as the reserve.

The key reason for multiplicity of PNEs is that now bidders are not allowed to place a bid $0 < b_i < r$. Changing one's bid from 0 to r may significantly affect the strategies of others, and then in turns changes her own best response constraints.

Consider an example where $n = 2$, $v_1 = v_2$, and reserve $r = v_1 - \epsilon$. Once bidder one bids r, bidder two has no incentive to participate. Similarly if bidder two bids r first, bidder one will not participate. So both $(r, 0)$ and $(0, r)$ are equilibria.

To compute the PNEs, one naive algorithm would be to enumerate the sets of bidders bidding 0 and r, and then check if it is a PNE when the rest bidders play their best responses. This naive algorithm is inefficient because one may need to enumerate exponential many possibilities. However,

we prove the following set of lemmas that find a monotonicity property of the bidders' strategies, thus enabling an efficient enumeration algorithm.

Formally, for any equilibrium B, define T_B as the set of bidders whose bids are larger than r, and define R_B as the set of bidders whose bids are equal to r. We call them *participants* of the equilibrium. Bidders that do not belong to T_B or R_B are not participants. Define S_B as the revenue of equilibrium B, from Observation 2, we have

- For all $i \in T_B$, $v_i > \frac{S_B^2}{S_B - r}$;

- For all $i \in R_B$, $S_B < v_i \leq \frac{S_B^2}{S_B - r}$;

- For all $i \notin T_B, i \notin R_B$, $v_i \leq S_B + r$.

Lemma 3. *For any equilibrium B, there exist $k_B \in [n]$ such that $T_B = \{1, 2, \ldots, k_B\}$ or $T_B = \emptyset$.*

Proof. We assume by contradiction that there exists i, j such that $i < j, i \notin T_B, j \in T_B$. So $v_j > S_B^2/(S_B - r)$, $v_i \leq S_B + r$. But $S_B^2/(S_B - r) > S_B + r$, it is a contradiction. $\square$

So $i \in T_B$ if and only if $i \in \{1, 2, \ldots, k_B\}$.

Note that if $n = 1$ and $v_1 > r$ the only equilibrium is $b_1 = r$.

Lemma 4. *For $n > 1$ and any equilibrium B, if $T_B = \emptyset$, then $S_B = |R_B|r$. Otherwise*

$$S_B = \frac{k_B - 1 + \sqrt{(k_B - 1)^2 + 4|R_B|r \sum_{i=1}^{k_B} \frac{1}{v_i}}}{2 \sum_{i=1}^{k_B} \frac{1}{v_i}}.$$

Proof. In equilibrium B, For any $i \in T_B$, by Observation 2, $b_i = S_B - S_B^2/v_i$. Add the equations together for $i = 1, 2, \ldots, k_B$, we get

$$S_B - |R_B|r = \sum_{i=1}^{k_B} b_i = k_B S_B - S_B^2 \sum_{i=1}^{k_B} \frac{1}{v_i}.$$

By solving the equation above, we prove the lemma. $\square$

The following lemma characterizes an important property for the set of all equilibria. It states that

Lemma 5. *In the APAPA, for any two equilibria B_1, B_2, if the number of participants are the same, i.e., $|T_{B_1}| + |R_{B_1}| = |T_{B_2}| + |R_{B_2}|$, then $T_{B_1} = T_{B_2}$ and $|R_{B_1}| = |R_{B_2}|$.*

As a corollary, if two equilibria B_1, B_2 share the same set of participants, i.e., $T_{B_1} \cup R_{B_1} = T_{B_2} \cup R_{B_2}$, then B_1 and B_2 are identical.

Proof. We assume by contradiction that $|R_{B_1}| > |R_{B_2}|$ (Note that by Lemma 3 $|T_{B_1}| = |T_{B_2}|$ indicates $T_{B_1} = T_{B_2}$). By Lemma 3 it is without loss of generality to assume that

$$B_1 = (b_1, b_2, \ldots, b_k, \underbrace{r, r, \ldots, r}_{m}),$$

$$B_2 = (b_1', b_2', \ldots, b_{k+l}', \underbrace{r, r, \ldots, r}_{m-l}),$$

where $l, m, k \in \mathbb{Z}_+, b_i, b_i' > r$.

On the one hand, we first prove that $S_{B_1} > S_{B_2}$.

Define $f(S) = \frac{S^2}{S-r}$. For equilibrium B_2, since $b_{k+l}' > r$, we have $v_{k+l} > f(S_{B_2})$. For equilibrium B_1, we know that $v_{k+l} \leq f(S_{B_1})$ (bidder $k + l$ bids r or does not participate. The latter case implies that $v_{k+l} \leq S_{B_1} + r < f(S_{B_1})$). So $f(S_{B_1}) > f(S_{B_2})$. Note that $f(S)$ is increasing when $S > 2r$, so we get $S_{B_1} > S_{B_2}$.

Then we prove that for any $i > 1$, $v_i \leq 2S_{B_1}$.

If $v_i > 2S_{B_1}$, from observation 2,

$$b_i = S_{B_1} - \frac{S_{B_1}^2}{v_i} > S_{B_1} - \frac{S_{B_1}^2}{2S_{B_1}} = \tfrac{1}{2} S_{B_1}$$

which implies that $b_1 + b_2 > S_{B_1}$, contradiction. So we have $v_i \leq 2S_{B_1}$. Similarly $v_i \leq 2S_{B_2}$.

Then we prove that for any $i > 1$, $b_i < b_i'$.

For any $i > 1$, define $g(S) = S - \frac{S^2}{v_i}$. $g(S)$ is a quadratic function which gets its maximum at $S = \frac{1}{2} v_i$. Since we already have $S_{B_1} > S_{B_2} > \frac{1}{2} v_i$, we get $g(S_{B_1}) < g(S_{B_2})$, i.e., $b_i < b_i'$.

On the other hand, for B_1, from Observation 2, we have

$$S_{B_1}^2 = v_1(S_{B_1} - b_1) = v_1(\textstyle\sum_{i=2}^{k} b_i + mr) \qquad (3)$$

Similarly we get

$$S_{B_2}^2 = v_1(\textstyle\sum_{i=2}^{k+l} b_i' + (m - l)r) \qquad (4)$$

Since we already have $b_i < b_i'$, $\forall 1 < i \leq k$ and also note that $b_i' > r$, $\forall k + 1 \leq i \leq k + l$, we get

$$\textstyle\sum_{i=2}^{k} b_i + mr < \sum_{i=2}^{k+l} b_i' + (m - l)r,$$

Put this into (3) and (4) we have $S_{B_1} < S_{B_2}$, contradiction. So the lemma is proved. $\square$

Given the value of any subset of bidders, by Lemma 5 there is a unique way partition them into two sets T and R, such that for all $i \in T$, $v_i > \frac{S^2}{S-r}$ and for all $i \in R$, $v_i \leq \frac{S^2}{S-r}$. (Here we ignore the condition that $v_i > S$.) We introduce Algorithm 1 to find the partition.

Lemma 6. *Given the valuations of any subset of bidders, Algorithm 1 returns a partition of the bidders into two sets T and R, such that for all $i \in T$, $v_i > \frac{S^2}{S-r}$ and for all $i \in R$, $v_i \leq \frac{S^2}{S-r}$.*

Algorithm 1: Partition

Input: $U = \{1, 2, \ldots, n'\}$ $(n' > 1)$
Output: set T, R and revenue S
1: $T \leftarrow U, R = \emptyset$
2: **for** $i = n'$ down to 0 **do**
3:     Compute S according to Lemma 4
    ($k_B \leftarrow |T|, |R_B| = |R|$).
4:     **if** $T = \emptyset$ **or** $v_i > \frac{S^2}{S-r}$ **then**
5:        return T, R and S
6:     **end if**
7:     $T = T \setminus \{i\}, R = R \cup \{i\}$
8: **end for**

Note that the partition given by Algorithm 1 is not enough for finding an equilibrium, since the returned set R includes bidders with value less than S. We need the following Algorithm 2 to check the existence of equilibrium:

Lemma 7. *Given values of all bidders, if there is no equilibrium Algorithm 2 returns "No Equilibrium"; otherwise Algorithm 2 returns the locally efficient equilibrium (see definition 2) with the minimum number of participants (where the revenue is also the minimum among all equilibria).*

Definition 2. *An equilibrium is locally efficient if the value of any participant is no less than the value of any non-participant.*

Algorithm 2: Finding the locally efficient Equilibrium with the minimum number of participants.

Input: $U = \{v_1, v_2, \ldots, v_n\}$
Output: Equilibrium
1: **if** $v_1 \leq r$ **then**
2:     **return** $(0, 0, \ldots, 0)$
3: **end if**
4: **for** $i = 1$ to n **do**
5:     Partition $\{1, \ldots, i\}(i > 1)$ by Algorithm 1 to get T,R and S (for $i = 1$, let $S \leftarrow r$).
6:     **if** $v_i \leq S$ **then**
7:         **return** "No Equilibrium"
8:     **end if**
9:     **if** $i = n$ or $v_{i+1} \leq S + r$ **then**
10:         $B \leftarrow (b_i = S - S^2/v_i \ \forall i \in T, b_i = r \ \forall i \in R, b_i = 0 \ otherwise)$
11:         **return** B
12:     **end if**
13: **end for**

To find all locally efficient equilibria, we can simply modify Algorithm 2 by letting it continue the for-loop when an equilibrium is found.

Proof of Theorem 2. By Lemma 5, fix the number of participants, any equilibrium has the same form to a locally efficient equilibrium. Given a locally efficient equilibrium B, we can derive all equilibria with the same number of participants as B. We construct an equilibrium $B' = (b_1', b_2', \ldots, b_n')$ as follows:

- For all i such that $v_i > S_B + r$, let $b_i' = b_i$. Suppose the number of these bidders is p, then these bidders are the first p bidders and they are all participants in B, i.e., $p \leq |T_B| + |R_B|$.
- For the remaining $n - p$ bidders, define Q as the set of bidders such that for all $i \in Q$, $S_B \leq v_i \leq S_B + r$. Note that if one of the $n - p$ remaining bidders is a participant in B — hence his value is no less than S_B — then he belongs to Q. So, $|T_B| + |R_B| - p \leq |Q|$. Choose any $|T_B| + |R_B| - p$ bidders from Q and set their bids to r in B'.
- For the remaining $n - |T_B| + |R_B|$ bidders, set their bids to 0 in B'.

It is not hard to check that B' is also an equilibrium. There are in total $\binom{|Q|}{|T_B|+|R_B|-m}$ equilibria derived from B. $\qquad\square$

Revenue Guarantee

A natural question is that, as the reserve increases from 0, how does the revenue changes? When r continuously increases from 0, it will first meet a point such that $v_k = S^2/(S - r)$ for some bidder k. Then as r continues to increasing, bidder k will bid r to follow, which leads a continuous increasing of the revenue, until $r = v_k$ and bidder k will choose to leave, which leads a suddenly decrease of the revenue. As a result, the shape of the reserve-to-revenue curve should be serrated. At each local optimum, the revenue $S = v_k$ is larger than the initial revenue ($r = 0$).

We then prove that the 2-approximation result still holds.

Theorem 3. *In the APAPA with reserve, the revenue of any equilibrium (if exists) is at least $v_2/2$.*

Proof. For any equilibrium B, if $k_B > 1$, from Lemma 4 we have

$$S_B \geq \frac{k_B - 1}{\sum_{i=1}^{k_B} v_i^{-1}} \geq \frac{k_B - 1}{v_1^{-1} + v_2^{-1} + (k_B - 2)/S_B}$$
$$\Longrightarrow S_B \geq \frac{v_1 v_2}{v_1 + v_2} \geq v_2/2$$

If $k_B + |R_B| > 1$ and $k_B \leq 1$, we assume by contradiction that $v_2 > 2S_B$, then $v_2 > 2S_B > 4r$

Since $b_2 = r$,

$$v_2 \leq \frac{S_B^2}{S_B - r} < \frac{v_2^2/4}{v_2/2 - r}$$

This implies that $v_2 < 4r$, contradiction.

It remains to consider the case where only bidder 1 participates, so $S_B = b_1 = r$.

As bidder 2 does not have any incentive to participate, we have $v_2 \leq S_B + r = 2r$, so $v_2 \leq 2S_B$.

By summing up, we prove the lemma. $\qquad\square$

No pure Nash equilibrium Now we show a 2-bidder example where no pure Nash equilibrium exists. $v_1 > v_2^2/r$ and $v_2 = 2r + \epsilon$. Then if: 1) Bidder two bids $b_2 = 0$, then bidder one will bid $b_1 = r$ and hence bidder twos best response is $b_2 = r$ in this case (since $b_1 + r < v_2 < (b_1 + r)^2/b_1$), for sufficiently small ϵ); 2) Bidder two bids $b_2 = r$, then bidder one will bid $b_1 = \sqrt{v_1 r} - r > r$ and hence bidder twos best response is $b_2 = 0$ in this case (since $v_2 < b_1 + r$). Also note that no matter $b_1 = 0$ or $b_1 \geq r$, bidder two never bids $b_2 > r$ in best response. Therefore, there is no pure Nash equilibrium in this example.

Experiments

To empirically evaluate how revenue changes as a function of reserve, we simulate a setting with $n = 10000$ bidders and show how revenue changes as the reserve grows from 0 to a reasonably large price. In particular, we evaluate
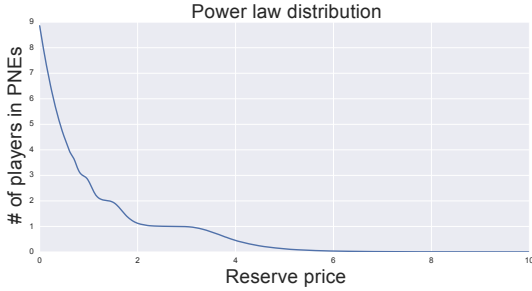
- how the (best equilibrium/worst equilibrium/equilibria on average) revenue varies;

- how the average number of players (who meet the reserve) at pure Nash equilibria varies.
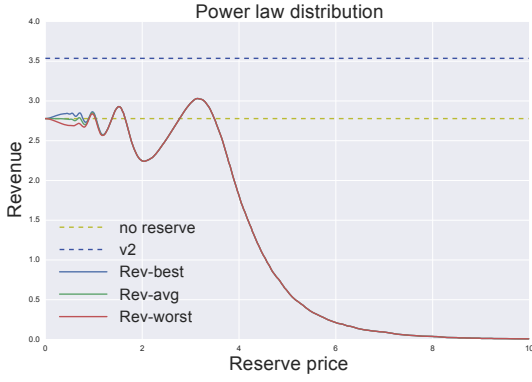
(a) Revenues change as the reserve grows.
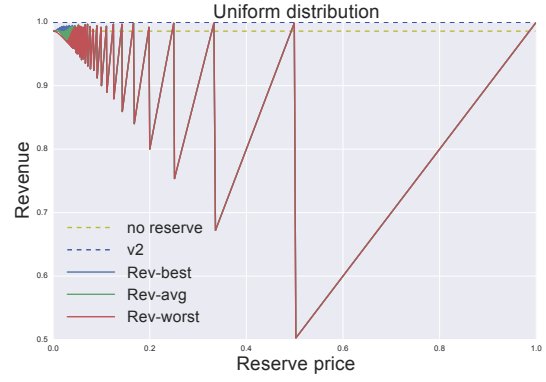
(b) Revenues change rapidly as the reserve grows.

(c) The number of players in PNEs decreases as the reserve grows.

(d) The number of players in PNEs decreases as the reserve grows.

Basic setups

We run simulations for bidders whose types are i.i.d. sampled from given valuation distributions. Here we consider two different distributions:

- Power law distribution with $\alpha = 7$ and $v_{\min} = 1$, i.e.,

$$f^{\text{power}}(v) = \alpha v^{-1-\alpha}, \ v \in [1, +\infty).$$

- $[0, 1]$ uniform distribution.

Power law distributions (Faloutsos, Faloutsos, and Faloutsos 1999; Mitzenmacher 2004; Gabaix 2008) have been used in many situations, such as city populations, incomes, etc., and to model online distributions. In particular, power law distribution is also a reasonable assumption on the valuation distribution for our model.

In contrast, for simulations using uniform distributions, they give intuitions on the structural properties of the pure equilibria. In fact, one can observe that when the several highest bids are very close, the revenue curve is very volatile.

For each sample (a value profile of n bidders) and reserve price r, we find all the pure Nash equilibria and calculate the best, worst, and average[4] revenue on them and the average number of players (with positive bidders) in these PNEs.

We repeat the sampling process for N times and take the average values of the statistics to draw the above charts,

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[4]By average, we mean average among all the PNEs with uniform weights.

where for power law distribution $N = 10000$ and for uniform distribution $N = 1000$ (since the latter converges fast).

Discussions

In the figures above, revenue for either distributions increases and decreases periodically. Moreover, each decrease on revenue corresponds to one bidder leaves the auction (bidding 0) due to the increase in reserve. Notably, at almost every peak (local maximum), the revenue is larger than the revenue of APAPA without reserve (dashed yellow line). In particular, the improvements at the last two peaks for power law distribution are quite significant, which suggests that local search might be a proper algorithm for revenue maximization in practice.

Another observation on the revenue for power law distribution (Fig 1(a)) is that the revenue variance (difference between the best and worst revenue) is quite small, and converges fast when the reserve price is more than 1. It suggests that although the APAPA with reserve has many PNEs, the revenue does not vary too much on different PNEs.

As we mentioned that revenue decrease corresponds to bidders leaving the auction, which can be verified by combining Fig 1(a) with Fig 1(c) (or Fig 1(b) with Fig 1(d)). For example, the number of players significantly drops from 2 to 1 when the reserve price grows from 1.5 to 2, where revenue drops from a local maximum to a local minimum. The fact matches our observation that bidder 2 will not be in the PNE if $2r > v_2$, where the mean value of v_2 is around 3.5 as shown in Fig 1(a).

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