

# A Simple Near-Linear Pseudopolynomial Time Randomized Algorithm for Subset Sum

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## Abstract

Given a multiset  $S$  of  $n$  positive integers and a target integer  $t$ , the SUBSET SUM problem asks to determine whether there exists a subset of  $S$  that sums up to  $t$ . The current best deterministic algorithm, by Koiliaris and Xu [SODA'17], runs in  $\tilde{O}(\sqrt{nt})$  time, where  $\tilde{O}$  hides poly-logarithm factors. Bringmann [SODA'17] later gave a randomized  $\tilde{O}(n+t)$  time algorithm using two-stage color-coding. The  $\tilde{O}(n+t)$  running time is believed to be near-optimal.

In this paper, we present a simple and elegant randomized algorithm for SUBSET SUM in  $\tilde{O}(n+t)$  time. Our new algorithm actually solves its counting version modulo prime  $p > t$ , by manipulating generating functions using FFT.

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## 1 Introduction

Given a multiset  $S$  of  $n$  positive integers and a target integer  $t$ , the SUBSET SUM problem asks to determine whether there exists a subset of  $S$  that sums up to  $t$ . It is one of Karp's original NP-complete problems [9], and is widely taught in undergraduate algorithm classes. In 1957, Bellman gave the well-known dynamic programming algorithm [2] in time  $O(nt)$ . Pisinger [12] first improved it to  $O(nt/\log t)$  on word-RAM models. Recently, Koiliaris and Xu gave a deterministic algorithm [10, 11] in time  $\tilde{O}(\sqrt{nt})$ , which is the best deterministic algorithm so far. Bringmann [4] later improved the running time to randomized  $\tilde{O}(n+t)$  using color-coding and layer splitting techniques. Abboud et al. [1] recently showed that SUBSET SUM has no  $O(t^{1-\epsilon}n^{O(1)})$  algorithm for any  $\epsilon > 0$ , unless the Strong Exponential Time Hypothesis (SETH) is false, so the  $\tilde{O}(n+t)$  time bound is likely to be near-optimal.

In this paper, we present a new randomized algorithm matching the  $\tilde{O}(n+t)$  running time by Bringmann [4]. The basic idea of our approach is quite straightforward. For prime  $p > t$ , we give an  $\tilde{O}(n+t)$  algorithm for  $\#_p$ SUBSET SUM, the counting version of SUBSET SUM problem modulo  $p$ . Then the decision version can be solved with high probability by randomly picking a sufficiently large prime  $p$ .

A closely related problem is  $\#$ KNAPSACK, which asks for the number of subsets  $S$  such that  $\sum_{s \in S} s \leq t$ . There are extensive studies on approximation algorithms for the  $\#$ KNAPSACK problem [6, 8, 13, 7]. Our algorithm can solve the modulo  $p$  version  $\#_p$ KNAPSACK in near-linear pseudopolynomial time for prime  $p > t$ .

Compared to the previous near-linear time algorithm for SUBSET SUM by Bringmann [4], our algorithm is simpler and more practical. The precise running time of our algorithm is  $O(n + t \log^2 t)$  with error probability  $O((n + t)^{-1})$ . If a faster algorithm for manipulating formal power series by Brent [3] is applied, it can be improved to  $O(n + t \log t)$  time (see Remark on Lemma 2), which is faster than Bringmann's algorithm by a factor of  $\log^4 n$ .

## 1.1 Main ideas of our algorithm

The SUBSET SUM instance can be encoded as a generating function  $A(x) = \prod_{i=1}^n (1 + x^{s_i})$ , where  $s_1, \dots, s_n$  are the input integers, and our goal is to compute the  $t$ -th coefficient of  $A(x)$  and see whether it is zero or not.

Instead of directly expanding  $A(x)$ , we consider its logarithm  $B(x) = \ln(A(x))$ . Using basic properties of the logarithm function and its power series, it's possible to compute the first  $t + 1$  coefficients of  $B(x)$  in  $\tilde{O}(t)$  time. Then we can recover the first  $t + 1$  coefficients of  $A(x) = \exp(B(x))$  in  $\tilde{O}(t)$  time using a simple divide and conquer algorithm with FFT (or a slightly faster algorithm by Brent [3]).

The coefficients involved in the algorithm could be exponentially large. To avoid dealing with high-precision numbers, we pick a prime  $p$  and perform arithmetic operations efficiently in the finite field  $\mathbb{F}_p$ , and in the end check whether the result is zero modulo  $p$ . By picking random  $p$  from a large interval, the algorithm succeeds with high probability.

## 2 Preliminaries

### 2.1 Subset sum problem

Given  $n$  (not necessarily distinct) positive integers  $s_1, s_2, \dots, s_n$  and a target sum  $t$ , the SUBSET SUM problem is to decide whether there exists a subset of indices  $I \subseteq \{1, 2, \dots, n\}$  such that  $\sum_{i \in I} s_i = t$ . We also consider the  $\#_p$ SUBSET SUM problem, which asks for the number of such subsets  $I$  modulo  $p$ . We use the word RAM model with word length  $w = \Theta(\log t)$  throughout this paper.

### 2.2 Polynomials and formal power series

#### Formal power series

Let  $R[x]$  denote the ring of polynomials over a ring  $R$ , and  $R[[x]]$  denote the ring of formal power series over  $R$ . A formal power series  $f(x) = \sum_{i=0}^{\infty} f_i x^i$  is a generalization of a polynomial with possibly an infinite number of terms. Polynomial addition and multiplication naturally generalize to  $R[[x]]$ . Composition  $(f \circ g)(x) = f(g(x)) = \sum_{i=0}^{\infty} f_i \left( \sum_{j=1}^{\infty} g_j x^j \right)^i$  is well-defined for  $f(x) = \sum_{i=0}^{\infty} f_i x^i \in R[[x]]$  and  $g(x) = \sum_{j=1}^{\infty} g_j x^j \in xR[[x]]$ . Here  $xR[[x]]$  (or  $xR[x]$ ) denotes the set of series in  $R[[x]]$  (or polynomials in  $R[x]$ ) with zero constant term.

#### Exponential and logarithm

We are familiar with the following two series in  $\mathbb{Q}[[x]]$ ,

$$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}, \quad (1)$$

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad (2)$$

satisfying

$$\exp(\ln(1 + f(x))) = 1 + f(x), \quad (3)$$

and

$$\ln((1 + f(x))(1 + g(x))) = \ln(1 + f(x)) + \ln(1 + g(x)) \quad (4)$$

for any  $f(x), g(x) \in x\mathbb{Q}[x]$ .

### Modulo $x^{t+1}$

Our algorithm only deals with the first  $t+1$  terms of any formal power series. For  $f(x), g(x) \in R[[x]]$ , we write  $f(x) \equiv g(x) \pmod{x^{t+1}}$  if  $[x^i]f(x) = [x^i]g(x)$  for all  $0 \leq i \leq t$ , where  $[x^i]f(x)$  denotes the  $i$ -th coefficient of  $f(x)$ .

As an example, define

$$\exp_t(x) = \sum_{i=0}^t \frac{x^i}{i!} \quad (5)$$

as a  $t$ -th degree polynomial in  $\mathbb{Q}[x]$ . Then  $\exp(f(x)) \equiv \exp_t(f(x)) \pmod{x^{t+1}}$  clearly holds for any  $f(x) \in x\mathbb{Q}[[x]]$ .

## 2.3 Modulo prime $p$

To avoid dealing with large fractions or floating-point numbers, we will work in the finite field  $\mathbb{F}_p = \{\bar{0}, \bar{1}, \dots, \overline{p-1}\}$  of prime order  $p = 2^{\Theta(\log t)}$ . Addition and multiplication in  $\mathbb{F}_p$  take  $O(1)$  time in the word RAM model. Finding the multiplicative inverse of a nonzero element in  $\mathbb{F}_p$  takes  $O(\log p)$  time using extended Euclidean algorithm [5, Section 31.2].

Our algorithm will regard polynomial coefficients as elements from  $\mathbb{F}_p$ . The coefficients can be rational numbers, but their denominators should not have prime factor  $p$ . Formally, let

$$\mathbb{Z}_{p\mathbb{Z}} = \{r/s \in \mathbb{Q} : r, s \text{ are coprime integers, } p \text{ does not divide } s\} \quad (6)$$

and apply the canonical homomorphism from  $\mathbb{Z}_{p\mathbb{Z}}[x]$  to  $\mathbb{F}_p[x]$ , determined by

$$r/s \mapsto \bar{s}^{-1}\bar{r}, \quad x \mapsto x. \quad (7)$$

We use  $\bar{A}$  or  $A \bmod p$  to denote  $A$ 's image in  $\mathbb{F}_p[x]$ .

From now on we assume  $p > t$ , so that  $\exp_t(x) \in \mathbb{Z}_{p\mathbb{Z}}[x]$  (see equation (5)), and let  $\overline{\exp_t(x)}$  denote its image in  $\mathbb{F}_p[x]$ .

## 2.4 Computing exponential using FFT

► **Lemma 1 (FFT).** *Given two polynomials  $f(x), g(x) \in \mathbb{F}_p[x]$  of degree at most  $t$ , one can compute their product  $f(x)g(x)$  in  $O(t \log t)$  time.*

**Proof.** The classic FFT algorithm [5, Chapter 30] can multiply  $f(x)$  and  $g(x)$ , regarded as polynomials in  $\mathbb{Z}[x]$ , in  $O(t \log t)$  time. Then take the remainder of each coefficient modulo  $p$ . ◀

Lemma 2 is a classical result on manipulating formal power series, and is the main building block of our algorithm.

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procedure COMPUTE( $l, r$ ) ▷ after COMPUTE( $l, r$ ) returns, all values  $g_1, \dots, g_r$  are ready
  if  $l < r$  then
     $m \leftarrow \lfloor (l + r)/2 \rfloor$ 
    COMPUTE( $l, m$ )
    for  $i \leftarrow m + 1, m + 2, \dots, r$  do
       $g_i \leftarrow g_i + i^{-1} \sum_{j=i}^m (i - j) f_{i-j} g_j$ 
    end for
    COMPUTE( $m + 1, r$ )
  end if
end procedure

procedure MAIN
  Initialize  $g_0 \leftarrow 1, g_i \leftarrow 0 (1 \leq i \leq t)$ 
  COMPUTE( $0, t$ )
end procedure

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■ **Figure 1** Algorithm for computing  $g_1, \dots, g_t$

► **Lemma 2** (Brent [3]). *Given a polynomial  $f(x) \in x\mathbb{F}_p[x]$  of degree at most  $t$  ( $t < p$ ), one can compute a polynomial  $g(x) \in \mathbb{F}_p[x]$  in  $\tilde{O}(t)$  time such that  $g(x) \equiv \overline{\exp_t}(f(x)) \pmod{x^{t+1}}$ .*

► **Remark.** Brent's algorithm [3] uses Newton's iterative method and runs in time  $O(t \log t)$ . Here we describe a simpler  $O(t \log^2 t)$  algorithm by standard divide and conquer. We present the algorithm as over  $\mathbb{Q}$  for notational simplicity.

**Proof.** Let  $f(x) = \sum_{i=1}^t f_i x^i$  and  $g(x) = \exp(f(x)) = \sum_{i=0}^{\infty} g_i x^i$ . Then  $g'(x) = g(x)f'(x)$ . Comparing the  $(i-1)$ -th coefficients on both sides gives a recurrence relation

$$g_i = i^{-1} \sum_{j=0}^{i-1} (i-j) f_{i-j} g_j \quad (8)$$

with initial value  $g_0 = 1$ . The desired coefficients  $g_1, \dots, g_t$  can be computed using the algorithm in Figure 1, which simply reorganizes the computation of recurrence formula (8) as a recursion.

To speed up this algorithm, define polynomial  $F(x) = \sum_{k=0}^{r-l} k f_k x^k$ ,  $G(x) = \sum_{j=0}^{m-l} g_{j+l} x^j$  and use FFT to compute  $H(x) = F(x)G(x)$  in  $O((r-l) \log(r-l))$  time after COMPUTE( $l, m$ ) returns. Then  $\sum_{j=l}^m (i-j) f_{i-j} g_j = [x^{i-l}]H(x)$ , and hence the **for** loop runs in  $O(r-m)$  time. The total running time is  $T(t) = 2T(t/2) + O(t \log t) = O(t \log^2 t)$ . ◀

### 3 Main algorithm

Recall that we are given  $n$  positive integers  $s_1, \dots, s_n$  and a target sum  $t$ . Consider the generating function  $A(x)$  defined by

$$A(x) = \prod_{i=1}^n (1 + x^{s_i}). \quad (9)$$

The number of subsets that sum up to  $t$  is  $[x^t]A(x)$ . The SUBSET SUM instance has a solution if and only if  $[x^t]A(x) \neq 0$ .

► **Lemma 3.** Suppose  $[x^t]A(x) \neq 0$ . Let  $p$  be a uniform random prime from  $[t+1, (n+t)^3]$ . With probability  $1 - O((n+t)^{-1})$ ,  $p$  does not divide  $[x^t]A(x)$ .

**Proof.** Notice that  $[x^t]A(x) \leq 2^n$ , so it has at most  $n$  prime factors. Since there are  $\Omega((n+t)^2)$  primes in the interval, the probability that  $p$  divides  $[x^t]A(x)$  is  $O((n+t)^{-1})$ . ◀

► **Lemma 4.** Let  $B(x) = \ln(A(x)) \in \mathbb{Q}[[x]]$ . For prime  $p > t$ , in  $\tilde{O}(t)$  time one can compute  $([x^r]B(x)) \bmod p$  for all  $0 \leq r \leq t$ .

**Proof.** By definition of  $B(x)$ ,

$$B(x) = \ln \left( \prod_{i=1}^n (1 + x^{s_i}) \right) = \sum_{i=1}^n \ln(1 + x^{s_i}) = \sum_{i=1}^n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} x^{s_i j}. \quad (10)$$

Let  $a_k$  be the size of the set  $\{j : s_j = k\}$ , and define polynomial

$$B_t(x) = \sum_{i=1}^n \sum_{j=1}^{\lfloor t/s_i \rfloor} \frac{(-1)^{j-1}}{j} x^{s_i j} = \sum_{k=1}^t \sum_{j=1}^{\lfloor t/k \rfloor} \frac{a_k (-1)^{j-1}}{j} x^{jk}. \quad (11)$$

Then  $[x^r]B_t(x) = [x^r]B(x)$  for all  $0 \leq r \leq t$ .

Note that the denominators  $j$  in (11) do not have prime factor  $p$ . After preparing the multiplicative inverses  $\bar{j}^{-1}$  for each  $1 \leq j \leq t$ , we can compute all  $([x^r]B_t(x)) \bmod p$  by simply iterating over  $k, j$  in equation (11), which only takes  $\sum_{k=1}^t \lfloor t/k \rfloor = O(t \log t)$  time. ◀

► **Lemma 5.** For prime  $p > t$ , one can compute  $([x^r]A(x)) \bmod p$  for all  $0 \leq r \leq t$  in  $\tilde{O}(t)$  time.

**Proof.** Let  $B(x) = \ln(A(x))$ . Then  $A(x) = \exp(B(x)) \equiv \exp_t(B_t(x)) \pmod{x^{t+1}}$ , where  $B_t(x) = \sum_{i=0}^t ([x^i]B(x))x^i$ . We use Lemma 4 to compute  $B_t(x)$ 's image  $\overline{B}_t(x) \in \mathbb{F}_p[x]$ , and then use Lemma 2 to compute the first  $t+1$  terms of  $\overline{\exp}_t(\overline{B}_t(x))$ , which give the values of  $([x^r]A(x)) \bmod p$  for all  $0 \leq r \leq t$ . ◀

► **Theorem 6.** The SUBSET SUM problem can be solved in time  $\tilde{O}(n+t)$  by a randomized algorithm with one-sided error probability  $O((n+t)^{-1})$ .

**Proof.** By sampling and using Miller-Rabin primality test [5, Section 31.8], we can pick a uniform random prime  $p$  from interval  $[t+1, (n+t)^3]$  in  $(\log(n+t))^{O(1)}$  time with  $O((n+t)^{-1})$  failure probability. Then the theorem immediately follows from Lemma 3 and Lemma 5. ◀

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