

Quantum Error-Correcting Codes for Qudit Amplitude Damping

Markus Grassl, Linghang Kong, Zhaohui Wei, Zhang-Qi Yin, Bei Zeng

Abstract

Traditional quantum error-correcting codes are designed for the depolarizing channel modeled by generalized Pauli errors occurring with equal probability. Amplitude damping channels model, in general, the decay process of a multilevel atom or energy dissipation of a bosonic system at zero temperature. We discuss quantum error-correcting codes adapted to amplitude damping channels for higher dimensional systems (qudits). For multi-level atoms, we consider a natural kind of decay process, and for bosonic systems, we consider the qudit amplitude damping channel obtained by truncating the Fock basis of the bosonic modes to a certain maximum occupation number. We construct families of single-error-correcting quantum codes that can be used for both cases. Our codes have larger code dimensions than the previously known single-error-correcting codes of the same lengths. Additionally, we present families of multi-error correcting codes for these two channels, as well as generalizations of our construction technique to error-correcting codes for the qutrit V and Λ channels.

Index Terms

amplitude damping channel, quantum codes, qudit

I. INTRODUCTION

For a q -level quantum system with Hilbert space \mathbb{C}^q , called a qudit, the most general physical operations (or quantum channels) allowed by quantum mechanics are completely positive, trace preserving linear maps which can be represented in the following Kraus decomposition form $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$, where the matrices E_k are called Kraus operators of the quantum channel \mathcal{N} satisfying the trace-preserving condition $\sum_k E_k^\dagger E_k = \mathbf{1}$.

In designing error-correcting codes for protecting messages carried by n qudits sent through a channel \mathcal{N} , it is usually assumed that the errors to be corrected are completely random, with no knowledge other than that they affect different qudits independently [16, 30]. The corresponding channel \mathcal{N} is the depolarizing channel which can

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be modeled by uniformly distributed error-operators given by generalized Pauli operators [17, 25, 31] $(X_q)^a(Z_q)^b$, for $a, b \in \{0, 1, \dots, q-1\}$, where $X_q|s\rangle = |s+1 \bmod q\rangle$, $Z_q|s\rangle = \omega^s|s\rangle$, and $\omega = \exp(2\pi i/q)$. When it is clear from the context, we may just write X and Z , dropping the index q .

However, if further information about the error process is available, more efficient codes can be designed. Indeed, in many physical systems, the noise is likely to be unbalanced between amplitude (X -type) errors and phase (Z -type) errors. Recently a lot of attention has been put into designing codes for this situation and into studying their fault tolerance properties [1, 12, 15, 20, 34, 40]. All these results use error models described by Kraus operators that are generalized Pauli operators, but for those error models, the X -type errors (i.e., non-diagonal Pauli matrices) happen with probability p_x which might be different from the probability p_z that Z -type errors (i.e., diagonal Pauli matrices) happen. The quantum channels described by this kind of noise are called asymmetric channels.

A closer look at the real physical process of amplitude damping noise shows that one needs to go even further, beyond Kraus operators of Pauli type. To be more precise, for $q = 2$, the qubit amplitude damping (AD) channel is given by the Kraus operators [9]

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (1)$$

Since the error model of the qubit AD channel is not described by Pauli-type Kraus operators, the known techniques dealing with Pauli errors result in codes with non-optimal parameters. Several new techniques for the construction of codes adapted to this type of noise with non-Pauli Kraus operators, and the qubit AD channel in particular, have been developed [9, 15, 27, 28, 36]. Systematic methods for constructing high performance single-error-correcting codes [27, 36] and multi-error-correcting codes [11] have been found.

In this paper, we discuss constructions of quantum codes for AD channels of general qudit systems. Unlike the qubit case, where the AD channel is unique, for qudit systems there are different AD channels associated with different physical systems. We will focus on two different models: multi-level atoms with a natural kind of decay process, and bosonic systems obtained by truncating the Fock basis of the bosonic modes to the maximum occupation number $q-1$ for a single bosonic mode.

II. THE AMPLITUDE DAMPING CHANNEL

For two-level atoms, the decay process at zero temperature is described by the Kraus operators A_0, A_1 as given in Eq. (1). For multi-level atoms, there are different kinds of decay processes at zero temperature. One natural decay process is the cascade structure Ξ , where the decay process is governed by the master equation [32, 35]

$$\frac{d\rho}{d\tau} = \sum_{1 \leq i \leq q-1} k_i (2\sigma_{i-1,i}^- \rho \sigma_{i-1,i}^+ - \sigma_{i-1,i}^+ \sigma_{i-1,i}^- \rho - \rho \sigma_{i-1,i}^+ \sigma_{i-1,i}^-). \quad (2)$$

Here $\{|i\rangle\}_{i=0}^{q-1}$ is a basis of the Hilbert space \mathbb{C}^q , $\sigma_{i-1,i}^- = |i-1\rangle\langle i|$ and $\sigma_{i-1,i}^+ = |i\rangle\langle i-1|$.

The solution to this master equation gives the Kraus expression

$$\Xi(\rho) = A_0 \rho A_0^\dagger + \sum_{0 \leq i < j \leq q-1} A_{ij} \rho A_{ij}^\dagger, \quad (3)$$

where $A_{ij} = \sqrt{\gamma_{ij}}|i\rangle\langle j|$ with positive coefficients γ_{ij} , and A_0 is a diagonal matrix with its elements determined by $A_0^\dagger A_0 + \sum_{0 \leq i < j \leq q-1} A_{ij}^\dagger A_{ij} = I$. Furthermore, when the decay time τ is small, γ_{ij} is of order τ^ℓ for any $j = i + \ell$, $\ell > 0$. As a consequence, A_0 is of order τ , and A_{ij} is of order $\tau^{\ell/2}$ for any $j = i + \ell$, $\ell > 0$. This is intuitively sound as for the cascade structure, the first order transition always happens from $|i+1\rangle$ to $|i\rangle$.

As an example, for three-level atoms, i.e., $q = 3$, we have

$$\begin{aligned} A_{01} &= \sqrt{\gamma_{01}}|0\rangle\langle 1|, \\ A_{02} &= \sqrt{\gamma_{02}}|0\rangle\langle 2|, \quad A_{12} = \sqrt{\gamma_{12}}|1\rangle\langle 2|, \\ A_0 &= |0\rangle\langle 0| + \sqrt{1 - \gamma_{01}}|1\rangle\langle 1| + \sqrt{1 - \gamma_{02} - \gamma_{12}}|2\rangle\langle 2|, \end{aligned}$$

where

$$\begin{aligned} \gamma_{01} &= 2k_2\tau + O(\tau^2), \\ \gamma_{02} &= 2k_1k_2\tau^2 + O(\tau^3), \\ \gamma_{12} &= 2k_1\tau + O(\tau^2), \end{aligned}$$

for $k_1 \neq k_2$. The values of γ_{ij} are slightly different for $k_1 = k_2$, but the order of γ_{ij} in terms of τ remains the same.

The channel \mathcal{A} describing energy dissipation of a bosonic system at zero temperature is discussed in [9]. The Kraus operators are given by

$$A_k = \sum_{r=k}^{q-1} \sqrt{\binom{r}{k}} \sqrt{(1-\gamma)^{r-k} \gamma^k} |r-k\rangle\langle r|, \quad (4)$$

where $q-1$ is the maximum occupation number of a single bosonic mode, and $k = 0, 1, \dots, q-1$. The parameter γ is of first order in terms of the decay time τ , i.e., $\gamma = c\tau + O(\tau^2)$. As a consequence, the non-identity part of A_0 is of order τ , and A_k is of order $\tau^{k/2}$ for $1 \leq k \leq d-1$.

For instance, for the qubit case, i.e., $q = 2$, we have the qubit amplitude channel given by Eq. (1). For $q = 3$, we have

$$\begin{aligned} A_0 &= |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1| + (1-\gamma)|2\rangle\langle 2|, \\ A_1 &= \sqrt{\gamma}|0\rangle\langle 1| + \sqrt{2\gamma(1-\gamma)}|1\rangle\langle 2|, \\ \text{and } A_2 &= \gamma|0\rangle\langle 2|. \end{aligned}$$

Note that for $q = 3$, the non-diagonal Kraus operators of the channel \mathcal{A} for bosonic systems are linear combinations of the Kraus operators of the channel Ξ . Hence codes correcting errors of the channel Ξ are also codes for the channel \mathcal{A} .

III. ERROR CORRECTION CRITERIA

A quantum error-correcting code Q is a subspace of $(\mathbb{C}^q)^{\otimes n}$, the space of n qudits. For a K -dimensional code space spanned by the orthonormal basis $|\psi_i\rangle$, $i = 1, \dots, K$, and a set of errors \mathcal{E} , there is a physical operation

correcting all elements $E_k \in \mathcal{E}$ if the error correction conditions [2, 26] are satisfied:

$$\forall i, j, k, l: \langle \psi_i | E_k^\dagger E_l | \psi_j \rangle = \delta_{ij} \alpha_{kl}, \quad (5)$$

where α_{kl} depends only on k and l . A code is said to be pure with respect to some set of errors \mathcal{E} if $\alpha_{kl} = 0$ for $k \neq l$. A K -dimensional code with length n is denoted by $((n, K))$.

For the AD channels Ξ and \mathcal{A} , if the decay time τ is small, we would like to correct the leading order errors that occur during amplitude damping [3, 4]. Similar as for the qubit case [16, Section 8.7], we will show below that in order to improve the fidelity of the transmission through the AD channel Ξ or \mathcal{A} from $1 - O(\tau)$ to $1 - O(\tau^{t+1})$, i.e., to correct t errors, it is sufficient to be able to detect t errors of type A_0 and to correct up to t errors of type A_{ij} with $j > i$ of total order $\tau^{t/2}$ for the channel Ξ (or to correct up to t errors of type A_i with $i > 0$ for the channel \mathcal{A}). We will then say that such a code corrects t amplitude damping errors since it improves the fidelity, just as much as a true t -error-correcting code would for the same channel. This is a direct consequence of the following sufficient condition for approximate error correction.

Theorem 1: Assume we are given a quantum channel with Kraus operators E_k that have a series expansion in terms of $\sqrt{\tau}$ for a parameter τ . A quantum code \mathcal{Q} with orthonormal basis $\{|c_i\rangle : i = 1, \dots, K\}$ corrects errors up to order $O(\tau^t)$ if the following conditions are fulfilled for all basis states $|c_i\rangle, |c_j\rangle$ and all error operators E_k, E_l :

$$\langle c_i | E_k^\dagger E_l | c_j \rangle = \delta_{ij} \nu_{kl} + O(\tau^{t+1}) \quad (6)$$

Proof: Assume we are given a quantum channel with Kraus operators $E_k(\tau)$ that depend on some small parameter τ . We expand the operators in terms of $\sqrt{\tau}$ as

$$E_k(\tau) = \sum_{m \geq 0} E_{km} \tau^{m/2}. \quad (7)$$

This leads to the following description of the channel:

$$\rho \mapsto \sum_k E_k \rho E_k^\dagger = \sum_k \sum_{m \geq 0} \sum_{\mu \geq 0} E_{km} \rho E_{k\mu}^\dagger \tau^{(m+\mu)/2}. \quad (8)$$

Given a quantum code \mathcal{Q} with orthonormal basis $\{|c_i\rangle : i = 1, \dots, K\}$, the conditions for perfect error correction are

$$\langle c_i | E_k^\dagger(\tau) E_l(\tau) | c_j \rangle = \delta_{ij} \alpha_{kl}. \quad (9)$$

Using the expansion of the Kraus operators in terms of $\sqrt{\tau}$, we get

$$\sum_{m, \mu \geq 0} \langle c_i | E_{km}^\dagger E_{l\mu} | c_j \rangle \tau^{(m+\mu)/2} = \delta_{ij} \alpha_{kl}. \quad (10)$$

We are looking for sufficient conditions such that the residual error with respect to ρ is of order $O(\tau^{t+1})$ for some $t > 0$.

We write each Kraus operator $E_k = B_k + C_k + D_k$ as sum of three terms, where

$$B_k = \sum_{m=0}^t E_{km} \tau^{m/2}, \quad (11)$$

$$C_k = \sum_{m=t+1}^{2t} E_{km} \tau^{m/2}, \quad (12)$$

$$D_k = \sum_{m>2t} E_{km} \tau^{m/2}. \quad (13)$$

Then the original channel \mathcal{N} can be written as

$$\mathcal{N}(\rho) = \sum_k (B_k + C_k + D_k) \rho (B_k + C_k + D_k)^\dagger \quad (14)$$

$$= \sum_k B_k \rho B_k^\dagger \quad (15)$$

$$+ \sum_k B_k \rho C_k^\dagger + C_k \rho B_k^\dagger \quad (16)$$

$$+ \sum_k B_k \rho D_k^\dagger + D_k \rho B_k^\dagger \quad (17)$$

$$+ \sum_k (C_k + D_k) \rho (C_k + D_k)^\dagger. \quad (18)$$

Condition (6) implies that

$$\langle c_i | B_k^\dagger B_l | c_j \rangle = \delta_{ij} \lambda_{kl}, \quad (19)$$

$$\langle c_i | B_k^\dagger C_l | c_j \rangle = \delta_{ij} \mu_{kl} + O(\tau^{t+1}). \quad (20)$$

In particular, the error operators B_k can be perfectly corrected. We first define the projection operator onto one of the spaces¹ spanned by $\{B_k | c_i\rangle : i = 1, \dots, K\}$:

$$P_{B_k} = \sum_i B_k | c_i \rangle \langle c_i | B_k^\dagger, \quad (21)$$

and the partial isometry that maps $B_k | c_i \rangle$ to $| c_i \rangle$:

$$U_{B_k} = \sum_i | c_i \rangle \langle c_i | B_k^\dagger. \quad (22)$$

We compute

$$U_{B_k} P_{B_k} = \left(\sum_i | c_i \rangle \langle c_i | B_k^\dagger \right) \left(\sum_j B_k | c_j \rangle \langle c_j | B_k^\dagger \right) \quad (23)$$

$$= \sum_{i,j} | c_i \rangle \langle c_i | B_k^\dagger B_k | c_j \rangle \langle c_j | B_k^\dagger \quad (24)$$

$$= \lambda_{kk} \sum_i | c_i \rangle \langle c_i | B_k^\dagger. \quad (25)$$

The last step follows from the fact that the error operators B_k can be perfectly corrected, which also determines

¹We may, without loss of generality, use linear combinations of the original error operators B_k such that these spaces become mutually orthogonal.

the constant λ_{kk} . Then the partial correction operator $\mathcal{R}_{\text{part}}$ is given by

$$\mathcal{R}_{\text{part}}(\rho) = \sum_k |\lambda_{kk}|^2 \sum_{i,j} |c_i\rangle\langle c_i| B_k^\dagger \rho B_k |c_j\rangle\langle c_j|. \quad (26)$$

For a general state

$$\rho_Q = \sum_{i,j} \alpha_{ij} |c_i\rangle\langle c_j| \quad (27)$$

in the quantum code Q , the term (15) of $\mathcal{N}(\rho_Q)$ reads

$$\mathcal{N}_{B_k}(\rho_Q) = \sum_k \sum_{i,j} \alpha_{ij} B_k |c_i\rangle\langle c_j| B_k^\dagger. \quad (28)$$

Since the error operators B_k can be perfectly corrected (implied by Eq. (19)), it can be shown that applying the partial recovery operator to $\mathcal{N}_{B_k}(\rho_Q)$ yields a state $\lambda\rho_Q$ that is proportional to the original state ρ_Q . Hence after partial recovery we have

$$\mathcal{R}_{\text{part}}(\mathcal{N}(\rho_Q)) = \lambda\rho_Q + \mathcal{S}(\rho_Q), \quad (29)$$

where the map \mathcal{S} is given by the application of the partial recovery operator to the terms given in (16), (17) and (18). The summands (17) and (18) are all of order $O(\tau^{t+1})$, so we can ignore them, but (16) contains terms of order $\tau^{l/2}$ for $t < l \leq 2t$. Applying the partial recovery operator to (16) and using (20) results in the state

$$\begin{aligned} & \mathcal{R}_{\text{part}} \left(\sum_k B_k \rho_Q C_k^\dagger + C_k \rho_Q B_k^\dagger \right) \\ &= \sum_k |\lambda_{kk}|^2 \sum_{i,j} |c_i\rangle\langle c_i| B_k^\dagger \left(\sum_l B_l \rho_Q C_l^\dagger + C_l \rho_Q B_l^\dagger \right) B_k |c_j\rangle\langle c_j| \\ &= \sum_{k,l} |\lambda_{kk}|^2 \sum_{i,j} \sum_{i',j'} \alpha_{i',j'} \left(|c_i\rangle\langle c_i| \underbrace{B_k^\dagger B_l}_{\lambda_{kl} \delta_{ii'}} |c_{i'}\rangle\langle c_{j'}| C_l^\dagger B_k |c_j\rangle\langle c_j| + |c_i\rangle\langle c_i| B_k^\dagger C_l |c_{i'}\rangle\langle c_{j'}| \underbrace{B_l^\dagger B_k}_{\lambda_{lk} \delta_{jj'}} |c_j\rangle\langle c_j| \right) \\ &= \sum_{k,l} |\lambda_{kk}|^2 \sum_{i',j'} \alpha_{i',j'} \left(\sum_j \lambda_{kl} \underbrace{\langle c_{j'}| C_l^\dagger B_k |c_j\rangle}_{\delta_{jj'} \mu_{kl}^* + O(\tau^{t+1})} |c_{i'}\rangle\langle c_j| + \sum_i \lambda_{lk} \underbrace{\langle c_i| B_k^\dagger C_l |c_{i'}\rangle}_{\delta_{ii'} \mu_{kl} + O(\tau^{t+1})} |c_i\rangle\langle c_{j'}| \right) \\ &= \rho_Q \sum_{k,l} |\lambda_{kk}|^2 (\lambda_{kl} \mu_{kl}^* + \lambda_{kl}^* \mu_{kl}) + O(\tau^{t+1}) \end{aligned} \quad (30)$$

which is, up to order $O(\tau^{t+1})$, proportional to the original state. ■

Note that in the proof of Theorem 1, we have split the error-operators accordingly based on their expansion (7) in terms of $\sqrt{\tau}$, see (11)–(13). Clearly, the high order parts D_k can be completely ignored. Only the errors B_k of approximately half the final order have to be corrected (19), while the errors C_k have to obey some kind of error detection criterion (20).

IV. STABILIZER AND ASYMMETRIC QUANTUM CODES

Before presenting our construction of quantum codes tailored to amplitude damping channels, we investigate the performance of traditional quantum error-correcting codes on these channels.

Stabilizer codes are a large class of quantum codes which contain many good quantum codes [16, 30]. A stabilizer code \mathcal{Q} with n qudits encoding k qudits has distance d if all errors of weight at most $d-1$ can be detected or have no effect on \mathcal{Q} , and we denote the parameters of \mathcal{Q} by $[[n, k, d]]_q$. Obviously a stabilizer code of distance $2t+1$ corrects t AD error as it corrects t arbitrary errors.

Calderbank-Shor-Steane (CSS) codes [6, 37] are a subclass of the stabilizer codes. It has been shown that CSS codes can be used to construct codes for the binary AD channel [16, Section 8.7]. The construction is based on so-called asymmetric quantum codes, which have a direct generalization to the qudit case [13]. The following theorem shows that those asymmetric CSS codes can also be used to obtain error correcting codes for qudit AD channels.

Theorem 2: An $[[n, k]]_q$ CSS code Q with pure X -distance $2t+1$ and pure Z -distance t corrects t AD error, i.e., errors up to order $O(\tau^t)$.

Proof: The generalized Pauli operators $X_q^k Z_q^l$ form a basis for all operators on a single qudit. Hence we can expand the error operators A_i in terms of tensor products of the generalized Pauli operators. The diagonal error operator A_0 of AD channels can be expanded in terms of the error operators Z_q^l , with the expansion coefficients of the operators Z_q^l , $l > 0$ being of first order in τ . The diagonal of the other error operators A_{ij} or A_i is zero. They can be expanded in terms of operators $Z_q^l X_q^k$, $k \neq 0$, with the expansion coefficients being of order $\sqrt{\tau}$.

Note that for CSS codes, X and Z errors can be corrected independently. The error operators B_k defined in (11) of the proof of Theorem 1 are of order at most $t/2$ in τ , and hence they contain no more than t X -errors and no more than $t/2$ Z -errors. As the code Q has X -distance $2t+1$ and Z -distance $t+1$, the error operators B_k can be corrected. Similarly, for the error operators C_k defined in (12), there are no more than $2t$ X -errors and no more than t Z -errors, which can be detected using Q . Hence, using Theorem 1, it follows that Q corrects all errors up to order $O(\tau^t)$. ■

V. CLASSICAL ASYMMETRIC CODES

In this section, we construct quantum codes correcting a single AD error using classical asymmetric codes. Codes for the qubit case have been presented in [27, 36]. Those codes are self-complementary, i.e., the basis states are of the form $|\psi_{\mathbf{u}}\rangle = \frac{1}{\sqrt{2}}(|\mathbf{u}\rangle + |\bar{\mathbf{u}}\rangle)$, where \mathbf{u} is an n -bit string, $\bar{\mathbf{u}} = \mathbf{1} \oplus \mathbf{u}$, and $\mathbf{1}$ is the all-one string.

For the non-binary case with $q > 2$, we consider a similar construction. Define $\bar{X} = X_q^{\otimes n}$, then the basis states are chosen as

$$|\psi_{\mathbf{u}}\rangle = \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} \bar{X}^l |\mathbf{u}\rangle. \quad (31)$$

For instance, for $q = 3$ and $n = 3$, we get $|\psi_{\mathbf{0}}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$.

The quantum code \mathcal{Q} is then spanned by $\{|\psi_{\mathbf{u}}\rangle\}$, where $\mathbf{u} \in \tilde{C}$ is some length- n string over the alphabet $\{0, 1, \dots, q-1\}$ (\tilde{C} is a classical code of length n). The advantage of this construction is that the code automatically satisfies the error-detection condition for a single Z_q^l error ($l = 1, 2, \dots, q-1$), as the code is stabilized by \bar{X} . Now consider a classical code with codewords $C = \{\mathbf{u} + \alpha\mathbf{1} : \mathbf{u} \in \tilde{C}, \alpha = 0, \dots, q-1\}$ and the corresponding quantum code spanned by $\{|\psi_{\mathbf{u}}\rangle : \mathbf{u} \in \tilde{C}\}$. The problem of correcting a single error for the qudit AD channels can then be reduced to finding certain classical codes.

The relevant classical channel is the classical asymmetric channel [23]. Let the alphabet be \mathbb{Z}_q with the ordering $0 < 1 < 2 < \dots < q-1$. A channel is called asymmetric if any transmitted symbol a is received as $b \leq a$. The mostly studied asymmetric channel, dating back to Varshamov [38], can be described by the following asymmetric distance $\Delta(\mathbf{x}, \mathbf{y})$.

Definition 3 (see [21]): Let $B = \{0, 1, \dots, q-1\} \subset \mathbb{Z}$. For $\mathbf{x}, \mathbf{y} \in B^n$, we define

- 1) $w(\mathbf{x}) := \sum_{i=1}^n x_i$.
- 2) $N(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n \max\{y_i - x_i, 0\}$.
- 3) $\Delta(\mathbf{x}, \mathbf{y}) := \max\{N(\mathbf{x}, \mathbf{y}), N(\mathbf{y}, \mathbf{x})\}$.

If \mathbf{x} is sent and \mathbf{y} is received, we say that $w(\mathbf{x} - \mathbf{y})$ errors have occurred (note that $x_i \geq y_i$ and hence each summand in $w(\mathbf{x} - \mathbf{y})$ is nonnegative). A code correcting t -errors is called a t -code.

Theorem 4 (see [21]): A code $C \subset B^n$ corrects r errors of the asymmetrical channel if and only if $\Delta(\mathbf{x}, \mathbf{y}) > r$ for all $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \neq \mathbf{y}$.

Our goal is to link these classical asymmetric codes to quantum AD codes. As discussed above, we start from the following definition.

Definition 5: A classical code C over the alphabet B is called self-complementary if for any $\mathbf{x} \in C$, $\mathbf{1} \oplus \mathbf{x} \in C$.

For any self-complementary code C , there exists another code \tilde{C} such that $C = \{\mathbf{u} + \alpha\mathbf{1} : \mathbf{u} \in \tilde{C}, \alpha = 0, \dots, q-1\}$ and $|C| = q|\tilde{C}|$. We may, for example, choose all $\mathbf{u} \in \tilde{C}$ such that the first digit is 0. From \tilde{C} we derive the quantum code \mathcal{Q} spanned by $\{|\psi_{\mathbf{u}}\rangle : \mathbf{u} \in \tilde{C}\}$ as given in Eq. (31). Our main result is given by the following theorem.

Theorem 6: If C is a classical (linear or non-linear) self-complementary code correcting a single error with respect to Definition 3, then \mathcal{Q} spanned by $\{|\psi_{\mathbf{u}}\rangle : \mathbf{u} \in \tilde{C}\}$ is a single-error-correcting code for the qudit AD channels Ξ and \mathcal{A} .

Proof: Let $E_{ij} = |i\rangle\langle j|$ with $i, j \in \{0, 1, \dots, q-1\}$ and $i < j$. For a small decay time τ , in order to improve the fidelity of the transmission through the qudit AD channel \mathcal{A} given by Eq. (4) from $1 - O(\tau)$ to $1 - O(\tau^2)$, it is sufficient to correct a single $E_{i,i+1}$ -error and detect one Z_q^l -error for $l = 1, 2, \dots, q-1$. The self-complementary form of $|\psi_{\mathbf{u}}\rangle$ given in Eq. (31) implies that $\bar{X}|\psi_{\mathbf{u}}\rangle = |\psi_{\mathbf{u}}\rangle$. In turn, this implies that $\langle \psi_{\mathbf{v}} | Z_q^l | \psi_{\mathbf{u}} \rangle = 0$ for any \mathbf{u}, \mathbf{v} and $l = 1, 2, \dots, q-1$, i.e., the error-detection condition for a single Z_q^l error is fulfilled.

Next consider a single operator $E_{i,i+1}$. Every state of the quantum code is a linear combination of states $|\mathbf{c}\rangle$ with $\mathbf{c} \in C$. Applying the operator $E_{i,i+1}$ to $|\mathbf{c}\rangle$ corresponds to a single asymmetric error. As the classical code C corrects a single asymmetric error, the distance $\Delta(\mathbf{u}, \mathbf{v})$ between any two codewords \mathbf{u} and \mathbf{v} is at least two. Therefore, the supports (set of basis states with non-zero coefficient in the superposition) of the states $|\psi_{\mathbf{u}}\rangle$ and

$E_{i,i+1}^{(\alpha)}|\psi_{\mathbf{v}}\rangle$ are disjoint for all positions α , where $E_{i,i+1}^{(\alpha)}$ denotes the operator $E_{i,i+1}$ acting at position α . Hence those states are mutually orthogonal. Finally note that for errors $E_{i,i+1}$ acting on the same position, the operator $E_{i,i+1}^\dagger E_{i,i+1}$ is diagonal and hence in the span of the operators Z_q^l , which can be detected. ■

Corollary 7: If there exists an $(n, K, 3)_q$ self-complementary code C , then there exists an $((n, K/q))_q$ quantum code correcting a single AD error.

Such codes have, e. g., been studied in [14]. For linear codes, we have the following corollary.

Corollary 8: If there exists an $[n, k + 1, 3]_q$ linear code C containing the all-one-vector $\mathbf{1} \in C$, then there exists an $[[n, k]]_q$ CSS code correcting a single AD error.

In the preceding corollaries we have used the notation $(n, K, d)_q$ for a classical code of length n with K codewords and minimum distance d over an alphabet with q elements, and the notation $[n, k, d]_q = (n, K = q^k, d)$ when the code is linear.

VI. SINGLE-ERROR-CORRECTING CODES: EVEN LENGTHS

We now use Theorem 6 to construct some families of good single-error-correcting AD codes. For this, we need to find some good self-complementary single-error-correcting classical asymmetric codes. The best known direct construction of single-error-correcting codes for the binary asymmetric channel is the so-called Varshamov-Tenengolts (VT)-Constantin-Rao (CR) code [10, 39], with a natural generalization to $q > 2$. These VT-CR codes are non-linear codes, in both the binary and non-binary cases.

For the binary case, many of these VT-CR codes are indeed self-complementary, and so they can be used to construct families of good single-error-correcting quantum AD codes [36]. As the VT-CR codes are nonlinear, the corresponding quantum codes are nonadditive codes. Unfortunately, for the non-binary case, the VT-CR codes are no longer self-complementary, so one needs some other constructions of good single-error-correcting for asymmetric channels.

We will use the idea of generalized concatenation, which has been discussed in the context of constructing binary AD codes in [36], and in the context of constructing (classical) asymmetric codes in [18]. This method will allow us to construct good self-complementary asymmetric linear codes for the non-binary case, which will lead to good single-error-correcting quantum codes for AD channels.

A. Qutrit Codes

First, we consider the case of $q = 3$. For the generalized concatenation construction, we choose the outer code as some ternary classical code over the alphabet $\{\tilde{0}, \tilde{1}, \tilde{2}\}$, and the inner codes as:

$$C_{\tilde{0}} = \{00, 11, 22\}, C_{\tilde{1}} = \{01, 12, 20\}, C_{\tilde{2}} = \{02, 10, 21\}. \quad (32)$$

Then we have the following result.

Theorem 9: For n even, generalized concatenation with an outer $[n/2, k, 3]_3$ code results in an $[n, n/2 + k]_3$ self-complementary linear code C . This code leads to an $[[n, n/2 + k - 1]]_3$ quantum stabilizer code \mathcal{Q} , correcting a single error for the channels Ξ and \mathcal{A} .

Proof: Note that $C_{\tilde{0}}$, $C_{\tilde{1}}$, and $C_{\tilde{2}}$ are all self-complementary codes correcting a single asymmetric error. Therefore, any outer ternary code will lead to a self-complementary ternary code C , and hence a quantum code \mathcal{Q} . A single amplitude damping error induces only a single error with respect to $\tilde{0}$, $\tilde{1}$, $\tilde{2}$. As the outer ternary code has distance 3, such an error can be corrected. ■

Note that with respect to the symbols $\tilde{0}$, $\tilde{1}$, $\tilde{2}$, the induced channel \mathcal{R}_3 is nothing but the ternary symmetric channel shown in Fig. 1.

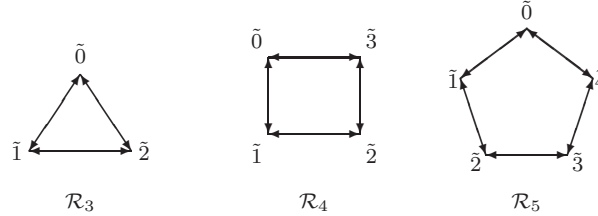


Fig. 1. The induced channel \mathcal{R}_3 for $q = 3$ (which is just the ternary symmetric channel), the induced channel \mathcal{R}_4 for $q = 4$, and the induced channel \mathcal{R}_5 for $q = 5$. The arrows indicate the possible transitions between symbols.

Example 10: For $n = 6$, take the outer code of length $n/2 = 3$ as $\{\tilde{0}\tilde{0}\tilde{0}, \tilde{1}\tilde{1}\tilde{1}, \tilde{2}\tilde{2}\tilde{2}\}$ with distance 3. Generalized concatenation yields a self-complementary ternary linear code of dimension 4. The corresponding quantum code \mathcal{Q} encodes $6/2 + 1 - 1 = 3$ qutrits. Both the best corresponding single-error-correcting quantum code $[[6, 2, 3]]_3$ and the best possible asymmetric CSS code $[[6, 2, \{3, 2\}]_3$ (see Corollary 8) encode only 2 qutrits.

B. The Case $q > 3$

For $q = 4$, we choose the inner codes as

$$\begin{aligned} C_{\tilde{0}} &= \{00, 11, 22, 33\}, & C_{\tilde{1}} &= \{01, 12, 23, 30\}, \\ C_{\tilde{2}} &= \{02, 13, 20, 31\}, & C_{\tilde{3}} &= \{03, 10, 21, 32\}. \end{aligned} \quad (33)$$

Similar as in Theorem 9, an outer code with distance three yields a self-complementary code from which a quantum AD code can be derived. However, in this case, the induced channel for the outer code is no longer symmetric. A single damping error will, for example, never map a codeword of the inner code $C_{\tilde{0}}$ to a codeword of $C_{\tilde{2}}$. So on the level of the outer code, there are no transitions between $\tilde{0}$ and $\tilde{2}$, or between $\tilde{1}$ and $\tilde{3}$. The induced quaternary channel \mathcal{R}_4 is shown in Fig. 1, where we see that errors only happen between ‘neighbors.’

The above constructions for $q = 3, 4$ have a direct generalization to general $q > 2$. For a given q , choose the outer code as some code over the alphabet $\{\tilde{0}, \tilde{1}, \dots, \widetilde{q-1}\}$. The q inner codes $C_{\tilde{0}}, C_{\tilde{1}}, \dots, C_{\widetilde{q-1}}$ are the double-repetition code $C_{\tilde{0}} = \{00, 11, \dots, (q-1)(q-1)\}$ and all its $q-1$ cosets $C_{\tilde{i}} = C_{\tilde{0}} \oplus (0i)$, i.e., we apply the rule that $0i \in C_{\tilde{i}}$. It is straightforward to check that each inner code has asymmetric distance 2, hence corrects a single asymmetric error. Similar as in the case of $q = 4$, a single damping error will only drive transitions between \tilde{i}, \tilde{j} for $\tilde{i} = \tilde{j} \pm \tilde{1}$. For instance, for $q = 5$, the induced channel \mathcal{R}_5 is shown in Fig. 1. In general, we will write the induced channel as \mathcal{R}_q for outer codes over $\{\tilde{0}, \tilde{1}, \dots, \widetilde{q-1}\}$.

Similar as Theorem 9, in general we have the following theorem.

Theorem 11: For n even, an outer $[n/2, k]_q$ code correcting a single error for the channel \mathcal{R}_q leads to an $[n, n/2 + k]_q$ self-complementary linear code C and hence an $[[n, n/2 + k - 1]]_q$ quantum code \mathcal{Q} , correcting a single error for the qudit AD channels Ξ and \mathcal{A} .

Note that the channel \mathcal{R}_q is no longer a symmetric channel, so outer codes of Hamming distance 3 are no longer expected to give the best codes. It turns out, however, that single-error-correcting codes for the channel \mathcal{R}_q are equivalent to single-symmetric-error correcting codes in Lee metric [5] (see also [24]), for which optimal linear codes are known (for a more detailed discussion, see [18]).

VII. SINGLE-ERROR-CORRECTING CODES: ODD LENGTHS

The construction of AD codes for even lengths given in Sec. VI based on generalized concatenation is relatively straightforward. The inner codes are just 1-codes of length 2 with q codewords and their cosets. In [18], codes of odd length were obtained using a mixed-alphabet code, treating one position differently. This does not directly translate to the situation considered here, as the resulting code has to be self-complementary.

Instead, we will use different inner codes, one of odd lengths and the length-two code from above. In particular, we can directly search for q mutually disjoint inner codes of length 3 which are 1-codes.

For $q = 4$, consider the following \mathbb{Z}_4 -linear code $C_{0'}$ of length 3 generated by $\{111, 002, 020\}$:

$$\begin{array}{r} 000 \ 111 \ 222 \ 333 \quad 002 \ 113 \ 220 \ 331 \\ 020 \ 131 \ 202 \ 313 \quad 022 \ 133 \ 200 \ 311. \end{array} \quad (34)$$

The code $C_{0'}$ has asymmetric distance 2, as well as the three cosets $C_{1'} = C_{0'} + 001$, $C_{2'} = C_{0'} + 010$, and $C_{3'} = C_{0'} + 100$. Applying generalized concatenation to the outer code $\{\tilde{0}\tilde{0}0', \tilde{1}\tilde{1}1', \tilde{2}\tilde{2}2', \tilde{3}\tilde{3}3'\}$ and the inner codes of length 2 and 3 for the first two and the third position, respectively, yields a self-complementary 1-code $[7, 5]_4$. The corresponding quantum code has parameters $[[7, 4]]_4$.

Note that the induced channel on the alphabet $\{0', 1', 2', 3'\}$ is no longer \mathcal{R}_4 , but the symmetric channel over \mathbb{Z}_4 . Therefore we have the following theorem for $q = 4$.

Theorem 12: For n odd, an outer $[(n-1)/2, k, 3]_4$ code leads to an $[n, (n+1)/2 + k]_4$ self-complementary linear 1-code C . The resulting quantum code $\mathcal{Q} = [[n, (n-1)/2 + k]]_4$ corrects a single error for the qudit AD channels Ξ and \mathcal{A} .

Proof: The inner codes $C_{\tilde{i}}$ of length two as well as the inner codes $C_{i'}$ of length three are self-complementary 1-codes. The outer code has distance 3 which ensures that a single error mixing the inner codes can be corrected. For the outer code, we always take the last coordinate to be of type s' , and all the other coordinates to be of type \tilde{s} , for $s = 0, \dots, 3$. Hence, the inner code for the last coordinate of the outer code has length three, while the other inner codes have length two. Therefore, for an outer $[(n-1)/2, k, 3]_4$ linear code, generalized concatenation yields an $[n, (n+1)/2 + k]_4$ self-complementary linear 1-code C , corresponding to an $[[n, (n-1)/2 + k]]_4$ quantum code. ■

We emphasize that the construction related to Theorem 12 is valid only for $q = 4$. For $q > 5$, the \mathbb{Z}_q -linear code

C_0 generated by $\{111, 013\}$ and its q cosets are all self-complementary codes with asymmetric distance 2. For this, note that $\Delta(\mathbf{x}, \mathbf{y}) = 1$ if and only if, up to permutation, $\mathbf{x} - \mathbf{y} \in \{(1, 0, 0), (1, -1, 0)\}$. For $q > 5$, the code C_0 does not contain such a vector. Hence we obtain the analogue result as in Theorem 12 for $q > 5$.

For $q = 3$ and $q = 5$, however, we cannot partition the trivial code $[3, 3]_q$ into q self-complementary codes $[3, 2]_q$ with asymmetric distance 2, substantiated by exhaustive search. However, we can use an inner code of length five, resulting in the following theorem.

Theorem 13: For $q = 3, 5$ and n odd, an outer $[(n-3)/2, k]_q$ code correcting a single error for the symmetric channel leads to an $[n, (n+1)/2 + k]_q$ self-complementary code \mathcal{C} and hence an $[[n, (n-1)/2 + k]_q$ quantum code \mathcal{Q} , correcting a single error for the qudit AD channel \mathcal{A} for $q = 3, 5$.

Proof: We can map the first digit of the outer code to q groups of codes of length 5. Again by exhaustive search, we find that for $q = 3, 5$, we cannot partition $[5, 5]_q$ into q self-complementary codes $[5, 4]_q$ with asymmetric distance 2, but we can get q codes $[5, 3]_q$. For $q = 3$, the code $C_{0'}$ is generated by $\{00011, 01201, 11111\}$, while for $q = 5$ it is generated by $\{00011, 00102, 11111\}$.

Then our construction can be described as follows. For an $[(n-3)/2, k]_q$ outer code, we use the length-5 code described above as the inner code for the first digit, and use length-2 code for the remaining $(n-5)/2$ digits, leading to a code with length $1 \times 5 + \frac{n-5}{2} \times 2 = n$. Similar as in the proof of Theorem 12, it follows that the resulting code corrects a single AD error. ■

In the nonlinear case, we can find larger codes. The results are summarized in Theorems 14, 15, and 16.

Theorem 14: For $q = 3$ and n odd, an outer $((n-3)/2, K, 3)_3$ code leads to an $(n, 33 \times 3^{(n-5)/2}K)_3$ self-complementary code. The resulting $((n, 11 \times 3^{(n-5)/2}K)_3$ quantum code corrects one error for the AD channels Ξ and \mathcal{A} .

Proof: An exhaustive search reveals that for $q = 3$, we can find three disjoint self-complementary codes of length 5 and asymmetric distance 2 with at most 33 codewords. Let

$$\begin{aligned} \tilde{C}_{0'} = \{00000, 00011, 00112, 00220, 01021, 01110, \\ 01202, 02022, 02101, 02120, 02211\} \end{aligned} \quad (35)$$

Then $C_{0'} = \{\mathbf{u} + \alpha \mathbf{1} \mid \mathbf{u} \in \tilde{C}_{0'}, \alpha = 0, 1, 2\}$, $C_{1'} = \{\mathbf{u} + 00001 \mid \mathbf{u} \in C_{0'}\}$, $C_{2'} = \{\mathbf{u} + 00002 \mid \mathbf{u} \in C_{0'}\}$. We construct the code similarly as in Theorem 13, i.e., the inner code for the first digit is $C_{i'}$, and for the remaining $(n-5)/2$ digits the inner code is $C_{\bar{i}}$. ■

For $q = 5$, we consider two constructions based on nonlinear codes.

Theorem 15: Then for $q = 5$ and n odd, an outer $((n-1)/2, K, 3)_5$ code leads to an $(n, 20 \times 5^{(n-3)/2}K)_5$ self-complementary code. The resulting $((n, 4 \times 5^{(n-3)/2}K)_5$ quantum code corrects one error for the AD channel \mathcal{A} .

Proof: Although we cannot partition $[3, 3]_5$, a weaker result can be obtained. Let

$$\begin{aligned}\tilde{C}_{0'} &= \{000, 002, 020, 022\}, & \tilde{C}_{1'} &= \{001, 004, 021, 024\}, \\ \tilde{C}_{2'} &= \{003, 011, 031, 033\}, & \tilde{C}_{3'} &= \{010, 023, 041, 043\}, \\ \tilde{C}_{4'} &= \{012, 014, 032, 034\}.\end{aligned}$$

Furthermore, set $C_{i'} = \{\mathbf{u} + \alpha\mathbf{1} \mid \mathbf{u} \in \tilde{C}_{i'}, \alpha = 0, 1, 2, 3, 4\}$.

Then we have a construction similar to Theorem 12, i.e., we use one copy of the length-3 inner code and $(n-3)/2$ copies of the length-2 inner code. ■

Theorem 16: For $q = 5$ and n odd, an outer $((n-3)/2, K, 3)_5$ code leads to an $(n, 295 \times 5^{(n-5)/2}K)_5$ self-complementary code. The resulting $((n, 59 \times 5^{(n-5)/2}K))_5$ quantum code can correct one error for the AD channel \mathcal{A} .

Proof: In this construction we use self-complementary codes of length 5. By non-exhaustive search, we can find a self-complementary code with 295 codewords. The 59 codewords of \tilde{C}'_0 are shown in Table I. From those we derive $C_{0'} = \{\mathbf{u} + \alpha\mathbf{1} \mid \mathbf{u} \in \tilde{C}'_0, \alpha = 0, 1, 2, 3, 4\}$ and $C_{i'} = \{\mathbf{u} + 0000i \mid \mathbf{u} \in C_{0'}\}$, $1 \leq i < 5$. Then we have

TABLE I
CODE CONSTRUCTION FOR $q = 5$

| | | | | |
|-------|-------|-------|-------|-------|
| 00000 | 00202 | 01241 | 02200 | 03110 |
| 00002 | 00220 | 01404 | 02203 | 03212 |
| 00013 | 00223 | 01412 | 02211 | 03231 |
| 00020 | 00244 | 02000 | 02223 | 03233 |
| 00031 | 00303 | 02002 | 02314 | 03300 |
| 00033 | 00311 | 02013 | 02321 | 03303 |
| 00044 | 00314 | 02021 | 02332 | 03342 |
| 00111 | 00330 | 02032 | 02424 | 03410 |
| 00114 | 00332 | 02034 | 02440 | 03412 |
| 00122 | 00424 | 02114 | 03041 | 03431 |
| 00141 | 00442 | 02130 | 03044 | 04234 |
| 00200 | 01133 | 02143 | 03102 | |

a construction similar to that in Theorem 13. ■

As shown in Table II, for many lengths, the construction based on Theorems 9, 13, and 14 outperforms both the best known quantum codes with distance 3, and the CSS codes of Corollary 8. The dimension of the asymmetric quantum codes (AQECC) is taken from [14]. Only when $n = 13$, our construction performs worse than CSS and AQECC. This might be due to the fact that the outer code we use here is the CSS code $[[5, 1, \{3, 2\}]]$, which is not efficient since we also have CSS code $[[4, 1, \{3, 2\}]]$.

VIII. MULTI-ERROR-CORRECTING CODES

For the binary case, multi-error-correcting amplitude damping codes are discussed in [11]. The basic idea is that for the encoding $|0_L\rangle = |01\rangle$, $|1_L\rangle = |10\rangle$, the amplitude damping channel simulates a binary erasure channel.

TABLE II

DIMENSION OF SINGLE-ERROR-CORRECTING QUANTUM AD CODES FROM THE $GF(3^2)$ CONSTRUCTION WITH DISTANCE 3, THE CSS CONSTRUCTION, ASYMMETRIC QUANTUM CODES (AQECC), AND THE GENERALIZED CONCATENATION CONSTRUCTION (GC).

| n | $GF(3^2)$ | CSS | AQECC | GC (linear) | GC (nonlinear) |
|-----|-----------|----------|----------|-------------|-----------------|
| 4 | 3^0 | 3^0 | 1 | 3 | 3 |
| 5 | 3^1 | 3^1 | 6 | 3^2 | 11 |
| 6 | 3^2 | 3^2 | 11 | 3^3 | 3^3 |
| 7 | 3^3 | 3^3 | 29 | 3^3 | 11×3^1 |
| 8 | 3^4 | 3^4 | 84 | 3^5 | 3^5 |
| 9 | 3^5 | 3^5 | 3^5 | 3^5 | 11×3^3 |
| 10 | 3^6 | 3^6 | 3^6 | 3^6 | 3^6 |
| 11 | 3^6 | 3^7 | 3^7 | 3^7 | 11×3^5 |
| 12 | 3^7 | 3^8 | 3^8 | 3^8 | 3^8 |
| 13 | 3^8 | 3^9 | 3^9 | 3^8 | 11×3^6 |
| 14 | 3^9 | 3^9 | 3^9 | 3^{10} | 3^{10} |
| 15 | 3^{10} | 3^{10} | 3^{10} | 3^{10} | 11×3^8 |
| 16 | 3^{11} | 3^{11} | 3^{11} | 3^{12} | 3^{12} |

So one can use erasure-correcting code as outer codes to build codes correcting amplitude damping errors. In this section we consider generalizations of this construction, for both the binary and non-binary cases.

It is mentioned in [11] that a possible generalization is to use $|001\rangle, |010\rangle, |100\rangle$ as the inner code, and a distance $t + 1$ quantum code as an outer code. However, it turns out that one can actually use $|001\rangle, |010\rangle, |100\rangle, |111\rangle$ as the inner code, and a distance $t + 1$ quantum code as an outer code. Here a single damping error will be treated as an erasure, and two damping errors or no damping can be treated as an error which is taken care of by the outer code.

To generalize this idea to the case $q > 2$, one can take a similar approach. In this section we consider the channel \mathcal{A} with Kraus operators given by Eq. (4). For $q = 3$, one can take the encoding $|0_L\rangle = |11\rangle, |1_L\rangle = |02\rangle, |2_L\rangle = |20\rangle$. Then the amplitude damping channel simulates a ternary erasure channel. So one can use erasure-correcting codes as outer codes to build codes correcting amplitude damping errors. Actually, using a similar idea as the construction based on $|001\rangle, |010\rangle, |100\rangle, |111\rangle$ for the binary case, one can use $|0_L\rangle = |00\rangle, |1_L\rangle = |20\rangle, |2_L\rangle = |11\rangle, |3_L\rangle = |02\rangle, \text{ and } |4_L\rangle = |22\rangle$ as the inner code, and still a distance $t + 1$ quantum code as an outer code. Here we give the general construction.

Assume the length of the inner code is m . We choose the set

$$S = \{|a_1 a_2 \dots a_m\rangle \mid a_1 + a_2 + \dots + a_m \text{ is even}\} \quad (36)$$

as the orthonormal basis of the code, and let

$$K = |S|. \quad (37)$$

If we have an outer code $[[n, k, t + 1]]_K$, we get an $[[nm, K^k]]_q$ code correcting t errors for \mathcal{A} . For $q = 3$, the code also corrects errors for the qutrit channel Ξ . We have the following theorem:

Theorem 17: The code constructed above corrects t errors for the channels \mathcal{A} and Ξ .

Proof: We want to prove that the condition in Theorem 1 holds. Let \mathcal{Q} be the outer $[[n, k, t + 1]]_K$ code. Take any two error operators E_ℓ and $E_{\ell'}$, and for any two vectors $|\psi_i\rangle$ and $|\psi_j\rangle$, we consider $\langle \psi_i | E_\ell^\dagger E_{\ell'} | \psi_j \rangle$. Suppose that on some inner code, E_ℓ has an odd number of errors while $E_{\ell'}$ has an even number of errors. Then $E_{\ell'} |\psi_j\rangle$ will be in the space spanned by S while $E_\ell |\psi_i\rangle$ will be in the space perpendicular to S . So $\langle \psi_i | E_\ell^\dagger E_{\ell'} | \psi_j \rangle = 0$ and the condition in Theorem 1 automatically holds. By symmetry, this argument also holds when E_ℓ is even and $E_{\ell'}$ is odd. Now assume that on each inner code, the number of errors corresponding to E_ℓ and $E_{\ell'}$ have the same parity. We consider the series expansion of the operator $E_\ell^\dagger E_{\ell'}$ with respect to τ . For this, we expand the tensor factor acting on the i -th particle such that A_{ij} corresponds to the term of order τ^j . Combined, we get

$$\begin{aligned} E_\ell^\dagger E_{\ell'} &= (A_{10} + A_{11}\tau + A_{12}\tau^2 + \dots) \\ &\otimes (A_{20} + A_{21}\tau + A_{22}\tau^2 + \dots) \\ &\otimes \dots \otimes (A_{m0} + A_{m1}\tau + A_{m2}\tau^2 + \dots) \end{aligned} \quad (38)$$

Note that each of $A_{10}, A_{20}, \dots, A_{m0}$ is either the identity operator or the zero operator, which can be proved if we take the limit $\tau \rightarrow 0$. For the terms of the form $\Omega(\tau^t)$, there are at most t non-identity terms on the inner codes, which will become 0 according to the error-detection criterion. Thus the condition in Theorem 1 is satisfied. ■

Finally we give an explicit expression for the dimension of the code K defined in (37) in terms of q and m .

Theorem 18:

$$K = \begin{cases} q^m/2 & \text{if } q \text{ is even.} \\ (q^m + 1)/2 & \text{if } q \text{ is odd.} \end{cases} \quad (39)$$

Proof: When q is even, we map $|a_1 a_2 \dots a_m\rangle$ to $|q - 1 - a_1, a_2 \dots a_m\rangle$, which is a one-to-one mapping from S to \bar{S} . When q is odd, let $i = \min\{j: a_j \neq 0\}$, and we map $|a_1 a_2 \dots a_m\rangle$ to $|a_1 \dots a_{i-1}, q - a_i, a_{i+1} \dots a_m\rangle$, which is a one-to-one mapping from $S \setminus \{|00 \dots 0\rangle\}$ to \bar{S} . So $|S| = |\bar{S}|$ when q is even, and $|S| = |\bar{S}| + 1$ when q is odd. Thus $|S| = q^m/2$ when q is even, and $|S| = (q^m + 1)/2$ when q is odd. ■

In Table III, we compare our codes with both stabilizer codes and AQECCs. We use the construction with $q = 3$ and $m = 2$, and compare the codes that corrects t errors. From $[[n/2, k, t + 1]]_5$ codes given (see also [29]), we can construct our $((n, K))_3$ codes where $K = 5^k$, and we compare them with stabilizer codes $[[n, k', 2t + 1]]_3$ that have dimension $K' = 3^{k'}$. It can be seen that our construction outperforms stabilizer codes, and our performance gets better in comparison for larger t . In most cases, our codes are better also better than the asymmetric CSS codes. Some of the CSS codes are optimal codes taken from [13], and others are based on the best known classical ternary codes [19]. For comparison, we also list an upper bound k''_{\max} on the maximal dimension of an AQECC based on the known bounds for classical codes. In most cases, this bound cannot be achieved.

TABLE III

DIMENSION OF QUANTUM CODES FROM OUR CONSTRUCTION THAT CORRECT t AD ERRORS COMPARED WITH STABILIZER CODES $[[n, k', 2t + 1]]_3$ AND PURE ASYMMETRIC CSS CODES WITH PARAMETERS $[[n, k'', \{2t + 1, t + 1\}]_3$.

| t | n | K | $\log_3 K$ | k' | k'' | k''_{\max} |
|-----|-----|-------|------------|------|-------------------------|--------------|
| 2 | 10 | 5 | 1.465 | 1 | $[[10, 1, \{5, 3\}]_3$ | 2 |
| 2 | 12 | 25 | 2.930 | 2 | $[[12, 3, \{5, 3\}]_3$ | 3 |
| 2 | 14 | 125 | 4.395 | 4 | $[[14, 4, \{5, 3\}]_3$ | 4 |
| 2 | 16 | 625 | 5.860 | 5 | $[[16, 5, \{5, 3\}]_3$ | 5 |
| 2 | 18 | 3125 | 7.325 | 6 | $[[18, 7, \{5, 3\}]_3$ | 7 |
| 2 | 20 | 15625 | 8.790 | 8 | $[[20, 9, \{5, 3\}]_3$ | 9 |
| 3 | 14 | 5 | 1.465 | N/A | $[[14, 0, \{7, 4\}]_3$ | 0 |
| 3 | 16 | 25 | 2.930 | 0 | $[[16, 1, \{7, 4\}]_3$ | 1 |
| 3 | 18 | 125 | 4.395 | 1 | $[[18, 3, \{7, 4\}]_3$ | 3 |
| 3 | 20 | 625 | 5.860 | 3 | $[[20, 4, \{7, 4\}]_3$ | 5 |
| 3 | 22 | 3125 | 7.325 | 4 | $[[22, 6, \{7, 4\}]_3$ | 6 |
| 3 | 24 | 15625 | 8.790 | 6 | $[[24, 6, \{7, 4\}]_3$ | 8 |
| 4 | 18 | 5 | 1.465 | N/A | N/A | – |
| 4 | 20 | 25 | 2.930 | N/A | $[[20, 0, \{9, 5\}]_3$ | 1 |
| 4 | 24 | 625 | 5.860 | 1 | $[[24, 4, \{9, 5\}]_3$ | 5 |
| 4 | 26 | 3125 | 7.325 | 2 | $[[26, 4, \{9, 5\}]_3$ | 6 |
| 4 | 28 | 15625 | 8.790 | 3 | $[[28, 5, \{9, 5\}]_3$ | 8 |
| 5 | 26 | 5 | 1.465 | N/A | $[[26, 0, \{13, 6\}]_3$ | 3 |
| 5 | 28 | 25 | 2.930 | 0 | $[[28, 1, \{11, 6\}]_3$ | 5 |
| 5 | 30 | 125 | 4.395 | 1 | $[[30, 2, \{11, 6\}]_3$ | 6 |
| 5 | 32 | 625 | 5.860 | 1 | $[[32, 4, \{11, 6\}]_3$ | 7 |
| 5 | 38 | 3125 | 7.325 | 4 | $[[38, 7, \{11, 6\}]_3$ | 12 |
| 5 | 40 | 15625 | 8.790 | 6 | $[[40, 8, \{12, 6\}]_3$ | 14 |
| 6 | 30 | 5 | 1.465 | N/A | ? | 0 |
| 6 | 36 | 25 | 2.930 | 0 | ? | 6 |
| 6 | 38 | 125 | 4.395 | 1 | $[[38, 2, \{13, 7\}]_3$ | 8 |
| 6 | 40 | 625 | 5.860 | 1 | $[[40, 3, \{13, 7\}]_3$ | 10 |
| 6 | 42 | 3125 | 7.325 | 2 | $[[42, 5, \{13, 7\}]_3$ | 11 |
| 6 | 44 | 15625 | 8.790 | 2 | $[[44, 6, \{13, 7\}]_3$ | 13 |

IX. APPLICATIONS TO OTHER AD CHANNELS

Our method can also be applied to some other amplitude damping processes. For instance, when $q = 3$, in addition to the channel Ξ , there are the following natural decay processes of three-level atoms: the Λ -pattern or the V pattern. They are illustrated together with the Ξ channel in Fig.2:

For the Λ -pattern, the master equation is

$$\begin{aligned} \frac{d\rho}{d\tau} = & k_1(2\sigma_{12}^- \rho \sigma_{12}^+ - \sigma_{12}^+ \sigma_{12}^- \rho - \rho \sigma_{12}^+ \sigma_{12}^-) \\ & + k_2(2\sigma_{02}^- \rho \sigma_{02}^+ - \sigma_{02}^+ \sigma_{02}^- \rho - \rho \sigma_{02}^+ \sigma_{02}^-), \end{aligned}$$

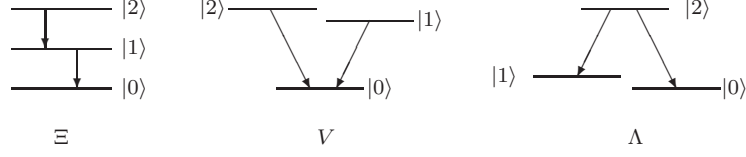


Fig. 2. Decay processes for three-level atoms with different level-structures.

where σ_{12}^- and σ_{12}^+ are the same as above, and

$$\sigma_{02}^- = |0\rangle\langle 2|, \quad \sigma_{02}^+ = |2\rangle\langle 0|. \quad (40)$$

By direct calculation, one can verify that the evolution of this master equation can be expressed by using the following Kraus operators:

$$\rho(\tau) = \sum_{i=0}^2 A_i \rho_0 A_i^\dagger, \quad (41)$$

where

$$\begin{aligned} A_0 &= \text{diag}\{1, 1, \sqrt{1 - \gamma_1 - \gamma_2}\}, \\ A_1 &= \sqrt{\gamma_1}|0\rangle\langle 2|, \\ A_2 &= \sqrt{\gamma_2}|1\rangle\langle 2|, \end{aligned} \quad (42)$$

and

$$\begin{aligned} \gamma_1 &= \frac{k_2}{k_1 + k_2} \left[1 - e^{-2(k_1 + k_2)\tau} \right] = 2k_2\tau + O(\tau^2), \\ \gamma_2 &= \frac{k_1}{k_1 + k_2} \left[1 - e^{-2(k_1 + k_2)\tau} \right] = 2k_1\tau + O(\tau^2). \end{aligned}$$

Both γ_1 and γ_2 are of first order in τ .

For the V -pattern, the Kraus expression has been found in [8], which is given as

$$\rho(\tau) = \sum_{i=0}^2 A_i \rho_0 A_i^\dagger, \quad (43)$$

where

$$\begin{aligned} A_0 &= \text{diag}\{1, \sqrt{1 - \gamma_1}, \sqrt{1 - \gamma_2}\}, \\ A_1 &= \sqrt{\gamma_1}|0\rangle\langle 1|, \\ A_2 &= \sqrt{\gamma_2}|0\rangle\langle 2|, \end{aligned}$$

and

$$\begin{aligned} \gamma_1 &= 1 - e^{-2k_1\tau} = 2k_1\tau + O(\tau^2), \\ \gamma_2 &= 1 - e^{-2k_2\tau} = 2k_2\tau + O(\tau^2). \end{aligned}$$

Again, both γ_1 and γ_2 are of first order τ .

We also introduce the classical counterparts of these two channels which are shown in Fig. 3. Here the arrows

indicate the allowed transitions. The classical channels \mathcal{L}_1 and \mathcal{L}_2 correspond to the amplitude damping channels V and Λ , respectively.

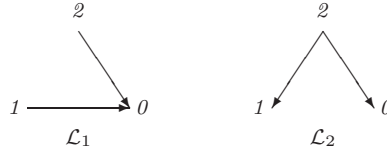


Fig. 3. Classical channels for trits: the arrows indicate the allowed transitions.

For the V -channel and the Λ -channel, we construct codes using similar ideas as shown above. For single-error-correcting codes, we can still use the idea of self-complementary codes based on the corresponding classical codes, which have been studied in [33] and [22]. Unfortunately, none of these constructions can be adapted to make self-complementary codes. For codes with short lengths, one can use numerical search.

For multi-error-correcting codes, we want to choose the basis of the inner code so that one single damping error will project the state to an orthogonal subspace. Let T_0 and T_1 be the set of binary strings of length m with even and odd parity, respectively:

$$T_i = \{x_1 x_2 \dots x_m \in \{0, 1\}^m \mid x_1 \oplus x_2 \oplus \dots \oplus x_m = i\}. \quad (44)$$

For the channel \mathcal{L}_1 , a binary 0 is mapped to the channel symbols 1 and 2, and a binary 1 is mapped to 0. For the channel \mathcal{L}_2 , a binary 0 is mapped to the channel symbols 0 and 1, and a binary 1 is mapped to 2. Then T_0 and T_1 are mapped to the set of codes S_0 and S_1 which are two sets of possible inner codes.

Example 19: For $m = 2$, we have

$$T_0 = \{00, 11\}, \quad T_1 = \{01, 10\}. \quad (45)$$

For the channel \mathcal{L}_1 , we get

$$S_0 = \{00, 11, 12, 21, 22\}, \quad S_1 = \{01, 02, 10, 20\}, \quad (46)$$

and for the channel \mathcal{L}_2 channel:

$$S_0 = \{00, 01, 10, 11, 22\}, \quad S_1 = \{01, 02, 10, 20\} \quad (47)$$

We have the following theorem.

Theorem 20: For any codeword $\mathbf{w} \in S_i$, $i = 0, 1$, when an error of the corresponding channel occurs, the resulting string \mathbf{v} will be in S_{1-i} .

Proof: We only prove the case when the channel is \mathcal{L}_1 , the proof for the channel \mathcal{L}_2 follows using the same argument. Let $\mathbf{w} = w_1 w_2 \dots w_m$ and $\mathbf{v} = v_1 v_2 \dots v_m$, and let the error occur at position t . Then w_t is either 1 or 2, while $v_t = 0$, and $w_k = v_k$, for all $k \neq t$. Suppose that in our construction, $w_1 \dots w_m$ corresponds to the binary string $a_1 \dots a_m$, and $v_1 \dots v_m$ corresponds to the binary string $b_1 \dots b_m$, then we have $a_t = 0$, $b_t = 1$, and $a_k = b_k$, for $k \neq t$. Since $\mathbf{a} \in T_i$, we know that $\mathbf{b} \in T_{1-i}$, and hence $\mathbf{v} \in S_{1-i}$. ■

Corollary 21: For any codeword $\mathbf{w} \in S_i$, $i = 0, 1$, when an odd number of errors of the corresponding channel

occur, the resulting string \mathbf{v} will be in S_{1-i} ; when an even number of errors of the corresponding channel occur, the resulting string \mathbf{v} will be in S_i .

With arguments similar as those in the proof of Theorem 17, for $i = 0, 1$, if we use the quantum code \mathcal{Q}_i spanned by $\{|\mathbf{u}\rangle : \mathbf{u} \in S_i\}$ as the inner code and an $[[n, k, t+1]]_{|S_i|}$ code as the outer code, we get an $[[nm, K^k]]$ code that corrects t errors for the corresponding quantum AD channel.

Finally we will give an explicit expression for $|S_i|$ in terms of m . To show its dependency on m , we write $\alpha_m = |S_0|$ and $\beta_m = |S_1|$.

Theorem 22:

$$\alpha_m = |S_0| = \frac{1}{2}(3^m + 1) \quad \text{and} \quad \beta_m = |S_1| = \frac{1}{2}(3^m - 1). \quad (48)$$

Proof: For any string $\mathbf{a} = a_1 a_2 \dots a_m \in T_0$, either $a_m = 0$ or $a_m = 1$. For $a_m = 0$, the strings in S_0 that \mathbf{a} maps to are just the strings $a_1 \dots a_{m-1}$ maps to, concatenated with a single symbol chosen from two options ($1, 2$ for \mathcal{L}_1 , and $0, 1$ for \mathcal{L}_2). For $a_m = 1$, the strings \mathbf{a} maps to are those that $a_1 \dots a_{m-1}$ maps to, concatenated with a fixed symbol. So we have the recurrence relation

$$\alpha_m = 2\alpha_{m-1} + \beta_{m-1}. \quad (49)$$

We also have

$$\alpha_m + \beta_m = 3^m. \quad (50)$$

Solving Eqs. (49) and (50) together with the initial condition $\alpha_1 = 1$ and $\beta_1 = 2$, we have

$$\alpha_m = \frac{1}{2}(3^m + 1) \quad \text{and} \quad \beta_m = \frac{1}{2}(3^m - 1), \quad (51)$$

which proves the theorem. ■

To achieve the maximal size of the constructed code, we should always choose $\{|\mathbf{u}\rangle : \mathbf{u} \in S_0\}$ as the inner code.

Example 23: We take $m = 2$, and the outer code is $[[5, 1, 3]]_5$ (see, e. g., [7]), which is

$$|k\rangle \mapsto \frac{1}{5\sqrt{5}} \sum_{p,q,r=0}^4 \omega^{k(p+q+r)+pr} |p+q+k\rangle \otimes |p+r\rangle \otimes |q+r\rangle \otimes |p\rangle \otimes |q\rangle, \quad (52)$$

where $\omega = \exp(2\pi i/5)$. We substitute $|0\rangle, |1\rangle, \dots, |4\rangle$ with $|00\rangle, |11\rangle, |12\rangle, |21\rangle$, and $|22\rangle$, respectively, and get a $((10, 5))_3$ code which corrects 2 errors of the channel V . If we substitute $|0\rangle, |1\rangle, \dots, |4\rangle$ with $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, and $|22\rangle$, we get a $((10, 5))_3$ 2-code for the channel Λ . In comparison, the best stabilizer code of length 10 which corrects 2 errors is $[[10, 1, 5]]_3$.

Other examples for which our construction outperforms stabilizer codes include the $((12, 25))_3$ and $((14, 125))_3$ 2-codes constructed from outer codes $[[6, 2, 3]]_5$ and $[[7, 3, 3]]_5$, while the best corresponding stabilizer codes are $[[12, 2, 5]]_3$ and $[[14, 4, 5]]_3$.

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