Computational Complexity of the Negative Decision Number of Graphs*

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Abstract

Let $G = (V, E)$ be a graph. A function $f : V \to \{-1, 1\}$ is called a *bad function* of G if $\sum_{u \in N_G(v)} f(u) \le 1$ for each $v \in V$, where $N_G(v)$ is the set of neighbors of v in G. The *negative decision number* of G, introduced by Wang, is the maximum value of $\sum_{v \in V} f(v)$ taken over all bad functions of G. In this paper, we comprehensively study the negative decision number from algorithmic, complexity, and graph-theoretic points of view. Our main results are as follows.

1. We prove that it is $\mathcal{NP}$-hard to compute the negative decision number of a given graph, even if the graph is bipartite. Moreover, it is $\mathcal{NP}$-complete to decide whether the negative decision number of a given bipartite graph is at least k, where k is *any* fixed integer (not necessarily positive). On the other hand, we show that the negative decision number can be computed in polynomial time for several special classes of graphs including trees.
2. For a below-upper-bound formulation of the problem of computing the negative decision number, we show an asymptotically tight approximation threshold of $\Theta(\log |V|)$. Specifically, it can be approximated within a factor of $O(\log |V|)$ in polynomial time, but cannot be approximated better than $c \log |V|$ for some constant $c > 0$ unless $\mathcal{NP} \subseteq DTIME(n^{O(\log \log n)})$.
3. The exact values of the negative decision number are determined for complete multipartite graphs, wheels, and fans.

1 Introduction

We generally follow the notation and terminologies of Diestel [6] in this paper. Let $G = (V, E)$ be a (simple and undirected) graph. For each $v \in V$, $N_G(v) = \{u \mid \{u, v\} \in E\}$ is the *open neighborhood* of v, $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v, and $d_G(v) = |N_G(v)|$ is the *degree* of v. If the graph G is clear in the context, we also write $N(v), N[v], d(v)$ instead of $N_G(v), N_G[v], d_G(v)$, respectively. Given a function $f : V \to \mathbb{R}$ and a subset of vertices $S \subseteq V$, let $f(S) = \sum_{v \in S} f(v)$. The *weight* of f is $f(V) = \sum_{v \in V} f(v)$.

A function $f : V \to \{-1, 1\}$ is called a *bad function* of G if $f(N(v)) \le 1$ for each $v \in V$. The *negative decision number* of G, denoted by $\beta_D(G)$, is the maximum weight of a bad function of G. The concepts of bad function and negative decision number are introduced by Wang [18],

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where some sharp bounds are proved and the exact values of this parameter for some special graph classes, including cliques, paths, and cycles, are determined. Some variants of the negative decision number have also been studied subsequently, including the negative k-subdecision number [10] and the lower negative decision number [19].

The negative decision number can be used to model the minimum number of "negative votes" in a social network that can force every individual in the network to have a "negative opinion" under certain rules [18]. It can also be regarded as the "dual" of another well-studied graph parameter called the *signed total domination number*, which is defined as the minimum weight of a function $f : V \to \{-1, 1\}$ that satisfies $f(N(v)) \geq 1$ for all $v \in V$ [20]. The *signed domination number* is similarly defined, with the condition $f(N(v)) \geq 1$ replaced by $f(N[v]) \geq 1$ [7]. Both the signed domination number and signed total domination number have been extensively studied in the literature (see e.g. [3, 7, 8, 14, 20, 21] and the references therein). The reader is also referred to the two excellent books by Hayes, Hedetniemi, and Slater [12, 13] for a comprehensive treatment on the theory of domination in graphs.

In this paper, we comprehensively investigate the negative decision number from the algorithmic, complexity, and graph-theoretic points of view. Our main results include the following:

1. We prove that it is $\mathcal{NP}$-hard to compute the negative decision number of a given graph, even if the graph is bipartite. Moreover, it is even $\mathcal{NP}$-complete to decide whether the negative decision number of a given bipartite graph is at least k, where k is *any* fixed integer (not necessarily positive). On the other hand, we show that the negative decision number can be computed in polynomial time for several special classes of graphs including trees.

2. We study approximation algorithms for computing the negative decision number of a given graph. Since this number can be negative, the normal concept of approximation cannot be directly applied here. We consider the *below-upper-bound approximation* of this problem, i.e., let the objective to be optimized be the difference between the weight of a bad function of G and the number of vertices in G, which is an obvious upper bound of $\beta_D(G)$. We obtain a tight approximation threshold of $\Theta(\log |V|)$ under the (standard) complexity assumption that $\mathcal{NP} \not\subseteq DTIME(n^{O(\log \log n)})$.

3. The exact values of the negative decision number are determined for complete multipartite graphs, fans, and wheels. Our result for complete multipartite graphs substantially generalizes those of complete graphs and bicliques given by Wang [18].

2 Complexity Issues of the Negative Decision Number

We first formally define the decision problem corresponding to the computation of the negative decision number of a graph as follows.

NEGATIVE DECISION NUMBER **(NDN)**
Instance: A graph $G = (V, E)$ and an integer k.
Question: Is $\beta_D(G) \geq k$?

2.1 $\mathcal{NP}$-completeness Result

In this subsection we show the $\mathcal{NP}$-completeness of the NEGATIVE DECISION NUMBER problem. The hardness of the following problem is needed in our proof.

VERTEX PACKING PROBLEM **(VPP)**

Instance: A graph $G = (V, E)$ and an integer k.

Question: Does there exist a subset of vertices $S \subseteq V$ such that $|S| \geq k$ and $|S \cap N_G(v)| \leq 1$ for each $v \in V$? Equivalently, can we choose at least k vertices such that for every vertex in V, at most one of its neighbors is chosen?

Proposition 1. ***VPP*** *is $\mathcal{NP}$-complete even on bipartite graphs.*

Proof. Clearly **VPP** is in $\mathcal{NP}$. We present a polynomial time reduction from the INDEPENDENT SET PROBLEM (**IS**), a classical $\mathcal{NP}$-complete problem [15]. Let $\{G = (V, E), k\}$ be an instance of **IS**. Construct a graph $G' = (V', E')$ as follows. Let $V' = X \cup Y \cup Z$, where $X = \{x_v \mid v \in V\}$, $Y = \{y_v \mid v \in V\}$, and $Z = \{z_e \mid e \in E\}$. Let $E' = \{x_v y_v \mid v \in V\} \cup \{x_v z_e \mid v \in e \in E\}$. It is easy to see that G' is bipartite. Let $k' = |V| + k$. We next show that the following two statements are equivalent, which will complete the reduction:

1. G has an independent set of size at least k.
2. There exists $S \subseteq V'$ such that $|S| \geq k'$ and $|S \cap N_{G'}(w)| \leq 1$ for each $w \in V'$.

(1) $\Rightarrow$ (2): Assume $T \subseteq V$ is an independent set in G with $|T| \geq k$. Define $S \subseteq V'$ as $S := Y \cup \{x_v \mid v \in T\}$. We have $|S| = |Y| + |T| \geq |V| + k = k'$. It is clear that for each $v \in X \cup Y$, at most one neighbor of v is in S. Let $z_e \in Z$ be an arbitrary vertex in Z, where $e = \{u, v\}$. Since T is an independent set in G, at most one of u and v is in T. Thus, at most one of x_u and x_v belongs to S, which proves that $|S \cap N_{G'}(w)| \leq 1$ for every $w \in V'$.

(2) $\Rightarrow$ (1): Suppose $S \subseteq V'$ satisfies that $|S| \geq k'$ and $|S \cap N_{G'}(w)| \leq 1$ for every $w \in V'$. We have $Y \cup Z \subseteq \bigcup_{x_v \in X} N_{G'}(x_v)$, i.e., each vertex in $Y \cup Z$ is the neighbor of some vertex in X. Therefore, $|S \cap (Y \cup Z)| \leq |S \cap \bigcup_{x_v \in X} N_{G'}(x_v)| \leq \sum_{x_v \in X} |S \cap N_{G'}(x_v)| \leq |X| = |Y|$. Now let $S' = (S \setminus Z) \cup Y$. We have $|S'| = |S \setminus (Y \cup Z)| + |Y| \geq |S \setminus (Y \cup Z)| + |S \cap (Y \cup Z)| = |S| \geq k'$ (note that $|S \cap (Y \cup Z)| \leq |Y|$). It is also clear that S' still satisfies $|S' \cap N_{G'}(w)| \leq 1$ for each $w \in V'$. Now take $T = \{v \in V \mid x_v \in S' \cap X\}$. If there exists some $e = \{u, v\} \in E$ with both endpoints in T, then both x_u and x_v are in S', indicating that $|S' \cap N_{G'}(z_e)| \geq 2$, a contradiction. Hence, T is an independent set of G, which has size at least $k' - |Y| = k' - |V| = k$. This finishes the reduction from **IS** to **VPP**, and concludes the $\mathcal{NP}$-completeness of the latter. $\square$

Theorem 1. NEGATIVE DECISION NUMBER *is $\mathcal{NP}$-complete, even on bipartite graphs.*

Proof. The problem is obviously in $\mathcal{NP}$. We will present a polynomial time reduction from **VPP** on bipartite graphs to it. Let $\{G = (V, E), k\}$ be an instance of **VPP**, where G is bipartite. Without loss of generality we may assume that G has no isolated vertex. We construct another graph $G' = (V', E')$ as follows. Let $V' = X \cup Y$, where $X = \{x_v \mid v \in V\}$ and $Y = \{y_{v,i} \mid v \in V; 1 \leq i \leq d_G(v) - 1\}$. Let $E' = \{x_u x_v \mid uv \in E\} \cup \{x_v y_{v,i} \mid v \in V; 1 \leq i \leq d_G(v) - 1\}$. It is easy to verify that G' is also bipartite. Let $k' = 2k + 2|E| - 2|V|$. We show that the following two statements are equivalent, which will complete our reduction:

1. There exists $S \subseteq V$ such that $|S| \geq k$ and $|S \cap N_G(w)| \leq 1$ for each $w \in V$.

2. There exists a bad function of G' of weight at least k'; that is, $\beta_D(G) \geq k'$.

(1) $\Rightarrow$ (2): Let $S \subseteq V$ be such that $|S| \geq k$ and $|S \cap N_G(w)| \leq 1$ for each $w \in V$. Define a function $f : V' \to \{-1, 1\}$ as follows: Let $f(v) = 1$ if $v \in Y \cup \{x_w \mid w \in S\}$, and $f(v) = -1$ if $v \in \{x_w \mid w \in V \setminus S\}$. The weight of f is

$$\begin{aligned}
& |Y| + |\{x_w \mid w \in S\}| - |\{x_w \mid w \in V \setminus S\}| \\
= & \sum_{w \in V} (d_G(w) - 1) + |S| - (|V| - |S|) \\
= & 2|S| + 2|E| - 2|V| \\
\geq & 2k + 2|E| - 2|V| \\
= & k'.
\end{aligned}$$

We next show that f is a bad function. Since each vertex in Y has only one neighbor in G', it is obvious that $f(N_{G'}(y)) \leq 1$ for every $y \in Y$. Now let x_w be an arbitrary vertex in X. By the definition of f and the property of S, we have

$$\begin{aligned}
f(N_{G'}(x_w)) &= (d_G(w) - 1) + |S \cap N_G(w)| - |N_G(w) \setminus S| \\
&= (d_G(w) - 1) + |S \cap N_G(w)| - (d_G(w) - |S \cap N_G(w)|) \\
&= 2|S \cap N_G(w)| - 1 \\
&\leq 1.
\end{aligned}$$

Therefore, f is a bad function of G of weight at least k'.

(2) $\Rightarrow$ (1): Let f be a bad function of G of weight at least k'. We first show that we can assume $f(y) = 1$ for every $y \in Y$. If this is not the case, there is some $y_{v,i} \in Y$ with $f(y_{v,i}) = -1$. We investigate the following two cases respectively:

- There exists $u \in N_G(v)$ such that $f(x_u) = 1$. In this case we swap the values of $f(y_{v,i})$ and $f(x_u)$; that is, we let $f(y_{v,i}) = 1$ and $f(x_u) = -1$. It is easy to see that this changing preserves the badness of f.

- $f(x_u) = -1$ for all $u \in N_G(v)$. In this case we change $f(y_{v,i})$ to 1. Since x_v already has $d_G(v)$ neighbors with function value -1 (the x_u's with $u \in N_G(v)$), it holds that $f(N_G(x_v)) \leq 1$ even if all $y_{v,i}$ vertices have value 1 under f (there are only $d_G(v) - 1$ such vertices due to our construction).

Therefore, we can assume without loss of generality that $f(y) = 1$ for all $y \in Y$. Define $S = \{v \in V \mid f(x_v) = 1\}$. We will prove that S satisfies the requirement in (1). The weight of f is

$$\begin{aligned}
& \sum_{y \in Y} f(y) + \sum_{v \in S} f(x_v) + \sum_{v \in V \setminus S} f(x_v) \\
= & \sum_{w \in V} (d_G(w) - 1) + |S| - (|V| - |S|) \\
= & 2|E| - 2|V| + 2|S|.
\end{aligned}$$

Since f is of weight at least k', we have $|S| \geq (k' + 2|V| - 2|E|)/2 = k$. It only remains to show that $|S \cap N_G(w)| \leq 1$ for each $w \in V$. Let w be an arbitrary vertex in V. As f is a bad function on G', we have

$$\begin{aligned}
1 &\geq f(N_{G'}(x_w)) \\
&= \sum_{1\leq i\leq d_G(w)-1} f(y_{w,i}) + \sum_{v\in N_G(w)} f(x_v) \\
&= (d_G(w)-1) + |\{v\in N_G(w) \mid f(x_v)=1\}| - |\{v\in N_G(w) \mid f(x_v)=-1\}| \\
&= (d_G(w)-1) + |S\cap N_G(w)| - (d_G(w) - |S\cap N_G(w)|) \\
&= 2|S\cap N_G(w)| - 1.
\end{aligned}$$

Hence, $|S\cap N_G(w)| \leq 1$.

We have proved the equivalence of (1) and (2), which completes the reduction and concludes the $\mathcal{NP}$-completeness of **NDN** on bipartite graphs. □

Corollary 1. *For any fixed integer k, it is $\mathcal{NP}$-complete to decide whether the negative decision number of a given bipartite graph is at least k.*

Proof. Let k be any fixed integer. By Theorem 1, deciding whether $\beta_D(G) \geq m$ is $\mathcal{NP}$-complete, where $G=(V,E)$ is an input bipartite graph, and m is an integer also given as input satisfying that $-|V| \leq m \leq |V|$. If $m \leq k$, we construct H by adding $k-m$ isolated vertices to G. It is easy to see that H is also bipartite, and $\beta_D(G) \geq m$ if and only if $\beta_D(H) \geq k$. If $m > k$, we construct H by adding m copies of C_6 on disjoint sets of vertices, together with $m+k$ isolated vertices, to G. Since C_6 is bipartite, the new graph H is also bipartite. Moreover, $\beta_D(C_6) = -2$ due to Theorem 5 in [18]. Thus, $\beta_D(G) \geq m$ if and only if $\beta_D(H) \geq m - 2m + (m+k) = k$. Hence Corollary 1 is proved. □

2.2 Approximation Behaviors

Since computing the negative decision number is $\mathcal{NP}$-hard in general, a natural approach is to design efficient approximation algorithms for it. The negative decision number of a graph, however, can be negative, so the normal concept of approximation ratio cannot be applied directly to this parameter. Also, any β-approximation for an instance with optimal value 0 needs to return a solution with value exactly 0, but this is $\mathcal{NP}$-hard owing to Corollary 1.

In the literature, when dealing with optimization problems with possibly negative objective values, a common approach is to standardize the objective to a nonnegative one, by considering the "distance" of the objective value from an obvious lower or upper bound instead. A famous example is the maximization version of the facility location problem [1]. Such "below-upper-bound" or "above-lower-bound" approximation, in some situation coincided with the concept of *differential approximation*, is also widely studied for other classical problems including set cover [11], bin packing [5], graph coloring [2], and the traveling salesman problem [17].

In this paper we consider the below-upper-bound formulation of the NEGATIVE DECISION NUMBER problem. Given a graph $G=(V,E)$, an obvious upper bound for $\beta_D(G)$ is $|V|$. Thus, we can define the following standardized optimization problem associated with the negative decision number:

BELOW-UPPER-BOUND APPROXIMATION FOR NEGATIVE DECISION NUMBER **(UpperNDN)**
Instance: A graph $G=(V,E)$.

Goal: Find a bad function f of G that minimizes $|V| - f(V)$.

Theorem 2. ***UpperNDN*** *can be approximated within a factor of* $O(\log |V|)$ *in polynomial time.*

Proof. Assume $G = (V, E)$ is an input graph of the **UpperNDN** problem. For any function $f : V \to \{-1, 1\}$, let $S^+(f) = \{v \in V \mid f(v) = 1\}$ and $S^-(f) = \{v \in V \mid f(v) = -1\}$. Note that the objective value of the problem is $|V| - f(V) = |V| - (|S^+(f)| - |S^-(f)|) = 2|S^-(f)|$. By the definition, f is a bad function of G if and only if for every $v \in V$, $|S^+(f) \cap N(v)| - |S^-(f) \cap N(v)| \leq 1$, or equivalently, by the fact that $|S^+(f) \cap N(v)| + |S^-(f) \cap N(v)| = d_G(v)$, $|S^-(f) \cap N(v)| \geq \frac{d_G(v)-1}{2}$. Therefore, the **UpperNDN** problem is reducible to finding a smallest subset of vertices ($S^-(f)$, in our notation) such that each vertex in the graph has at least a prescribed number of neighbors ($\frac{d_G(v)-1}{2}$, in our case) in this subset. This is a special case of the *total vector domination problem* which allows a polynomial time $O(\log |V|)$ approximation [4]. □

The performance guarantee of $O(\log |V|)$ is asymptotically tight as suggested by the following theorem. Note that the **UpperNDN** problem is similar to the total q-domination studied by Cicalese et al. [4] with $q = 1/2$, where they prove similar hardness results. However, the slight difference between the two problems makes their argument inapplicable to our case. An important observation is that the (unique) neighbor of any degree-1 vertex must be included in a total $\frac{1}{2}$-dominating set, while this is not the case for our problem.

Theorem 3. *There is a constant* $c > 0$ *such that, unless* $NP \subseteq DTIME(n^{O(\log \log n)})$, *no polynomial time* $c \log |V|$*-approximation algorithm exists for* ***UpperNDN****.*

Proof. By the analysis in the proof of Theorem 2, **UpperNDN** is equivalent to the following problem: Given a graph $G = (V, E)$, find a *negative set* of G of smallest cardinality, where a set $S \subseteq V$ is called a *negative set* if and only if $|S \cap N_G(v)| \geq \frac{d_G(v)-1}{2}$ for each vertex $v \in V$. In what follows, when referring to **UpperNDN**, we will always mean this formulation.

Fix ϵ with $0 < \epsilon < 1$, and let $c = (1-\epsilon)/8$. Suppose that there exists a polynomial time $c \ln |V|$-approximation algorithm for **UpperNDN**. We will use it to design a $(1-\epsilon)\ln n$-approximation algorithm for the problem of computing the minimum total dominating set of a given graph G of order n, which implies $\mathcal{NP} \subseteq DTIME(n^{O(\log \log n)})$ by the result of Feige [9], and thus concludes the proof. (A total dominating set of a graph $G = (V, E)$ is a subset of V such that each vertex in V has at least one neighbor in this subset. The minimum size of a total dominating set of G is denoted by $\gamma_t(G)$.)

Let $G = (V, E)$ be a graph for which we wish to compute the minimum total dominating set, where $|V| = n$. Without loss of generality we assume that $n \geq 10$ and $\gamma_t(G) \geq 1$. Construct a new graph $G' = (V', E')$ as follows. Let $V' = \left(\bigcup_{i=1}^{n^2} V^{(i)}\right) \cup \left(\bigcup_{i=1}^{n^2} V^{(i)^*}\right) \cup A \cup B \cup C$, where:

- $V^{(i)} = \{v^{(i)} \mid v \in V\}$ for each $1 \leq i \leq n^2$;
- $V^{(i)^*} = \{v^{(i)^*} \mid v \in V\}$ for each $1 \leq i \leq n^2$;
- $A = \{a_i \mid 1 \leq i \leq 2n^2\}$;
- $B = \{b_i \mid 1 \leq i \leq 2n^2\}$;
- $C = \{c_i \mid 1 \leq i \leq 2n^2\}$.

$$\begin{aligned}
\text{Let } E' &= \{u^{(i)}v^{(i)} \mid uv \in E; 1 \le i \le n^2\} \\
&\cup \{v^{(i)}v^{(i)*} \mid v \in V; 1 \le i \le n^2\} \\
&\cup \{a_i a_j \mid 1 \le i < j \le 2n^2\} \\
&\cup \{a_i b_i \mid 1 \le i \le 2n^2\} \\
&\cup \{b_i c_i \mid 1 \le i \le 2n^2\} \\
&\cup \{v^{(i)}a_j \mid v \in V; 1 \le i \le n^2; (\lceil \frac{i}{n} \rceil - 1)n + 1 \le j \le (\lceil \frac{i}{n} \rceil - 1)n + d_G(v) - 1\}.
\end{aligned}$$

(The conditions in defining the last set of edges may seem awkward. In fact, any construction that satisfies the following two conditions can be used instead: (1) every vertex $v^{(i)}$ is adjacent to exactly $d_G(v) - 1$ vertices in A, and (2) every vertex in A is adjacent to at most n^2 vertices in $\cup_{i=1}^{n^2} V^{(i)}$.)

Note that $|V'| = 2n^2 \cdot |V| + 6n^2 = 2n^3 + 6n^2 \le n^4$ when $n \ge 10$. In addition, it is not difficult to verify the following:

- For every $1 \le i \le 2n^2$, $N_{G'}(c_i) = \{b_i\}$ and $d_{G'}(c_i) = 1$;
- For every $1 \le i \le 2n^2$, $N_{G'}(b_i) = \{a_i, c_i\}$ and $d_{G'}(b_i) = 2$;
- For every $1 \le i \le 2n^2$, $N_{G'}(a_i) = \{b_i\} \cup (A \setminus \{a_i\}) \cup \{v^{(j)} \mid v \in V; 1 \le j \le n^2; (\lceil \frac{j}{n} \rceil - 1)n + 1 \le i \le (\lceil \frac{j}{n} \rceil - 1)n + d_G(v) - 1\}$, and $d_{G'}(a_i) \le 1 + (2n^2 - 1) + n^2 = 3n^2$.
- For every $1 \le i \le n^2$, $N_{G'}(v^{(i)}) = \{u^{(i)} \mid u \in N_G(v)\} \cup \{v^{(i)*}\} \cup \{a_j \mid (\lceil \frac{i}{n} \rceil - 1)n + 1 \le j \le (\lceil \frac{i}{n} \rceil - 1)n + d_G(v) - 1\}$, and $d_{G'}(v^{(i)}) = 2d_G(v)$;
- For every $1 \le i \le n^2$, $N_{G'}(v^{(i)*}) = \{v^{(i)}\}$ and $d_{G'}(v^{(i)*}) = 1$.

Let $neg(G')$ denote the smallest size of a negative set of G'. We next show that $neg(G') = n^2\gamma_t(G) + 2n^2$.

On one hand, let $S \subseteq V$ be a total dominating set of G of size $\gamma_t(G)$. Let $S' = A \cup \{v^{(i)} \mid v \in S, 1 \le i \le n^2\}$. We have $|S'| = n^2 \cdot |S| + |A| = n^2\gamma_t(G) + 2n^2$. Next we prove that S' is a negative set of G'. It is clear that the condition $|S' \cap N_{G'}(v)| \ge \frac{d_{G'}(v)-1}{2}$ (which we will call the *generic condition* hereafter) holds for all $v \in C \cup B$. For any $a_i \in A$, $|S' \cap N_{G'}(a_i)| \ge |A| - 1 = 2n^2 - 1 \ge \frac{3n^2-1}{2} \ge \frac{d_{G'}(a_i)-1}{2}$, and thus the generic condition holds for a_i. For any $v \in V$, since S is a total dominating set of G, it contains at least one neighbor of v in G. Thus, for each i, at least one $u^{(i)}$ with $u \in N_G(v)$ is contained in S'. By our construction, $v^{(i)}$ is adjacent to $u^{(i)}$ and another $d_G(v) - 1$ vertices in A. Therefore,

$$|S' \cap N_{G'}(v^{(i)})| \ge d_G(v) > \frac{2d_G(v) - 1}{2} = \frac{d_{G'}(v^{(i)}) - 1}{2},$$

and hence the generic condition holds for all $v^{(i)}$ with $v \in V$ and $1 \le i \le n^2$. Finally, each vertex $v^{(i)*}$ has degree 1 and surely satisfies the generic condition. This shows that S' is indeed a negative set of G'. Hence, $neg(G') \le |S'| = n^2\gamma_t(G) + 2n^2$.

On the other hand, assume S' is a negative set of G' of size $k = neg(G')$. We show that we can modify S' to obtain a new negative set S'' satisfying that:

1. $A \subseteq S''$;

2. $S'' \cap \left(B \cup C \cup \left(\bigcup_{i=1}^{n^2} V^{(i)^*}\right)\right) = \emptyset$;

3. $|S''| \leq k$.

For the first condition, since $N_{G'}(b_i) = \{a_i, c_i\}$ for each $1 \leq i \leq 2n^2$, at least one of a_i and c_i must be in S'. Hence, we have $|S' \cap (A \cup C)| \geq 2n^2 = |A|$. Now we remove all vertices in $B \cup C$ from S' and add all vertices of A to S', i.e., let $S' \leftarrow (S' \setminus (B \cup C)) \cup A = (S' \setminus (A \cup B \cup C)) \cup A$. It is clear that the new set is still a negative set of G', and has size no more than that of S'. Call the new set S'_0. If there is some $v^{(i)^*} \in S'_0$, we try to eliminate it from S'_0 without violating the property of a negative set. Since the only neighbor of $v^{(i)^*}$ is $v^{(i)}$, we check the set of neighbors of $v^{(i)}$ other than $v^{(i)^*}$, i.e., $\{u^{(i)} \mid u \in N_G(v)\}$. If all vertices in this set are contained in S'_0, then the generic condition for $v^{(i)}$ is still satisfied even if $v^{(i)^*}$ is removed from S'_0 (since $d_{G'}(v^{(i)}) = 2d_G(v)$ and $|\{u^{(i)} \mid u \in N_G(v)\}| = d_G(v)$), so in this case we simply delete it. If some vertex, say $u^{(i)}$, is not in S'_0, then we delete $v^{(i)^*}$ from S'_0 and add $u^{(i)}$ to S'_0. It is easy to see that the newly obtained set is still a negative set of G'. Keep doing this until all vertices in $\bigcup_{i=1}^{n^2} V^{(i)^*}$ are no longer in S'_0; call the final set S''. Then S'' is a negative set of G' satisfying all the three conditions listed above.

For each $1 \leq i \leq n^2$, let $S^{(i)} = \{v \in V \mid v^{(i)} \in S''\}$. We will prove that $S^{(i)}$ is a total dominating set of G. To see this, assume to the contrary that there exists some $v \in V$ such that no neighbor of v is in $S^{(i)}$, which means that no $u^{(i)}$ vertices with $u \in N_G(v)$ is in S''. Then we have $|S'' \cap N_{G'}(v^{(i)})| = |A \cap N_{G'}(v^{(i)})| = d_G(v) - 1 < \frac{2d_G(v)-1}{2} = \frac{d_{G'}(v^{(i)})-1}{2}$, contradicting with the generic condition. Thus, every $S^{(i)}$ is a total dominating set of G. Pick the smallest one of them and call it S. Then we have:

$$neg(G') = k \geq |S''| = |A| + \sum_{i=1}^{n^2} |S^{(i)}| \geq |A| + n^2 \cdot |S| \geq 2n^2 + n^2\gamma_t(G).$$

Combining the two directions, we obtain that $neg(G') = 2n^2 + n^2\gamma_t(G)$. In fact, from the above analysis we also see that, given a negative set of G' of size k, we can find in polynomial time a total dominating set of G of size at most $(k - |A|)/n^2 < k/n^2$ by taking the smallest $S^{(i)}$ defined before.

Now we apply the polynomial time $c \ln |V'|$-approximation algorithm for **UpperNDN** on the graph G', which will produce a negative set of G' of size at most $c \ln |V'| \cdot neg(G')$. According to our previous analysis, this in turn gives us a total dominating set of G of size at most

$$\begin{aligned}
\frac{1}{n^2} \cdot c \ln |V'| \cdot neg(G') &\leq \frac{1}{n^2} \cdot c \ln(n^4)(n^2\gamma_t(G) + n^2) \\
&= 4c \ln n(\gamma_t(G) + 1) \\
&\leq 8c \ln n \cdot \gamma_t(G) \\
&= (1 - \epsilon) \ln n \cdot \gamma_t(G).
\end{aligned}$$

Thus we obtain a polynomial time $(1 - \epsilon) \ln n$-approximation algorithm for the Total Dominating Set problem, implying that $\mathcal{NP} \subseteq DTIME(n^{O(\log \log n)})$ [9]. This completes the proof of Theorem 3. □

2.3 Polynomial Time Algorithms for Special Instances

In this short subsection we remark that the negative decision number can be computed in polynomial time for some special classes of graphs. When we want to compute the exact value of the negative decision number of a graph, we can equivalently solve the **UpperNDN** problem on this graph (and then use n minus twice the returned value to get the negative decision number). As noted before, **UpperNDN** is a special case of the total vector domination problem [4]. Thus, all the polynomial time algorithms for total vector domination on special graphs can be directly applied to the negative decision number. Cicalese et al. [4] pointed out that the total vector domination problem on strongly chordal graphs is polynomial time solvable using the algorithm of Liao and Chang [16]. (In fact the M-domination problem considered by Liao and Chang [16] is slightly different from the total vector domination problem, in that the former considers the closed neighborhood of v as the set of vertices that dominate v, whereas the latter uses the open neighborhood instead. Nevertheless, an easy modification will make the algorithm work also for total vector domination.) Since directed path graphs, interval graphs, block graphs and trees are all subclasses of strongly chordal graphs, we obtain the following:

Corollary 2. *The negative decision number can be computed in polynomial time for the following classes of graphs: strongly chordal graphs, directed path graphs, interval graphs, block graphs, and trees.*

3 Negative Decision Number in Particular Graph Classes

In this section, we determine the negative decision number of complete multipartite graphs, wheels and fans. Our result for complete multipartite graphs substantially generalizes those of complete graphs (since a complete graph of order n can be regarded as a complete n-partite graph) and bicliques (complete bipartite graphs) given by Wang [18].

Let $K_{n_1,n_2,\ldots,n_k}$ denote the complete k-partite graph with vertex set $V = V_1 \cup V_2 \cup \ldots \cup V_k$ and edge set E, where $V_i = \{v_{i,j} \mid 1 \le j \le n_i\}$ and $E = \{v_{i,j}v_{i,j'} \mid i \ne i'\}$. We always assume that $k \ge 2$ and $n_i \ge 1$ for each $i \in \{1, 2, \ldots, k\}$.

Theorem 4. *Suppose $G = K_{n_1,n_2,\ldots,n_k}$. Let t be the number of i's such that n_i is odd; that is, $t = |\{i \mid 1 \le i \le k; n_i \text{ is odd}\}|$. Then,*

$$\beta_D(G) = \begin{cases} -1 & \textit{if } t \textit{ is odd and } t \ge 3; \\ 1 & \textit{if } t = 1; \\ 2 & \textit{if } t = k = 2; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let f be any bad function of G. Observe that $f(N(v_{i,1})) = \sum_{i' \ne i} f(V_{i'})$ for each $i \in \{1, 2, \ldots, k\}$. We investigate the following cases.

1. t is odd. Without loss of generality we assume that $n_1, n_2, \ldots, n_t$ are odd, while $n_{t+1}, \ldots, n_k$ are even. For every $i \in \{1, 2, \ldots, t\}$, we have $\sum_{i' \ne i} f(V_{i'}) = f(N(v_{i,1})) \le 0$, since f is a bad function and the degree of $v_{i,1}$ is even. For every $i \in \{t+1, \ldots, k\}$, we have $\sum_{i' \ne i} f(V_{i'}) = f(N(v_{i,1})) \le 1$. Summing up the k inequalities for all $i \in \{1, 2, \ldots, k\}$, we get $(k-1)\sum_{1 \le i \le k} f(V_i) \le k - t$. Noting that $\sum_{1 \le i \le k} f(V_i) = f(V)$, we have

$$f(V) \le \frac{k-t}{k-1}. \tag{1}$$

We further consider two subcases.

(a) $t = 1$. Equation (1) gives $f(V) \leq 1$, and so $\beta_D(G) \leq 1$. On the other hand, we can obtain a bad function f^* of weight 1 by defining $f^*(v_{i,j}) = (-1)^{j-1}$ for all $1 \leq i \leq t$ and $1 \leq j \leq n_i$. (It is easy to verify that $f^*(V_1) = 1$ and $f^*(V_i) = 0$ for $i > 1$.) Therefore, $\beta_D(G) = 1$.

(b) $t \geq 3$. Equation (1) then gives $f(V) < 1$. Since in this case the number of vertices in V is odd, we have $f(V) \leq -1$. On the other hand, we can obtain a bad function f^* of weight -1 by defining $f^*(v_{i,j}) = (-1)^{i+j-1}$ for all $1 \leq i \leq t$ and $1 \leq j \leq n_i$. (It is easy to verify that $f^*(V_i) = (-1)^i$ for $1 \leq i \leq t$, and $f^*(V_i) = 0$ for $t+1 \leq i \leq k$.) Therefore, $\beta_D(G) = -1$.

2. t is even. Without loss of generality we assume that $n_1, n_2, \ldots, n_t$ are odd, while $n_{t+1}, \ldots, n_k$ are even. For every $i \in \{1, 2, \ldots, t\}$, we have $\sum_{i' \neq i} f(V_{i'}) = f(N(v_{i,1})) \leq 1$. For every $i \in \{t+1, \ldots, k\}$, we have $\sum_{i' \neq i} f(V_{i'}) = f(N(v_{i,1})) \leq 0$, since the degree of $v_{i,1}$ is even.

   Summing up the k inequalities for all $i \in \{1, 2, \ldots, k\}$, we get $(k-1)f(V) \leq t$. Hence,

$$f(V) \leq \frac{t}{k-1}. \tag{2}$$

   We further consider two subcases.

(a) $t = k = 2$. Equation (2) shows $f(V) \leq 2$, and so $\beta_D(G) \leq 2$. On the other hand, we can obtain a bad function f^* of weight 2 by defining $f^*(v_{i,j}) = (-1)^{j-1}$ for all $1 \leq i \leq t$ and $1 \leq j \leq n_i$. (It is easy to verify that $f^*(V_1) = f^*(V_2) = 1$ and $f^*(V_i) = 0$ for $i > 2$.) Therefore, $\beta_D(G) = 2$.

(b) $t < k$ or $t = k \geq 4$. If $t < k$, Equation (2) gives $f(V) \leq 1$. If $t = k \geq 4$, Equation (2) indicates $f(V) \leq \frac{t}{t-1} < 2$. Since in this case the number of vertices in V is even, we have $f(V) \leq 0$. On the other hand, we can obtain a bad function f^* of weight 0 by defining $f^*(v_{i,j}) = (-1)^{i+j-1}$ for all $1 \leq i \leq t$ and $1 \leq j \leq n_i$. (It is easy to verify that $f^*(V_i) = (-1)^i$ for $1 \leq i \leq t$, and $f^*(V_i) = 0$ for $t+1 \leq i \leq k$.) Therefore, $\beta_D(G) = 0$.

This completes the proof of Theorem 4. □

For every integer $n \geq 3$, let W_n denote the wheel graph with vertex set $V = \{v_0, v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_0 v_i \mid 1 \leq i \leq n\}$. Thus, W_n is obtained by adding a vertex v_0 to the cycle C_n with vertex set $\{v_1, \ldots, v_n\}$ and joining v_0 with all the original cycle vertices.

Theorem 5. *For any integer $n \geq 3$,*

$$\beta_D(W_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \ (\textit{mod } 4); \\ -1 & \textit{if } n \equiv 2 \ (\textit{mod } 4); \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $V = \{v_0, v_1, \ldots, v_n\}$ be the vertex set of W_n. Consider the following function $f : V \to \{-1, 1\}$:

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 3 \pmod 4; \\ -1 & \text{if } i \equiv 1 \text{ or } 2 \pmod 4. \end{cases}$$

It is easy to verify that f has weight 1 when $n \equiv 0 \pmod 4$, has weight -1 when $n \equiv 2 \pmod 4$, and has weight 0 when n is odd. Furthermore, we have $f(N(v_0)) = f(\{v_1, \dots, v_n\}) \le 0$, and for each $v_i, 1 \le i \le n$, at least one of its three neighbors has value -1, indicating that $f(N(v_i)) \le 1$. Hence, f is a bad function of W_n that achieves the bounds in the theorem.

We next show that these bounds are best possible. Let f be an arbitrary bad function of W_n. We consider two cases.

1. $f(v_0) = -1$.

   We have $f(N(v_0)) \le 0$ if n is even, and $f(N(v_0)) \le 1$ if n is odd. Thus, $f(V) \le -1$ if n is even, and $f(V) \le 0$ if n is odd. This case is completed.

2. $f(v_0) = 1$.

   Similar to the previous case, we have $f(V) \le 1$ if n is even, and $f(V) \le 2$ if n is odd. We need to show that $f(V) \le -1$ if $n \equiv 2 \pmod 4$, and $f(V) \le 0$ if n is odd.

   - $n \equiv 2 \pmod 4$.
     Assume $f(V) = 1$. Then
     $$f(v_1) + f(v_2) + \ldots + f(v_n) = 0. \tag{3}$$
     For each $i \in \{1, \dots, n\}$, let $i^+ = i + 1$ if $i < n$, and $i^+ = 1$ if $i = n$. Similarly, let $i^- = i - 1$ if $i > 1$, and $i^- = n$ if $i = 1$. Clearly $N(v_i) = \{v_{i^+}, v_{i^-}, v_0\}$. Since $f(N(v_i)) \le 1$ and $f(v_0) = 1$, we have $f(v_{i^-}) + f(v_{i^+}) \le 0$. Summing up the inequalities for all $i \in \{1, \dots, n\}$ gives $f(v_1) + f(v_2) + \ldots + f(v_n) \le 0$. Combined with Equa. (3), we obtain
     $$f(v_{i^-}) + f(v_{i^+}) = 0, \text{ for each } i \in \{1, \dots, n\}. \tag{4}$$
     Furthermore, there exists $i \in \{1, 2, \dots, n\}$ such that $f(v_i) = 1$. By the symmetry of the structure of W_n, we may assume w.l.o.g. that $f(v_1) = 1$. By Equa. (4), we have $f(v_3) = -1, f(v_5) = 1, \dots, f(v_{n-1}) = 1$ (since $n \equiv 2 \pmod 4$). But then $f(v_{n^+}) = f(v_{n^-}) = 1$, a contradiction. Therefore, $f(V) < 1$. As W_n has odd number of vertices, we must have $f(V) \le -1$. This case is complete.
   - n is odd.
     We use notation i^+ and i^- as in the previous case. Since $f(N(v_i)) \le 1$ and $f(v_0) = 1$, we have $f(v_{i^-}) + f(v_{i^+}) \le 0$ for each $i \in \{1, \dots, n\}$. Summing up the inequalities for all $i \in \{1, \dots, n\}$ gives $f(v_1) + f(v_2) + \ldots + f(v_n) \le 0$, implying that $f(V) \le 1$. As W_n has even number of vertices, we have $f(V) \le 0$. The second case is complete.

The proof of Theorem 5 is thus finished. □

For every integer $n \ge 2$, let F_n denote the fan graph with vertex set $V = \{v_0, v_1, v_2, \dots, v_n\}$ and edge set $E = \{v_i v_{i+1} \mid 1 \le i \le n - 1\} \cup \{v_0 v_i \mid 1 \le i \le n\}$. That is, F_n is obtained by adding a vertex to the path P_n and connecting this vertex with all the original path vertices.

Theorem 6. *For any integer $n \ge 2$, we have*

$$\beta_D(F_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \ (\textit{mod } 4); \\ -1 & \textit{if } n \equiv 2 \ (\textit{mod } 4); \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. $\beta_D(F_2) = -1$ is obvious. (Note that $F_2 = K_3$.) In the following we assume $n \geq 3$. We first construct bad functions that achieve these bounds, and then prove that they are the best possible.

If $n \equiv 0 \pmod 4$, we define f as:

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 1 \pmod 4; \\ -1 & \text{if } i \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

It is easy to verify that f is a bad function of G of weight 1.

If $n \equiv 2 \pmod 4$, we define f as:

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 3 \pmod 4; \\ -1 & \text{if } i \equiv 1 \text{ or } 2 \pmod 4. \end{cases}$$

It is easy to verify that f is a bad function of G of weight -1.

If n is odd, we define f as:

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \text{ or } 2 \pmod 4; \\ -1 & \text{if } i \equiv 0 \text{ or } 3 \pmod 4. \end{cases}$$

It is easy to verify that f is a bad function of G of weight 0.

It remains to prove that these bounds cannot be improved. By Theorem 5, it suffices to show that $\beta_D(F_n) \leq \beta_D(W_n)$ for every $n \geq 3$. Let f be a bad function of F_n that has weight $\beta_D(F_n)$. We prove that f is also a bad function of W_n (we identify the vertex set of F_n with that of W_n). For $i \in \{2, 3, \ldots, n-1\} \cup \{0\}$, it holds that $f(N_{W_n}(v_i)) = f(N_{F_n}(v_i)) \leq 1$. If $i = 1$ or n, we have $f(N_{F_n}(v_i)) \leq 0$, since v_i has even number of neighbors in F_n. Thus, $f(N_{W_n}(v_i)) \leq 1$ for $i = 1$ or n (in W_n, v_1 and v_n become neighbors of each other). Therefore, f is also a bad function of W_n, which implies that $\beta_D(W_n) \geq \beta_D(F_n)$. This completes the proof of Theorem 6. $\square$

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