

A Simplicial Approach for Discrete Fixed Point Theorems

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Abstract We present a new discrete fixed point theorem based on a variation of the direction-preserving maps over simplicial structures. We show that the result is more general than the recent discrete fixed point theorem of Iimura et al. (*J. Math. Econ.* 41(8):1030–1036, 2005) by deriving it from ours. The simplicial approach applied in the development of the new theorem reveals a clear structural comparison with the classical approach for the continuous case.

Keywords Sperner's lemma · Fixed point theorem · Direction-preserving map · Simplicial structure

1 Introduction

There have recently been a sequence of works related to fixed point theorems in a discrete disguise, inspired by the work of Iimura [2] who introduced a crucial concept of direction-preserving maps that may be considered as a discrete counter-part of continuity. Iimura, Murota and Tamura [1] corrected the proof of Iimura for the definition domains of the maps. With a different technique, Chen and Deng intro-

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duced another discrete fixed point theorem in order to achieve the optimal algorithmic bound for finding a discrete fixed point for all finite dimensions [3]. In [4], van der Laan, Talman and Yang designed a combinatorial algorithm for the discrete zero point problem. Friedl, Ivanyosy, Santha and Verhoeven defined the black-box Sperner problem and obtained an upper bound of $O(\sqrt{n})$ for the two-dimensional case [5]. On the other hand, Chen and Deng [6] showed that the two theorems, that of Iimura, Murota and Tamura [1], as well as that of Chen and Deng [3], cannot directly derive each other.

The discrete nature of the fixed point theorem has been noticed previously, mainly due to the proof techniques of Sperner's lemma [7]. The simplicial structure has since played an important role in establishing various continuous fixed point theorems, and in developing algorithmic solutions, starting with Scarf [8] in the tradition of the Lemke-Howson algorithm [9], and further developed by Eaves, Saigal, Todd, van der Laan and Talman [10–14], to name a few.

Although the recent studies on discrete fixed point theorems have started with the hypercubic approach by Iimura [2], as well as in [1], and [3], the simplicial approach has also come back from several fronts to deal with the discrete versions, with the path-following approach in [15], and also the divide-&-conquer approach in [5, 16] which are in essence originated from the degree theory of Brouwer [17].

While Iimura's concept of direction preserving is coordinate-wise for every two points on a same unit hypercube, van der Laan, Talman, and Yang introduced a much stronger concept of simplicially local gross direction preserving by requiring the function values to be within a right angle for any two points on a same simplex. They derived discrete zero point theorems, as well as proofs of convergence for both a convergence condition and an antipodal condition [15]. In solving this problem, they adapted the $2n$ -ray vector labeling algorithm of van der Laan and Talman [14]. Both their proofs and algorithms are closely related to the path-following paradigm, started in the Lemke-Howson algorithm [9].

In this article, we derive a general discrete fixed point theorem based on a combination of simplicial structures and an associated variation of Iimura's direction-preserving maps. Our approach can be viewed as a discrete approach of Brouwer's degree theory, an approach we started to use in the hypercubic case [3]. A work closely related to ours is that of Friedl, Ivanyosy, Santha, and Verhoeven [5] who derived a matching bound for finding a Sperner triangle in $O(\sqrt{n})$ time which matches the lower bound of Crescenzi and Silvestri [18] (mirroring an earlier result of Hirsch, Papadimitriou and Vavasis on the computation of two-dimensional approximate fixed points [19]).

The recent effort in the direct formulation of discrete fixed point theorems has also been made in response to the challenges in the complexity characterization of related problems. The recent work in characterizing the computational complexity of Nash equilibria, by Daskalakis, Goldberg and Papadimitriou [20], Chen and Deng [21], Daskalakis and Papadimitriou [22], Chen and Deng [23], has been based on another innovative formulation of the 2-dimensional (or 3-dimensional) discrete fixed point problem, where a fixed point is defined as a collection of four [24] (or eight [20]) corners of a unit square (or unit cube). It's difficult to efficiently generalize such a formulation to high-dimensional spaces, since a hypercube has an exponential number of corners, which is computationally infeasible. Instead, a simplicial definition

has been necessary in extending those results to a non-approximability theorem obtained recently in [25].

We first introduce notations and definitions with a review of the previous work of Iimura, Murota and Tamura [1]. We then define the concept of simplicial direction-preserving maps, which is followed by the primitive discrete fixed point theorem. In Sect. 3, we present the discrete Brouwer’s fixed point theorem for simplicial direction-preserving maps, with the theorem of Iimura, Murota and Tamura [1] derived as a direct corollary. Finally, we conclude in Sect. 4 with discussion and remarks.

2 Basic Definitions and Known Results

We use e^k to denote the k th unit vector of \mathbb{Z}^d , where $e_k^k = 1$ and $e_i^k = 0$ for all $i : 1 \leq i \neq k \leq d$.

Throughout the paper, we use “map” to denote a point-to-point correspondence from a finite set $X \subset \mathbb{R}^d$ to \mathbb{R}^d , and use “direction function” (or simply “function”) to denote a point-to-value correspondence from set $X \subset \mathbb{R}^d$ to $\{0, \pm e^1, \dots, \pm e^{d-1}, \pm e^d\} \subset \mathbb{Z}^d$.

Let \mathcal{F} be a map on $X \subset \mathbb{R}^d$, f be a direction function on X , and $C \subset X$. We use $\mathcal{F}(C)$ and $f(C)$ to denote $\{\mathcal{F}(r), r \in C\}$ and $\{f(r), r \in C\}$, respectively.

2.1 Hypercubic Direction-Preserving Maps and Functions

Informally speaking, a map \mathcal{F} (or a direction function f) on a finite point set $X \subset \mathbb{Z}^d$ is hypercubic direction-preserving if for any two neighboring points in X , their directions given by \mathcal{F} (or f) are not opposite. The neighborhood relation considered here is defined by the infinity norm.

Definition 1 (Hypercubic direction-preserving maps) Let X be a finite subset of \mathbb{Z}^d . Map \mathcal{F} from X to \mathbb{R}^d is said to be hypercubic direction-preserving on X if for any two points $r^1, r^2 \in X$ with $\|r^1 - r^2\|_\infty \leq 1$, we have $(\mathcal{F}_i(r^1) - r_i^1)(\mathcal{F}_i(r^2) - r_i^2) \geq 0$, for all $i : 1 \leq i \leq d$.

Definition 2 (Hypercubic direction-preserving functions) Let X be a finite subset of \mathbb{Z}^d . A direction function f from X to $\{0, \pm e^1, \dots, \pm e^{d-1}, \pm e^d\}$ is said to be hypercubic direction-preserving if for any two points $r^1, r^2 \in X$ such that $\|r^1 - r^2\|_\infty \leq 1$, we have $\|f(r^1) - f(r^2)\|_\infty \leq 1$.

Point $r \in X$ is called a *fixed point* of map \mathcal{F} (or function f), if $\mathcal{F}(r) = r$ (or $f(r) = 0$). Iimura, Murota and Tamura proved in [1] that every hypercubic direction-preserving map from an integrally convex set X to \bar{X} must have a fixed point.¹ Here we use \bar{X} to denote the convex hull of finite set $X \subset \mathbb{Z}^d$.

¹Although the original theorem of [1] deals with point-to-set correspondences, it is essentially equivalent to the point-to-point version presented here.

Definition 3 (Integrally convex sets) Finite set $X \subset \mathbb{Z}^d$ is *integrally convex* if for all $x \in \overline{X}$, $x \in \overline{X} \cap N(x)$ where $N(x) = \{r \in \mathbb{Z}^d \mid \|r - x\|_\infty < 1\}$.

Theorem 1 [1] *Let $X \subset \mathbb{Z}^d$ be an integrally convex set, then any hypercubic direction-preserving map \mathcal{F} from X to \overline{X} has a fixed point in X .*

2.2 Simplicial Direction-Preserving Maps and Functions

We now introduce simplicial direction-preserving maps and functions based on simplicial structures. We follow the main idea from our extended abstract originally published in the Proceedings of the 12th Annual International Conference on Computing and Combinatorics [16], held at Taipei, on August 15–18, 2006. We note that a more general concept, called simplicially local gross direction-preserving functions, was also introduced by van der Laan, Talman and Yang in [15].

Let X be a finite set in \mathbb{R}^d . Here we only consider non-degenerate cases where $\overline{X} \subset \mathbb{R}^d$ is a convex d -polytope. For standard definitions concerning polytopes, readers are referred to [26] for details.

Definition 4 Given a finite set $X \subset \mathbb{R}^d$, we let $\overline{X} \subset \mathbb{R}^d$ denote its convex hull. A simplicial decomposition \mathcal{S} of \overline{X} is a finite collection of simplices satisfying:

1. $C = \bigcup_{S \in \mathcal{S}} S$;
2. For any $S \in \mathcal{S}$, if S' is a face of S , then $S' \in \mathcal{S}$; and
3. For any two simplices $S_1, S_2 \in \mathcal{S}$, if $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2$ is a face of both S_1 and S_2 .

Furthermore, a simplicial decomposition \mathcal{S} of set X is a simplicial decomposition of \overline{X} such that for every $S \in \mathcal{S}$, we have $V_S \subset X$, where V_S is the vertex set of simplex S .

Given a simplicial decomposition \mathcal{S} of X , we use $F_S \subset \mathcal{S}$ to denote the set of $(d - 1)$ -simplices on the boundary of \overline{X} , and $B_X \subset X$ to denote the set of points in X on the boundary of \overline{X} :

$$F_S = \{(d - 1)\text{-simplex } S \in \mathcal{S} \mid S \subset F \text{ and } F \text{ is a facet of } \overline{X}\} \quad \text{and}$$

$$B_X = \{r \in X \mid r \in F \text{ and } F \text{ is a facet of } \overline{X}\}.$$

Definition 5 (Simplicial direction-preserving maps) A simplicial direction-preserving map is a triple $M = (\mathcal{F}, X, \mathcal{S})$. Here X is a finite set in \mathbb{R}^d and \mathcal{S} is a simplicial decomposition of X . Map \mathcal{F} from X to \mathbb{R}^d satisfies the property that, for any two points $r^1, r^2 \in X$, if there exists a simplex $S \in \mathcal{S}$ such that $r^1, r^2 \in V_S$, then $(\mathcal{F}_i(r^1) - r_i^1)(\mathcal{F}_i(r^2) - r_i^2) \geq 0$, for all $i : 1 \leq i \leq d$.

Definition 6 (Simplicial direction-preserving functions) A triple $G = (f, X, \mathcal{S})$ is said to be a simplicial direction-preserving function, if X is a finite set in \mathbb{R}^d , \mathcal{S} is a simplicial decomposition of X , and direction function $f : X \rightarrow \{0, \pm e^1, \pm e^2, \dots, \pm e^d\}$ satisfies the property that, for any two points $r^1, r^2 \in X$, if there exists $S \in \mathcal{S}$ such that $r^1, r^2 \in V_S$, then $\|f(r^1) - f(r^2)\|_\infty \leq 1$.

In other words, for any two neighboring points in X , their directions given by map \mathcal{F} (or function f) cannot be opposite. The only difference with the hypercubic model is that the neighborhood relation is now defined by the simplices in the simplicial decomposition \mathcal{S} instead of the unit hypercubes in \mathbb{Z}^d .

Remark We note that there is a possibility that \mathcal{F} is hypercubic direction-preserving on $X \subset \mathbb{R}^d$, but none of its simplicial decompositions \mathcal{S} can make $M = (\mathcal{F}, X, \mathcal{S})$ simplicial direction-preserving. Therefore, our result in Sect. 3 is limited in the sense that it covers only the cases when a direction-preserving simplicial decomposition exists, and presented. However, for $X \subset \mathbb{R}^d$ of certain type (e.g., integrally convex sets [1]), there always exists an easy-to-find \mathcal{S} such that $M = (\mathcal{F}, X, \mathcal{S})$ is simplicial direction-preserving (see Lemma 3 in Sect. 3.3 for the formal statement).

Also note that the concept of simplicially local gross directional-preserving functions in [15] is only defined over subsets of \mathbb{Z}^d , while ours is more general, which is defined over subsets of \mathbb{R}^d .

2.3 The Primitive Discrete Fixed Point Theorem

We are now ready to state the primitive discrete fixed point theorem in the tradition of Brouwer’s degree theory [3, 5, 17].

Definition 7 (Bad simplices) Let $G = (f, X, \mathcal{S})$ be a simplicial direction-preserving function with $X \subset \mathbb{R}^d$. For $t : 0 \leq t < d$, a t -simplex $S \in \mathcal{S}$ is *bad* relative to G if $f(V_S) = \{e^1, e^2, \dots, e^{t+1}\}$, where V_S is the vertex set of S .

We let N_G denote the number of *bad* $(d - 1)$ -simplices in $F_{\mathcal{S}}$.

Lemma 1 [3, 5, 17] *For every simplicial direction-preserving function $G = (f, X, \mathcal{S})$, if there exists no fixed point in X , then N_G is even.*

Proof Firstly, one can show that for every $(d - 1)$ -simplex $S \in \mathcal{S}$, if $S \in F_{\mathcal{S}}$, then there exists exactly one d -simplex in \mathcal{S} containing S . Otherwise, there are exactly two such simplices. Using this property, the parity of N_G is the same as the one of the following summation:

$$\sum_{d\text{-simplex } S^d \in \mathcal{S}} |\{\text{bad } (d - 1)\text{-simplices (relative to } G) \text{ in } S^d\}|.$$

As G is direction-preserving and has no fixed point, the number of bad $(d - 1)$ -simplices in S^d is either 0 or 2. Therefore, the summation above must be even. \square

We now get the primitive theorem as a simple corollary of Lemma 1.

Theorem 2 (The primitive discrete fixed point theorem [3, 5, 17]) *Let $G = (f, X, \mathcal{S})$ be a simplicial direction-preserving function. If N_G , i.e. the number of bad $(d - 1)$ -simplices on the boundary of \overline{X} , is odd, then G must have a fixed point $r \in X$ such that $f(r) = 0$.*

3 Main Results

Our main result is a constructive proof for the following discrete fixed point theorem concerning simplicial direction-preserving maps.

Theorem 3 (The discrete Brouwer’s fixed point theorem) *For every simplicial direction-preserving map $M = (\mathcal{F}, X, \mathcal{S})$ such that \mathcal{F} maps X to \overline{X} , there must exist a fixed point in X .*

We will employ the primitive discrete fixed point theorem to prove this theorem, which can be recognized as a discrete version of Brouwer’s fixed point theorem. In Sect. 3.3, we will use it to derive the theorem of Iimura, Murota and Tamura as a corollary.

On the other hand, it is not very surprising that, one can apply Brouwer’s fixed point theorem to obtain a quick proof of this discrete version. However, our proof is completely combinatorial and is based on an interesting boundary characterization for simplicial direction-preserving maps (as will be shown in Lemma 2 below). The advantage of such a constructive proof is that, for many structured sets $X \subset \mathbb{R}^d$, (e.g., hypergirds [3]), one can employ the boundary characterization to derive a divide-&-conquer algorithm for finding a discrete fixed point.

3.1 Preliminaries

Let X be a finite set in \mathbb{R}^d , \mathcal{S} be a simplicial decomposition of X , and S be a $(d - 1)$ -simplex in $F_{\mathcal{S}}$. We say S is visible from point $r \notin \overline{X}$, if $e_S \cdot (r - r_S) > 0$ for some point $r_S \in S$, where e_S is the unique vector of length 1 that is outgoing and perpendicular to S .

It is easy to check that, for all $r \in \overline{X}$ and $r_S \in S$, $e_S \cdot (r - r_S) \leq 0$.

Construction 1 (Extension of simplicial decomposition) Let $X \subset \mathbb{R}^d$ be a finite set and \mathcal{S} be a simplicial decomposition of X . For every point $r \notin \overline{X}$, we can add new simplices into \mathcal{S} and build a simplicial decomposition \mathcal{S}' of set $X' = X \cup \{r\}$ as follows. For every $(d - 1)$ -simplex $S \in F_{\mathcal{S}}$ visible from r , we add d -simplex $\text{conv}(S, r)$ and all its faces into \mathcal{S} .

One can check that \mathcal{S}' is a simplicial decomposition of X' , and $\mathcal{S} \subset \mathcal{S}'$.

Given a simplicial direction-preserving map $M = (\mathcal{F}, X, \mathcal{S})$, we can convert it into a direction-preserving function $G = (f, X, \mathcal{S})$ as follows.

Construction 2 Given a simplicial direction-preserving map $M = (\mathcal{F}, X, \mathcal{S})$, we can build a simplicial direction-preserving function $G = (f, X, \mathcal{S})$ as follows. For every $r \in X$, if $\mathcal{F}(r) = r$, then $f(r) = 0$. Otherwise, let $i : 1 \leq i \leq d$ be the smallest integer such that $\mathcal{F}_i(r) - r_i \neq 0$, then $f(r) = \text{sign}(\mathcal{F}_i(r) - r_i) \cdot e^i$.

3.2 The Key Lemma

We shall prove the key lemma, Lemma 2, through a sequence of definitions, structures, and their properties.

Lemma 2 *Let $M = (\mathcal{F}, X, \mathcal{S})$ be a simplicial direction-preserving map where \mathcal{F} is from X to \overline{X} , and $G = (f, X, \mathcal{S})$ be the direction function built from M using Construction 2. Then either f has a fixed point in B_X or N_G is odd.*

From Construction 2, every fixed point of f must also be a fixed point of \mathcal{F} . Therefore, Theorem 3 follows directly from Theorem 2 and Lemma 2.

We now outline the proof of Lemma 2.

Proof sketch Let $n = \max_{r \in X, 1 \leq i \leq d} |r_i|$, then we can scale down set X to get $X' \subset (-1, 1)^d$ by setting

$$X' = \left\{ \frac{r}{(n+1)}, r \in X \right\}.$$

We also get a simplicial decomposition \mathcal{S}' of X' from \mathcal{S} , using the one-to-one correspondence between X and X' . We let \mathcal{F}' denote the map from X' to \overline{X}' where

$$\mathcal{F}'(r) = \frac{\mathcal{F}((n+1)r)}{n+1}.$$

Let G' be the function constructed from map $M' = (\mathcal{F}', X', \mathcal{S}')$ using Construction 2, then it is easy to check that $N_G = N_{G'}$. Therefore, we only need to prove the lemma for maps $M = (\mathcal{F}, X, \mathcal{S})$ with $X \subset (-1, 1)^d$. From now on, we always assume that $X \subset (-1, 1)^d$.

If f has a fixed point $r \in B_X$, then the lemma is proven. Otherwise, we need to extend (by applying Construction 1 for a number of times) $G = (f, X, \mathcal{S})$ to be a new function $G^* = (f^*, X^*, \mathcal{S}^*)$ such that

$$X \subset X^*, \quad \overline{X^*} = [-1, 1]^d, \quad \text{and} \quad \mathcal{S} \subset \mathcal{S}^*.$$

We will describe the construction of G^* in Sect. 3.2.1. In Sect. 3.2.2, we will show that, if G is simplicial direction-preserving, then its extension G^* is also simplicial direction-preserving. Finally, in Sect. 3.2.3, we will prove the following two properties about functions G and G^* :

Property 1 N_{G^*} is odd; and

Property 2 $N_G \equiv N_{G^*} \pmod{2}$.

Clearly, Lemma 2 follows directly from these two properties. □

3.2.1 The Construction of Function G^*

Before describing the construction of G^* , it should be emphasized that, for any point $r \in X$, the value of f at r (as described in Construction 2) is decided by the first

non-zero component of $\mathcal{F}(r) - r$. Therefore, to evaluate $f(r)$, we need to check the components of $\mathcal{F}(r) - r$, in the order of indices from lower to higher, until a non-zero component is found. However, when constructing G^* , we first extend G along the d -th coordinate, then along the $(d - 1)$ -st, \dots , and finally along the first coordinate. The use of reversed orders in these two constructions is very important, as will become clear in the proof of Proposition 4.

We start with some notations used in the construction of function G^* . For point $r \in \mathbb{R}^d$, $k : 1 \leq k \leq d$ and $c \in \mathbb{R}$, we use $r[k \rightarrow c]$ to denote the point $r' \in \mathbb{R}^d$ such that $r'_k = c$ and $r'_i = r_i$ for all $i : 1 \leq i \neq k \leq d$. For $X \subset \mathbb{R}^d$, we use $X[k \rightarrow c]$ to denote $\{r[k \rightarrow c], r \in X\}$.

Definition 8 (Extension along the k th coordinate) Let $G = (f, X, \mathcal{S})$ be a triple such that: X is a finite subset of $[-1, 1]^d$; every point $r \in X$ satisfies $|r_k| < 1$; \mathcal{S} is a simplicial decomposition of X ; and $f : X \rightarrow \{0, \pm e^1, \dots, \pm e^d\}$ is a direction function² on X . Triple $G' = (f', X', \mathcal{S}')$ is said to be an extension of G along the k th coordinate if the following properties are satisfied:

1. \mathcal{S}' is a simplicial decomposition of X' where

$$X' = X \cup X[k \rightarrow +1] \cup X[k \rightarrow -1]; \tag{1}$$

2. $\mathcal{S} \subset \mathcal{S}'$; and
3. For any $r \in X$, $f'(r) = f(r)$, $f'(r[k \rightarrow +1]) = -e^k$, $f'(r[k \rightarrow -1]) = +e^k$.

Proposition 1 guarantees the existence of G' .

Proposition 1 For every triple $G = (f, X, \mathcal{S})$ which satisfies all the conditions in Definition 8, there exists a triple $G' = (f', X', \mathcal{S}')$ which is an extension of G along the k th coordinate.

Proof According to Definition 8, it suffices to construct a simplicial decomposition \mathcal{S}' for X' (as defined in (1)) satisfying $\mathcal{S} \subset \mathcal{S}'$.

Actually, \mathcal{S}' can be built as follows. We sort all the points in $X' - X$ under the lexicographic order,³ add them into X one by one, and extend the original decomposition \mathcal{S} step by step using the method described in Construction 1. □

We now define function G^* as an extension of G .

Definition 9 G^* is said to be an extension of G , if there is a sequence $G^{d+1}, G^d, \dots, G^2, G^1$ such that $G^{d+1} = G$, $G^* = G^1$ and G^i is an extension of G^{i+1} along the i th coordinate, for all $i : 1 \leq i \leq d$.

²Here we only require f to be a direction function from X to $\{0, \pm e^1, \dots, \pm e^d\}$, and G is *not* necessary to be simplicial direction-preserving.

³The lexicographic order over \mathbb{R}^d is defined as follows: For any two $r^1 \neq r^2 \in \mathbb{R}^d$, $r^1 < r^2$ if there exists $i : 1 \leq i \leq d$ such that $r^1_i < r^2_i$ and $r^1_j = r^2_j$ for all $j : 1 \leq j < i$.

The existence of G^* (and also the sequence $G = G^{d+1}, G^d, \dots, G^2, G^1 = G^*$) is guaranteed by Lemma 1.

3.2.2 Function G^* is Simplicial Direction-Preserving

We let $G^k = (f^k, X^k, S^k)$ for all $k : 1 \leq k \leq d$. Before proving G^* is simplicial direction-preserving, we need the following propositions about G^k .

Proposition 2 *If $r \in \overline{X^k}$, $r' \in [-1, 1]^d$ and $r_i = r'_i$ for all $i : 1 \leq i < k$, then $r' \in \overline{X^k}$.*

The following two corollaries can be derived from Proposition 2.

Corollary 1 $\overline{X^*} = [-1, 1]^d$.

Definition 10 Vector $v \in \mathbb{R}^d$ is *outgoing* (with respect to \overline{X}) at $r \in X$, if for any $\epsilon > 0$, we have $r + \epsilon v \notin \overline{X}$.

Corollary 2 *If $v = e^{k-1}$ or $-e^{k-1}$ is outgoing (with respect to $\overline{X^k}$) at $r \in X^k$, then*

$$r + \epsilon v + \sum_{i=k}^d c^i e^i \notin \overline{X^k}, \quad \text{for all } \epsilon > 0, c^i \in \mathbb{R}.$$

The following proposition concerning extensions along the k th coordinate, where $1 \leq k \leq d$, is easy to prove.

Proposition 3 *Let $G = (f, X, S)$ be a triple that satisfies all the conditions in Definition 8, and $G' = (f', X', S')$ be an extension of G along the k th coordinate. For any $r \in X$, if $+e^k$ (or $-e^k$) is not outgoing (with respect to \overline{X}) at r , then no pair (r', S) , where $r' \in X' - X$ and $S \in S' - S$, can satisfy $r'_k = +1$ (or -1) and $r, r' \in V_S$.*

Now we can show that, when G is simplicial direction-preserving, any of its extensions G^* must be simplicial direction-preserving.

Proposition 4 *Let $M = (\mathcal{F}, X, S)$ be a simplicial direction-preserving map and \mathcal{F} maps X to \overline{X} . Let $G = (f, X, S)$ be the simplicial direction-preserving function constructed from M using the method in Construction 2, then any extension G^* of G is also simplicial direction-preserving.*

Proof We will use induction on k to show that $G^k = (f^k, X^k, S^k)$ is a simplicial direction-preserving function. The case for $k = d + 1$ is trivial. For case $k \leq d$, if G^k is not direction-preserving, then there must exist a simplex $S \in S^k - S^{k+1}$ and $r^1, r^2 \in V_S$ such that $f^k(r^1)$ and $f^k(r^2)$ are opposite. As G^{k+1} is direction-preserving, one of these two points belongs to X^{k+1} and the other is added into X^k when G^{k+1} is extended along the k th coordinate.

Without loss of generality, we assume $r^1 \in X^{k+1}$ and $r_k^2 = 1$. As a result,

$$f^k(r^2) = -e^k \quad \text{and} \quad f^k(r^1) = +e^k,$$

as $f^k(r^1)$ is opposite to $f^k(r^2)$. The value of $f^k(r^1)$ implies $r^1 \in X$ and $f(r^1) = +e^k$. According to the way we construct G from M , $\mathcal{F}(r^1)$ must satisfy

$$\mathcal{F}_i(r^1) = r_i^1 \quad \text{for all } i : 1 \leq i < k \quad \text{and} \quad \mathcal{F}_k(r^1) > r_k^1.$$

From Corollary 2, we see that $+e^k$ is not outgoing (with respect to $\overline{X^{k+1}}$) at $r^1 \in X^{k+1}$, since $\mathcal{F}(r^1) \in \overline{X} \subset \overline{X^{k+1}}$. Therefore, Proposition 3 asserts that no pair (r^2, S) , where $r^2 \in X^k - X^{k+1}$ and $S \in \mathcal{S}^k - \mathcal{S}^{k+1}$, can satisfy $r_k^2 = +1$ and $r^1, r^2 \in V_S$. This contradicts with our assumption. \square

3.2.3 Proofs of Properties 1 and 2

Finally, we finish the proof of Lemma 2 by proving Properties 1 and 2 as stated in the proof sketch.

Property 1 Let $G = (f, X, \mathcal{S})$ be a triple such that: $X \subset (-1, 1)^d$ is finite; \mathcal{S} is a simplicial decomposition of X ; and $f : X \rightarrow \{0, \pm e^1, \dots, \pm e^d\}$ is a direction function⁴ on X . If $G^* = (f^*, X^*, \mathcal{S}^*)$ is an extension of G , then N_{G^*} is odd.

Proof We will use induction on d . The base case for $d = 1$ is trivial. For case $d \geq 2$, we first prove that every bad $(d - 1)$ -simplex $S \in \mathcal{S}^*$ (relative to G^*) on the boundary of $[-1, 1]^d$ satisfies $r_d = -1$ for any $r \in V_S$. Let $S \in F_{\mathcal{S}^*}$ be such a simplex, then there exists $r \in V_S$ satisfies $f^*(r) = +e^d$ according to the definition of badness. Thus r is added into X when it is extended along the d th coordinate. We have $r_d = -1$ and $-1 < r_i < 1$ for all $i : 1 \leq i < d$. This shows that S is covered by the hyperplane H which passes $0[d \rightarrow -1]$ and is perpendicular to $+e^d$.

Let N be the number of bad $(d - 1)$ -simplices (relative to G^*) in H , then we only need to prove N is odd. Let $X' = P(X)$ and $X'' = P(H \cap X^*)$, where projection $P(r) = (r_1, r_2, \dots, r_{d-1}) \in \mathbb{R}^{d-1}$, then \mathcal{S}' and \mathcal{S}'' are simplicial decompositions of X' and X'' , respectively, where

$$\begin{aligned} \mathcal{S}' &= \{P(S) \mid S \in \mathcal{S}^* \text{ and } V_S \subset X[d \rightarrow -1]\}, \\ \mathcal{S}'' &= \{P(S) \mid S \in \mathcal{S}^* \text{ and } S \subset H\}. \end{aligned}$$

Let f' be an arbitrary direction function on X' , then we extend it to be f'' on X'' as follows: For any $r \in X''$, $f''(r) = f'(r)$ if $r \in X'$; and

$$f''(r) = f^*((r_1, r_2, \dots, r_{d-1}, -1))$$

otherwise. It's easy to check that $G'' = (f'', X'', \mathcal{S}'')$ is an extension of $G' = (f', X', \mathcal{S}')$ in the $(d - 1)$ -dimensional space.

Using arguments which are similar to those in the proof of Lemma 1, the parity of N is the same as the number of bad $(d - 2)$ -simplices on the boundary of G'' . We know that the latter is odd according to the induction hypothesis. Thus, Property 1 is proven. \square

⁴Again, $G = (f, X, \mathcal{S})$ is not necessary to be simplicial direction-preserving.

Property 2 Let $M = (\mathcal{F}, X, \mathcal{S})$ be a simplicial direction-preserving map and \mathcal{F} maps X to \overline{X} . If $G = (f, X, \mathcal{S})$ (that is, the function constructed from M using the method in Construction 2) has no fixed point on the boundary B_X , then any extension G^* of G satisfies $N_{G^*} \equiv N_G \pmod{2}$.

Proof With arguments which are similar to those in the proof of Lemma 1, we have $N_G \equiv R^1 \pmod{2}$ and $N_{G^*} \equiv R^2 \pmod{2}$ where

$$R^1 = \sum_{S^d \in \mathcal{S}} |\{\text{bad } (d - 1)\text{-simplices (relative to } G) \text{ in } S^d\}|,$$

$$R^2 = \sum_{S^d \in \mathcal{S}^*} |\{\text{bad } (d - 1)\text{-simplices (relative to } G^*) \text{ in } S^d\}|$$

(here we use S^d to denote d -simplices in \mathcal{S} and \mathcal{S}^*). As $\mathcal{S} \subset \mathcal{S}^*$, we have

$$R^2 - R^1 = \sum_{S^d \in \mathcal{S}^* - \mathcal{S}} |\{\text{bad } (d - 1)\text{-simplices (relative to } G^*) \text{ in } S^d\}|.$$

On the other hand, since G has no fixed point in B_X , every d -simplex $S^d \in \mathcal{S}^* - \mathcal{S}$ satisfies $0 \notin f^*(V_{S^d})$. As G^* is simplicial direction-preserving according to Proposition 4, the number of bad $(d - 1)$ -simplices in S^d is either 0 or 2. Thus we have $R^2 - R^1 \equiv 0 \pmod{2}$, and Property 2 is proven. \square

3.3 Theorem 1 as a Corollary of Theorem 3

Finally, we prove the fixed point theorem of Iimura, Murota, and Tamura as a corollary of Theorem 3.

Lemma 3 (Property of integrally convex sets [1]) *For every integrally convex set $X \subset \mathbb{Z}^d$, there exists a simplicial decomposition \mathcal{S} of \overline{X} such that, for every $x \in \overline{X}$, letting $S_x \in \mathcal{S}$ be the smallest simplex containing x , then all the vertices of S_x belong to $N(x) = \{r \in \mathbb{Z}^d \mid \|r - x\|_\infty < 1\}$.*

Let \mathcal{F} be a hypercubic direction-preserving map from integrally convex set $X \subset \mathbb{Z}^d$ to \overline{X} , and \mathcal{S} be a simplicial decomposition of set X which satisfies the condition in Lemma 3, then one can check that $M = (\mathcal{F}, X, \mathcal{S})$ is a simplicial direction-preserving map from X to \overline{X} . By Theorem 3, we know that there is a fixed point of \mathcal{F} in X .

Moreover, the argument above shows that the theorem of Iimura, Murota, and Tamura can be greatly strengthened. Actually, map \mathcal{F} is not necessary to be hypercubic direction-preserving. Being simplicial direction-preserving relative to some simplicial decomposition of X is sufficient to ensure the existence of a fixed point in X .

4 Concluding Remarks

In this paper, we generalize the concept of direction-preserving maps and characterize a new class of discrete maps (and functions) over simplicial structures. The

primitive discrete fixed point theorem is then stated, which is based on the counting of bad $(d - 1)$ -simplices on the boundary. It is then applied to prove the discrete Brouwer's fixed point theorem (Theorem 3) which is more general than the one of Iimura, Murota and Tamura.

A stronger notion of simplicially local gross direction-preserving functions was introduced by van der Laan, Talman and Yang [15]. They derived discrete fixed point theorems, stronger in the requirement of the functions, using the adapted $2n$ -ray labeling algorithm in a path-following fashion. Ours can be employed to derive a divide-&-conquer algorithm, originated from our earlier work on the algorithmic complexity of the discrete fixed point problem [3]. Ours is more general in terms of the relaxed boundary condition for the function values, in the tradition of Brouwer's degree theory approach [17].

It should be noticed that, in the simplicial approach, the property of a map $M = (\mathcal{F}, X, \mathcal{S})$ being simplicial direction-preserving heavily relies on the simplicial decomposition \mathcal{S} of X in the triple. For example, given an \mathcal{F} on X , there might exist two different decompositions \mathcal{S}_1 and \mathcal{S}_2 of X , such that, $(\mathcal{F}, X, \mathcal{S}_1)$ is simplicial direction-preserving, while $(\mathcal{F}, X, \mathcal{S}_2)$ is not.

Besides, when proving Theorem 1 as a corollary of Theorem 3, we use the property that for any hypercubic direction-preserving \mathcal{F} over an integrally convex set $X \subset \mathbb{Z}^d$, there exists a decomposition \mathcal{S} of X , such that, $(\mathcal{F}, X, \mathcal{S})$ is simplicial direction-preserving. However, this conversion is not always possible for general sets $X \subset \mathbb{Z}^d$. Actually, there exist sets $X \subset \mathbb{Z}^d$ such that even if \mathcal{F} is hypercubic direction-preserving over X , no decomposition \mathcal{S} of X can make $(\mathcal{F}, X, \mathcal{S})$ simplicial direction-preserving. A counterexample the readers may want to check is when $X \subset \mathbb{Z}^2$ is a long and skew triangle.

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References

1. Iimura, T., Murota, K., Tamura, A.: Discrete fixed point theorem reconsidered. *J. Math. Econ.* **41**(8), 1030–1036 (2005)
2. Iimura, T.: A discrete fixed point theorem and its applications. *J. Math. Econ.* **39**(7), 725–742 (2003)
3. Chen, X., Deng, X.: On algorithms for discrete and approximate Brouwer fixed points. In: Proceedings of the 37th Annual ACM Symposium on Theory of Computing, pp. 323–330 (2005)
4. van der Laan, G., Talman, D., Yang, Z.: Solving discrete zero point problems. *Math. Program.* **108**(1), 127–134 (2006)
5. Friedl, K., Ivanyosy, G., Santha, M., Verhoeven, F.: On the black-box complexity of Sperner's lemma. In: Proceedings of the 15th International Symposium on Fundamentals of Computation Theory, pp. 245–257 (2005)
6. Chen, X., Deng, X.: Lattice embedding of direction-preserving correspondence over integrally convex set. In: Proceedings of the 2nd International Conference on Algorithmic Aspects in Information and Management, pp. 53–63 (2006)
7. Sperner, E.: Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes. *Abh. Math. Semin. Univ. Hamburg* **6**, 265–272 (1928)
8. Scarf, H.: The approximation of fixed points of a continuous mapping. *SIAM J. Appl. Math.* **15**(5), 1328–1343 (1967)
9. Lemke, C., Howson, J.T.: Equilibrium points of bimatrix games. *J. Soc. Ind. Appl. Math.* **12**, 413–423 (1964)

10. Eaves, B., Scarf, H.: The solutions of systems of piecewise linear equations. *Math. Oper. Res.* **1**, 1–27 (1976)
11. Eaves, B.: Homotopies for computation of fixed points. *Math. Program.* **3**, 1–22 (1972)
12. Eaves, B., Saigal, R.: Homotopies for computation of fixed points on unbounded regions. *Math. Program.* **3**, 225–237 (1972)
13. Todd, M.: *Computation of Fixed Points and Applications*. Springer, Berlin (1976)
14. van der Laan, G., Talman, D.: A class of simplicial restart fixed point algorithms without an extra dimension. *Math. Program.* **20**, 33–48 (1981)
15. van der Laan, G., Talman, D., Yang, Z.: A vector labeling method for solving discrete zero point and complementarity problems. *SIAM J. Optim.* **18**(1), 290–308 (2007)
16. Chen, X., Deng, X.: A simplicial approach for discrete fixed point theorems. In: *Proceedings of the 12th Annual International Computing and Combinatorics Conference*, pp. 3–12 (2006)
17. Brouwer, L.: Über Abbildung von Mannigfaltigkeiten. *Math. Ann.* **71**, 97–115 (1912)
18. Crescenzi, P., Silvestri, R.: Sperner’s lemma and robust machines. *Comput. Complex.* **7**(2), 163–173 (1998)
19. Hirsch, M., Papadimitriou, C., Vavasis, S.: Exponential lower bounds for finding Brouwer fixed points. *J. Complex.* **5**, 379–416 (1989)
20. Daskalakis, C., Goldberg, P., Papadimitriou, C.: The complexity of computing a Nash equilibrium. In: *Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, pp. 71–78 (2006)
21. Chen, X., Deng, X.: 3-Nash is PPAD-complete. *ECCC Report*, TR05-134 (2005)
22. Daskalakis, C., Papadimitriou, C.: Three-player games are hard. *ECCC Report*, TR05-139 (2005)
23. Chen, X., Deng, X.: Settling the complexity of two-player Nash equilibrium. In: *Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science*, pp. 261–272 (2006)
24. Chen, X., Deng, X.: On the complexity of 2d discrete fixed point problem. In: *Proceedings of the 33rd International Colloquium on Automata, Languages and Programming*, pp. 489–500 (2006)
25. Chen, X., Deng, X., Teng, S.H.: Computing Nash equilibria: Approximation and smoothed complexity. In: *Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science*, pp. 603–612 (2006)
26. Ziegler, G.: *Lectures on Polytopes*. Springer, New York (1982)