Theoretical Computer Science 562 (2015) 46–56

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

On t-extensions of the Hankel determinants of certain automatic sequences

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ARTICLE INFO

Article history:
Received 7 February 2014
Received in revised form 17 August 2014
Accepted 13 September 2014
Available online 21 September 2014
Communicated by G. Ausiello

Keywords:
Hankel determinant
t-Hankel determinant
Permutation
Involution
Thue–Morse sequence
Period-doubling sequence
Regular paperfolding sequence

ABSTRACT

In 1998, Allouche, Peyrière, Wen and Wen considered the Thue–Morse sequence, and proved that all the Hankel determinants of the period-doubling sequence are odd integers. We speak of t-extension when the entries along the diagonal in the Hankel determinant are all multiplied by t. We prove that the t-extension of each Hankel determinant of the period-doubling sequence is a polynomial in t, whose leading coefficient is the *only one* to be an odd integer. Our proof makes use of the combinatorial set-up developed by Bugeaud and Han, which appears to be very suitable for this study, as the parameter t counts the number of fixed points of a permutation. Finally, we prove that all the t-extensions of the Hankel determinants of the regular paperfolding sequence are polynomials in t of degree less than or equal to 3.

1. Introduction

The Thue–Morse sequence (or the infinite Thue–Morse word, the Thue–Morse morphism) is widely studied in Theoretical Computer Science (see, for example, [17,9,19]) and has many applications in different fields. In 1998, Allouche, Peyrière, Wen and Wen considered the Thue–Morse sequence, and proved that all the Hankel determinants of the period-doubling sequence (which is derived from the Thue–Morse sequence) are odd integers [4]. This result allowed Bugeaud [5] to prove that the irrationality exponents of the Thue–Morse–Mahler numbers are exactly 2.

In the present paper we are interested in trying to understand better why the late determinants are odd integers. We speak of *t-extension* when the entries along the diagonal in the Hankel determinant are all multiplied by t. We prove that the t-extension of each Hankel determinant of the period-doubling sequence is a polynomial in t, whose leading coefficient is the *only one* to be an odd integer. Clearly, our result generalizes the APWW theorem. The proof makes use of the combinatorial set-up developed by Bugeaud and Han [6].

The t-extension of the Hankel determinants introduced in the paper is a new concept for studying the *automatic sequences*. As another example, we prove that all the t-extensions of the Hankel determinants of the regular paperfolding sequence are polynomials in t of degree less than or equal to 3.

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[1] The first author was supported by the National Basic Research Program of China Grants 2011CBA00300, 2011CBA00301, the National Natural Science Foundation of China Grants 61033001, 61361136003.

http://dx.doi.org/10.1016/j.tcs.2014.09.032
0304-3975/

Hankel determinant is a very classical mathematical subject widely studied in Linear Algebra, Combinatorics, Number Theory and Algorithmics (see, for example, [14,20,15,11,10]). Let x be an indeterminate. We identify each sequence

$$\mathbf{c} = (c_0, c_1, c_2, \ldots)$$

with its generating function

$$C = C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

In general, the constant term c_0 will be equal to 1. For $k \geq 1$ and $p \geq 0$ let

$$H_k^p(C) = H_k^p(\mathbf{c}) := \begin{vmatrix} c_p & c_{p+1} & \cdots & c_{p+k-1} \\ c_{p+1} & c_{p+2} & \cdots & c_{p+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p+k-1} & c_{p+k} & \cdots & c_{p+2k-2} \end{vmatrix} \tag{1}$$

be the (p,k)-order Hankel determinant of the series $C(x)$ or of the sequence $\mathbf{c} = (c_0, c_1, c_2, \ldots)$. We write $H_k(C) := H_k^0(C)$ for short. The Thue–Morse sequence $\mathbf{e} = (1, -1, -1, 1, \ldots)$ can be defined by the generating function

$$P_2(x) = \sum_{k=0}^{\infty} e_k x^k = \prod_{k=0}^{\infty} \left(1 - x^{2^k}\right). \tag{2}$$

Then, the *period-doubling sequence* $\mathbf{d} = (1, 0, 1, 1, 1, 0, \ldots)$ is derived from the Thue–Morse sequence by defining

$$d_k = \frac{1}{2}|e_k - e_{k+1}| \quad (k \geq 0). \tag{3}$$

The result obtained by Allouche, Peyrière, Wen and Wen can be stated as follows [4].

Theorem 1. *For every positive integer k the Hankel determinant $H_k(\mathbf{d})$ of the period-doubling sequence $\mathbf{d}$ is an odd integer. In other words,*

$$H_k(\mathbf{d}) \equiv 1 \pmod 2. \tag{4}$$

Coons [7] considered the series

$$G_{0,0}(x) := \sum_{n=0}^{\infty} \frac{x^{2^n - 1}}{1 - x^{2^n}} \tag{5}$$

and proved that all the Hankel determinants $H_k(G_{0,0})$ of the power series $G_{0,0}(x)$ are odd integers. As shown in [6], Coons's result is essentially equivalent to Theorem 1.

Let t be a parameter. We speak of *t-extension* when the entries along the diagonal in the (p,k)-order Hankel determinant are all multiplied by t. In other words, we define the *t-Hankel determinant* of the formal power series $C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ (or of the sequence $\mathbf{c} = (c_0, c_1, c_2 \ldots)$) by

$$H_k^p(\mathbf{c}, t) := H_k^p(C, t) := \begin{vmatrix} c_p t & c_{p+1} & \cdots & c_{p+k-1} \\ c_{p+1} & c_{p+2} t & \cdots & c_{p+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p+k-1} & c_{p+k} & \cdots & c_{p+2k-2} t \end{vmatrix}. \tag{6}$$

Obviously, the above t-Hankel determinant (6) is a polynomial in t of degree less than or equal to k, which is equal to the traditional Hankel determinant (1) when $t = 1$. Again, we write $H_k(C, t) := H_k^0(C, t)$. Our main result is stated as follows.

Theorem 2. *For every positive integer k the t-Hankel determinant $H_k(\mathbf{d}, t)$ of the period-doubling sequence $\mathbf{d}$ is a polynomial in t of degree k, whose leading coefficient is the only one to be an odd integer. In other words,*

$$H_k(\mathbf{d}, t) \equiv t^k \pmod 2. \tag{7}$$

In the following table we reproduce the first few values of the t-Hankel determinants of the period-doubling sequence $\mathbf{d}$. We see that all the coefficients are even integers, except the coefficient of t^k. When $t = 1$ we recover Theorem 1.

k	$H_k(\mathbf{d}, t)$	$H_k(\mathbf{d}, t) \pmod 2$	$H_k(\mathbf{d}, 1)$
0	1	1	1
1	t	t	1
2	t^2	t^2	1
3	$t^3 - 2t$	t^3	−1
4	$t^4 - 4t^2$	t^4	−3
5	$t^5 - 6t^3 + 2t^2 + 4t$	t^5	1
6	$t^6 - 8t^4 + 4t^3 + 12t^2 - 8t$	t^6	1
7	$t^7 - 12t^5 + 10t^4 + 24t^3 - 24t^2$	t^7	−1
8	$t^8 - 16t^6 + 16t^5 + 48t^4 - 64t^3$	t^8	−15

Actually, Theorem 1 has three proofs. The first one is due to Allouche, Peyrière, Wen and Wen by using determinant manipulation [4], which consists of proving sixteen recurrence relations between determinants. The second one is a combinatorial proof derived by Bugeaud and Han [6]. The third proof is very short by using the Jacobi continued fraction algebra [13]. For proving Theorem 2 it seems the method used in the second proof is more suitable, as the parameter t counts the number of fixed points of permutations (see Eq. (23)).

Some basic notations and properties on permutations and involutions are collected in Section 2, including the statement of the key combinatorial result, namely, Theorem 5. The proof of the main result (Theorem 2) is found in Section 3, after proving Theorem 5.

The *regular paperfolding sequence* $\mathbf{r} = (1, 1, 0, 1, 1, 0, 0, \ldots)$ can be defined by the generating function [1,2]

$$G_{0,2}(x) = \sum_{n\geq 0} r_n x^n = \sum_{n=0}^{\infty} \frac{x^{2^n-1}}{1-x^{2^{n+2}}}. \tag{8}$$

Coons and Vrbik conjectured [8] and Guo, Wu and Wen [12] proved the following result.

Theorem 3. *The parities of the Hankel determinants of the regular paperfolding sequence* $\mathbf{r}$ *are periodic of period* 10. *More precisely, we have*

$$\big(H_k(\mathbf{r}) \pmod 2\big)_{k\geq 0} = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^{\omega}, \tag{9}$$

where u^{ω} is the infinite sequence obtained by repeating u an infinity of times (see, for example, [18, p. 14]).

Our second result is stated next.

Theorem 4. *For every positive integer k the t-Hankel determinant $H_k(\mathbf{r}, t)$ of the regular paperfolding sequence* $\mathbf{r}$ *is a polynomial in t of degree less than or equal to* 3.

Theorem 4 is proved in Section 4. In the following table we reproduce the first few values of the t-Hankel determinants of the regular paperfolding sequence $\mathbf{r}$. We see that all the $H_k(\mathbf{r}, t)$'s are polynomials of degree less than or equal to 3.

k	$H_k(\mathbf{r}, t)$	k	$H_k(\mathbf{r}, t)$
0	1	5	$-t^3 + 2t^2 + 2t - 2$
1	t	6	$2t^2 - 2t - 4$
2	-1	7	$3t^3 - 6t^2 - 7t + 6$
3	$-2t$	8	$-9t^2 + 12t + 16$
4	$-t^2 + 2t + 1$	9	$-15t^3 + 20t^2 + 46t - 40$

As earlier mentioned, Theorem 2 is a t-extension of Theorem 1. However, Theorem 3 cannot be obtained from Theorem 4 by specializing $t = 1$. The following problem remains unsolved.

Problem. *Find a true t-extension of Theorem 3. In other words, find a property of the t-Hankel determinants of the regular paperfolding sequence, which implies relation (9) when $t = 1$.*

2. Permutations and involutions

A combinatorial set-up, based on permutations and involutions, for studying the Hankel determinants of the period-doubling sequence was introduced in [6]. We propose a refinement of such a combinatorial set-up for studying t-Hankel determinants. The following infinite sets of integers play an important role.

$$
\begin{aligned}
N &= \mathbb{N} = \{0, 1, 2, 3, \ldots\}, \\
J &= \left\{(2n+1)2^{2k} - 1 \;\middle|\; n, k \in N\right\} = \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \ldots\}, \\
J^* &= \left\{(2n+1)2^{2k} - 1 \;\middle|\; n, k \in N, k > 0\right\} = \{3, 11, 15, 19, 27, 35, \ldots\}, \\
K &= N \setminus J = \left\{(2n+1)2^{2k+1} - 1 \;\middle|\; n, k \in N\right\} = \{1, 5, 7, 9, 13, 17, \ldots\}, \\
L &= N \setminus J^* = K \cup \{2n \mid n \in N\} = \{0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 12, \ldots\}, \\
P &= \left\{k \;\middle|\; k \equiv 0, 3 \pmod 4\right\} = \{0, 3, 4, 7, 8, 11, 12, 15, 16, \ldots\}, \\
Q &= \left\{k \;\middle|\; k \equiv 1, 2 \pmod 4\right\} = \{1, 2, 5, 6, 9, 10, 13, 14, 17, \ldots\}.
\end{aligned}
$$

For each infinite set A let $A|_m$ be the finite set composed of the smallest m integers in A.

Let $\mathfrak{S}_m = \mathfrak{S}_{\{0,1,\ldots,m-1\}}$ be the set of all permutations on $N|_m$. A permutation is represented by the product of its disjoint cycles. For example, the permutation $\sigma = (0, 5)(1)(2, 6, 3)(4, 8)(7)$ is an element from $\mathfrak{S}_9$. An *involution* is a permutation σ such that $\sigma = \sigma^{-1}$. Equivalently, a permutation σ is an involution if each cycle of σ is either a fixed point (b) or a *transposition* (c, d). For instance, $\sigma = (0, 5)(1)(2, 6)(3)(4, 8)(7) \in \mathfrak{S}_9$ is an involution.

Definition 1. For each set B, a transposition (c, d) is said "*in B*" if $c + d \in B$. In this case, we write $(c, d) \in B$.

Definition 2. For a non-negative integer k and two sets of positive integers A, B such that A is finite, let $\mu(A, k, B)$ be the number of involutions σ in $\mathfrak{S}_A$ having exactly k transpositions such that all transpositions of σ are in B.

The following key result is useful for proving Theorem 2 (see Section 3).

Theorem 5. *For $m \geq 1$ and $k \geq 0$, we have*

$$\mu(N|_m, k, J) \equiv \begin{cases} 1 \pmod 2, & \text{if } k = 0; \\ 0 \pmod 2, & \text{if } k \geq 1. \end{cases} \tag{10}$$

The proof of Theorem 5 is given in Section 3, with the help of several lemmas stated in the end of this section.

Lemma 6. *For $m \geq 1$ and $k \geq 0$ we have*

$$\mu(N|_m, k, J) = \mu(P|_m, k, L) \tag{11}$$

and

$$\mu\big(P|_m, k, J^*\big) = \mu\big(Q|_m, k, J^*\big). \tag{12}$$

Proof. We define two transformations:

$$
\begin{aligned}
\beta : N \to P; \qquad & \ell \mapsto \begin{cases} 2\ell, & \text{if } \ell \text{ is even;} \\ 2\ell + 1, & \text{if } \ell \text{ is odd;} \end{cases} \\
\delta : P \to Q; \qquad & \ell \mapsto \begin{cases} \ell + 1, & \text{if } \ell \text{ is even;} \\ \ell - 1, & \text{if } \ell \text{ is odd.} \end{cases}
\end{aligned}
$$

The transformation β is a bijection of $N|_m$ onto $P|_m$, and can be extended to the set of all involutions on $N|_m$ by applying β on every letter of the involutions. For example

$$\beta\big((7)(0, 5), (6, 3), (1), (8, 2), (4)\big) = (15)(0, 11)(12, 7)(3)(16, 4)(8).$$

We now claim that, for any $c, d \in N|_m$, the transposition (c, d) is in J if and only if $(\beta(c), \beta(d))$ is in L. The proof of this claim works by distinguishing the parities of c and d: (i) if c and d are even, then $\beta(c) = 2c$ and $\beta(d) = 2d$, so that $\beta(c) + \beta(d)$ is even and is in L; (ii) if c and d are odd, then $\beta(c) = 2c + 1$ and $\beta(d) = 2d + 1$, so that $\beta(c) + \beta(d)$ is even and is in L; (iii) if $c + d \in J$ and one of the integers c, d is even, the other being odd. Then,

$$\beta(c) + \beta(d) = 2c + 2d + 1 = 2 \times \big((2n+1)2^{2k} - 1\big) + 1 = (2n+1)2^{2k+1} - 1 \in L.$$

The converse is proved in the same manner. Thus, Eq. (11) holds.

The transformation δ is a bijection of $P|_m$ onto $Q|_m$, and can be extended to the set of all involutions on $P|_m$ by applying δ on every letter of the involutions. For example

$$\delta\big((15)(0, 11)(12, 7)(4)(16, 3)(8)\big) = (14)(1, 10)(13, 6)(5)(17, 2)(9).$$

If the transposition (c, d) is in J^* and $c, d \in P$, then one of the integers c, d is even, the other being odd. Hence,

$$\delta(c) + \delta(d) = c - 1 + d + 1 = c + d \in J^*.$$

Thus, Eq. (12) is proved. □

Lemma 7. *For each $k \ge 0$ we have*

$$\mu(N|_{2n}, k, J^*) \equiv \begin{cases} 0 \pmod 2, & \text{if } k \text{ is odd;} \\ \mu(P|_n, k/2, J^*) \pmod 2, & \text{if } k \text{ is even.} \end{cases} \tag{13}$$

Proof. It is easy to see that, if $c + d \in J^*$, then $c + d \equiv 3 \pmod 4$. Thus, both c and d belong either to P or to Q. Hence,

$$\mu(N|_{2n}, k, J^*) = \sum_{i+j=k} \mu(P|_n, i, J^*)\, \mu(Q|_n, j, J^*) \tag{14}$$

$$= \sum_{i+j=k} \mu(P|_n, i, J^*)\, \mu(P|_n, j, J^*). \tag{15}$$

The last identity holds by Lemma 6. When $k = 2\ell + 1$ is odd, the right-hand side of Eq. (15) is equal to

$$2\sum_{i=0}^{\ell} \mu(P|_n, i, J^*)\, \mu(P|_n, 2\ell + 1 - i, J^*) \equiv 0 \pmod 2.$$

When $k = 2\ell$ is even, we have

$$\begin{aligned} &\sum_{i+j=2\ell} \mu(P|_n, i, J^*)\, \mu(P|_n, j, J^*) \\ &= 2\sum_{i=0}^{\ell-1} \mu(P|_n, i, J^*)\, \mu(P|_n, 2\ell - i, J^*) + \mu(P|_n, \ell, J^*)\, \mu(P|_n, \ell, J^*) \\ &\equiv \mu(P|_n, k/2, J^*) \pmod 2. \end{aligned}$$

This completes the proof. □

In what follows, the notation $a \equiv b$ means that the integers a and b are congruent modulo 2 when nothing else is specified.

Lemma 8. *For $m \ge 1$ and $k \ge 1$ we have*

$$\sum_{i=0}^{k} \mu(P|_m, i, J^*) \binom{m - 2i}{2k - 2i} \equiv \mu(P|_m, k, L) \pmod 2. \tag{16}$$

Proof. Recall that $\mu(A, k, B)$ is the number of involutions σ in $\mathfrak{S}_A$ having exactly k transpositions such that all transpositions of σ are in B. For two disjoint sets of integers B_1 and B_2, we define $\mu(A, k_1, k_2, B_1, B_2)$ to be the number of involutions σ in $\mathfrak{S}_A$ having exactly k_1 transpositions in B_1 and k_2 transpositions in B_2 such that all transpositions are in $B_1 \cup B_2$. So that $\mu(A, 0, k_2, B_1, B_2) = \mu(A, k_2, B_2)$.

Let i and j be two non-negative integers such that $0 \le i \le j \le k$. Consider the set $\mathfrak{I}_j$ of involutions σ on $P|_m$ having exactly j transpositions in J^* and $k - j$ transpositions in L and no other transposition. Then, the cardinality of $\mathfrak{I}_j$ is equal to $\mu(P|_m, j, k - j, J^*, L)$. A *marked involution with i colored transpositions* is obtained from an involution $\sigma \in \mathfrak{I}_j$ by coloring i transpositions among the j transpositions in J^*. Let $\mathfrak{I}_{i,j}$ be the set of all those marked involutions with i colored transpositions. The cardinality of $\mathfrak{I}_{i,j}$ is equal to $\binom{j}{i}$ $\mu(P|_m, j, k - j, J^*, L)$. Hence, the cardinality of the set of all marked involutions $\mathfrak{I}_{i,\bullet} = \mathfrak{I}_{i,i} + \mathfrak{I}_{i,i+1} + \cdots + \mathfrak{I}_{i,k}$, where the plus sign "+" means the disjoint union, is equal to

$$\sum_{j=i}^{k} \binom{j}{i} \mu(P|_m, j, k - j, J^*, L). \tag{17}$$

On the other hand, the marked involutions in $\mathfrak{I}_{i,\bullet}$ can be enumerated as follows. Consider the involutions on $P|_m$ that have exactly i transpositions in J^*, which are said to be colored. There are $\mu(P|_m, i, J^*)$ such involutions. Then randomly choose $2k - 2i$ letters from the remaining $m - 2i$ original fixed points on $P|_m$, to generate another $k - i$ transpositions, which are either in J^* or in L. We get a marked involution which has exactly $i + (k - i) = k$ transpositions. Hence, the cardinality of the set $\mathfrak{I}_{i,\bullet}$ is equal to

$$\mu\big(P|_m, i, J^*\big)\binom{m-2i}{2k-2i}(2k-2i-1)(2k-2i-3)\cdots 3\cdot 1. \tag{18}$$

Hence, the two quantities (17) and (18) are equal. We have successively

$$\begin{aligned}
&\sum_{i=0}^{k}\mu\big(P|_m, i, J^*\big)\binom{m-2i}{2k-2i}\\
&\quad\equiv \sum_{i=0}^{k}\mu\big(P|_m, i, J^*\big)\left[\binom{m-2i}{2k-2i}(2k-2i-1)(2k-2i-3)\cdots(3)(1)\right]\\
&\quad= \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\mu\big(P|_m, j, k-j, J^*, L\big)\\
&\quad= \sum_{j=0}^{k}\left(\sum_{i=0}^{j}\binom{j}{i}\right)\mu\big(P|_m, j, k-j, J^*, L\big)\\
&\quad= \sum_{j=0}^{k}2^j\mu\big(P|_m, j, k-j, J^*, L\big)\\
&\quad\equiv \mu\big(P|_m, 0, k, J^*, L\big)\\
&\quad= \mu(P|_m, k, L).
\end{aligned}$$

This completes the proof. $\square$

Remark. The only property on L and J^* actually required for the proof of Lemma 8 is that they are complementary. In other words, we have the following more general statement.[2]

Lemma 9. *For any finite set $A \subset N$, any set $B \subset N$ and any positive integer $k \geq 1$ we have*

$$\sum_{i=0}^{k}\mu(A, i, B)\binom{\#A-2i}{2k-2i} \equiv \mu(A, k, N\setminus B) \pmod 2,$$

where $\#A$ denotes the cardinality of the set A.

3. Proofs of Theorems 5 and 2

Firstly, we establish two lemmas about congruences for binomial coefficients.

Lemma 10. *For $n, k \geq 0$ we have*

$$\sum_{i+j=k}\binom{n}{2i}\binom{n}{2j} \equiv \begin{cases} 0 \pmod 2, & \text{if } k \text{ is odd;}\\ \binom{n}{k} \pmod 2, & \text{if } k \text{ is even.}\end{cases} \tag{19}$$

Proof. If $k = 2\ell + 1$ is odd, then

$$\sum_{i+j=2\ell+1}\binom{n}{2i}\binom{n}{2j} = 2\sum_{i=0}^{\ell}\binom{n}{2i}\binom{n}{4\ell+2-2i} \equiv 0 \pmod 2.$$

If $k = 2\ell$ is even, then

[2] This remark is kindly pointed out by one anonymous referee.

$$\sum_{i+j=2\ell}\binom{n}{2i}\binom{n}{2j}=2\sum_{i=0}^{\ell-1}\binom{n}{2i}\binom{n}{4\ell-2i}+\binom{n}{2\ell}\binom{n}{2\ell}$$
$$\equiv\binom{n}{k}\pmod 2.$$

This completes the proof. □

Lemma 11. *For $n, m, k \geq 0$ such that $n+m$ is odd, we have*

$$\sum_{i+j=k}\binom{n}{2i}\binom{m}{2j}\equiv\binom{n+m}{2k}\pmod 2. \tag{20}$$

Proof. We have

$$\binom{n+m}{2k}=\sum_{i+j=2k}\binom{n}{i}\binom{m}{j}\quad\text{[Vandermonde's identity]}$$
$$=\sum_{i+j=k}\binom{n}{2i}\binom{m}{2j}+\sum_{i+j=k-1}\binom{n}{2i+1}\binom{m}{2j+1}.$$

Since $\binom{2a}{2b+1}$ is even for any positive integers a and b [16],

$$\binom{n}{2i+1}\binom{m}{2j+1}\equiv 0\pmod 2, \tag{21}$$

if n or m is even. This is true because $n+m$ is odd. Eq. (20) holds. □

Secondly, we prove Theorem 5 by induction.

Proof of Theorem 5. When $k=0$, the quantity $\mu(N|_m, k, J)$ counts the involutions σ without any transposition. It means that every letter of σ is a fixed point, so that $\mu(N|_m, 0, J)=1$.

To prove identity (10) for $k\geq 1$ proceed by induction on m. Clearly $\mu(N|_m, k, J)\equiv 0 \pmod 2$ for $k\geq 1$ and $m=1,2$. Notice that any transposition of type $(even, even)$ or (odd, odd) is in J since J contains all even integers. Let $k_1+k_2=k$. An involution σ having exactly k transpositions in J can be generated from an involution τ having exactly k_1 transpositions in J^* by adding k_2 transpositions in $J\setminus J^*=\{2n \mid n\in N\}$. The latter k_2 transpositions are of type $(even, even)$ or (odd, odd), and are easy to count by using binomial coefficients. Two cases are to be considered.

(i) When $m=2n$ is even and $k\geq 1$ we have

$$\begin{aligned}\mu(N|_{2n}, k, J)&=\sum_{k_1+k_2=k}\mu(N|_{2n}, k_1, J^*)\sum_{i+j=k_2}\left[\binom{n-k_1}{2i}(2i-1)(2i-3)\cdots 1\binom{n-k_1}{2j}(2j-1)(2j-3)\cdots 1\right]\\&\equiv\sum_{k_1+k_2=k}\mu(N|_{2n}, k_1, J^*)\sum_{i+j=k_2}\binom{n-k_1}{2i}\binom{n-k_1}{2j}\pmod 2.\end{aligned}$$

If k is odd, then one of the k_1, k_2 is odd and the other is even. By Lemma 10 and Lemma 7, $\mu(N|_{2n}, k, J)\equiv 0 \pmod 2$. If $k=2\ell$ is even, then

$$\begin{aligned}\mu(N|_{2n}, k, J)&=\sum_{k_1+k_2=2\ell}\mu(N|_{2n}, k_1, J^*)\sum_{i+j=k_2}\binom{n-k_1}{2i}\binom{n-k_1}{2j}\\&\equiv\sum_{k_1+k_2=\ell}\mu(N|_{2n}, 2k_1, J^*)\sum_{i+j=2k_2}\binom{n-2k_1}{2i}\binom{n-2k_1}{2j}\quad\text{[by Lemma 7]}\\&\equiv\sum_{k_1+k_2=\ell}\mu(P|_n, k_1, J^*)\binom{n-2k_1}{2k_2}\quad\text{[by Lemmas 7 and 10]}\end{aligned}$$

$$\begin{aligned}
&\equiv \mu(P|_n, \ell, L) \quad [\text{by Lemma 8}] \\
&= \mu(N|_n, k/2, J) \quad [\text{by Lemma 6}] \\
&\equiv 0 \pmod 2 \quad [\text{by induction}].
\end{aligned}$$

(ii) When $m = 2n+1$ is odd and $k \geq 1$, we successively have

$$\begin{aligned}
&\mu(N|_{2n+1}, k, J) \\
&= \sum_{k_1+k_2=k} \mu\big(N|_{2n+1}, k_1, J^*\big) \sum_{i+j=k_2} \left[\binom{n+1-k_1}{2i}(2i-1)(2i-3)\cdots 1 \binom{n-k_1}{2j}(2j-1)(2j-3)\cdots 1\right] \\
&\equiv \sum_{k_1+k_2=k} \mu\big(N|_{2n+1}, k_1, J^*\big) \sum_{i+j=k_2} \binom{n+1-k_1}{2i}\binom{n-k_1}{2j} \\
&\equiv \sum_{k_1+k_2=k} \mu\big(N|_{2n+1}, k_1, J^*\big) \binom{2n+1-2k_1}{2k_2},
\end{aligned}$$

where the last identity is obtained by using Lemma 11. As mentioned in the proof of Lemma 7, two integers c and d such that $c+d \in J^*$ belong either to P or to Q. If $m \equiv 1 \pmod 4$, then $N|_{2n+1} = P|_{n+1} + Q|_n$ (recall that the plus sign "+" means the disjoint union) and

$$\mu\big(N|_{2n+1}, k_1, J^*\big) = \sum_{r+s=k_1} \mu\big(P|_{n+1}, r, J^*\big)\mu\big(Q|_n, s, J^*\big).$$

If $m \equiv 3 \pmod 4$, then $N|_{2n+1} = P|_n + Q|_{n+1}$ and

$$\begin{aligned}
\mu\big(N|_{2n+1}, k_1, J^*\big) &= \sum_{r+s=k_1} \mu\big(Q|_{n+1}, r, J^*\big)\mu\big(P|_n, s, J^*\big) \\
&= \sum_{r+s=k_1} \mu\big(P|_{n+1}, r, J^*\big)\mu\big(Q|_n, s, J^*\big),
\end{aligned}$$

where the last identity is obtained by Lemma 6. Hence

$$\begin{aligned}
&\mu(N|_{2n+1}, k, J) \\
&\equiv \sum_{k_1+k_2=k} \left[\sum_{r+s=k_1} \mu\big(P|_{n+1}, r, J^*\big)\mu\big(Q|_n, s, J^*\big)\right]\binom{2n+1-2k_1}{2k_2}.
\end{aligned}$$

Applying Lemmas 6 and 11 to the above quantity we get

$$\begin{aligned}
&\mu(N|_{2n+1}, k, J) \\
&\equiv \sum_{k_1+k_2=k} \left[\sum_{r+s=k_1} \mu\big(P|_{n+1}, r, J^*\big)\mu\big(P|_n, s, J^*\big) \sum_{i+j=k_2} \binom{n+1-2r}{2i}\binom{n-2s}{2j}\right] \\
&= \sum_{r+s+i+j=k} \mu\big(P|_{n+1}, r, J^*\big)\binom{n+1-2r}{2i}\mu\big(P|_n, s, J^*\big)\binom{n-2s}{2j} \\
&= \sum_{k_1+k_2=k} \left[\sum_{r+i=k_1} \mu\big(P|_{n+1}, r, J^*\big)\binom{n+1-2r}{2i} \sum_{s+j=k_2} \mu\big(P|_n, s, J^*\big)\binom{n-2s}{2j}\right] \\
&\equiv \sum_{k_1+k_2=k} \mu(P|_{n+1}, k_1, L)\mu(P|_n, k_2, L) \quad [\text{by Lemma 8}] \\
&\equiv \sum_{k_1+k_2=k} \mu(N|_{n+1}, k_1, J)\mu(N|_n, k_2, J) \quad [\text{by Lemma 6}] \\
&\equiv 0 \pmod 2 \quad [\text{by induction}].
\end{aligned}$$

This completes the proof. □

Lastly, Theorem 2 on the t-extensions of the Hankel determinants of the period-doubling sequence is proved as follows. Keep in mind the infinite set

$$J = \{(2n+1)2^{2k} - 1 \mid n, k \in N\} = \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \ldots\},$$

and the period-doubling sequence $\mathbf{d} = (1, 0, 1, 1, 1, 0, \ldots)$ defined by:

$$d_k = \frac{1}{2}|e_k - e_{k+1}| \quad (k \ge 0).$$

In [3] Allouche et al. proved the following result (see also [6]).

Lemma 12. *For $k \ge 0$, the integer d_k is equal to 1 if, and only if, k is in J.*

Proof of Theorem 2. It is well known that the Leibniz formula expresses the determinant of a square matrix $A = (a_{ij})_{i,j=0,\ldots,k-1}$ in terms of permutations:

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} a_{0,\sigma(0)} a_{1,\sigma(1)} \cdots a_{k-1,\sigma(k-1)}, \tag{22}$$

where $\mathrm{inv}(\sigma)$ is the number of inversions of σ defined by

$$\mathrm{inv}(\sigma) = \#\{(i, j) \mid 0 \le i < j \le k-1, \sigma(i) > \sigma(j)\}.$$

Let k be a positive integer and $D(x)$ be the generating function of the period-doubling sequence

$$D(x) = \sum_{n \ge 0} d_n x^n = 1 + x^2 + x^3 + x^4 + x^6 + \cdots$$

By (22), the t-Hankel determinant $H_k(D, t)$ is equal to

$$\sum_{\sigma \in \mathfrak{S}_k} t^{\mathrm{fix}(\sigma)} (-1)^{\mathrm{inv}(\sigma)} d_{0+\sigma(0)} d_{1+\sigma(1)} \cdots d_{k-1+\sigma(k-1)}, \tag{23}$$

where $\mathrm{fix}(\sigma)$ is the number of fixed points of σ defined by

$$\mathrm{fix}(\sigma) = \#\{i \mid 0 \le i \le k-1, \sigma(i) = i\}.$$

By Lemma 12 the product

$$d_{0+\sigma(0)} d_{1+\sigma(1)} \cdots d_{k-1+\sigma(k-1)} \tag{24}$$

is equal to 1 if $i + \sigma(i) \in J$ for $i = 0, 1, \ldots, k-1$, and is equal to 0 otherwise. Let σ be a permutation such that $\sigma \ne \sigma^{-1}$. We have $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$ and $\mathrm{fix}(\sigma) = \mathrm{fix}(\sigma^{-1})$. Accordingly, they have the same contribution to summation (23), and can be deleted. Hence

$$H_k(D, t) \equiv \sum_{\sigma} t^{\mathrm{fix}(\sigma)} d_{0+\sigma(0)} d_{1+\sigma(1)} \cdots d_{k-1+\sigma(k-1)} \pmod 2, \tag{25}$$

where the sum is over the set of all involutions σ on $N|_k$. Notice that each of the fixed points of an involution σ produces a 1 factor in the product (24) as all even numbers are in J. By (25) and Theorem 5,

$$H_k(D, t) \equiv \sum_{i=0}^{\lfloor k/2 \rfloor} t^{k-2i} \mu(N|_k, i, J) \equiv t^k \mu(N|_k, 0, J) = t^k.$$

This completes the proof. □

4. Regular paperfolding sequence

We define the infinite set

$$R = \{(4k+1)2^n - 1 \mid n, k \in N\} = \{0, 1, 3, 4, 7, 8, 9, 12, 15, 16, \ldots\}.$$

Notice that, for each integer m in the set R, there are unique integers n and k such that $(4k+1)2^n - 1 = m$. Recall the regular paperfolding sequence $\mathbf{r} = (r_k)_{k \ge 0}$ defined by (8). The following lemma is a well-known description of the regular paperfolding sequence (see, for example, [2, Theorem 6.5.2]). Its proof is included for the sake of completeness.

Lemma 13. *For each $k \ge 0$ the integer r_k is equal to 1 if and only if k is in R, and is equal to 0 otherwise.*

Proof. By definition of (8), we have

$$\begin{aligned} G_{0,2}(x) &= \sum_{n\geq 0} r_n x^n = \sum_{n=0}^{\infty} \frac{x^{2^n-1}}{1-x^{2^{n+2}}} \\ &= \sum_{n=0}^{\infty} x^{2^n-1}\left(\sum_{k\geq 0}\left(x^{2^{n+2}}\right)^k\right) \\ &= \sum_{n,k\geq 0} x^{4k\cdot 2^n+2^n-1}. \end{aligned}$$

Thus the lemma holds. □

Proof of Theorem 4. As discussed in Section 3, the t-Hankel determinant $H_k(\mathbf{r},t)$ is equal to

$$\sum_{\sigma\in\mathfrak{S}_k} t^{\mathrm{fix}(\sigma)}(-1)^{\mathrm{inv}(\sigma)} r_{0+\sigma(0)} r_{1+\sigma(1)}\cdots r_{k-1+\sigma(k-1)}. \tag{26}$$

By Lemma 13 the product

$$r_{0+\sigma(0)} r_{1+\sigma(1)}\cdots r_{k-1+\sigma(k-1)} \tag{27}$$

is equal to 1 if $i+\sigma(i)\in R$ for $i=0,1,\cdots,k-1$, and is equal to 0 otherwise.

Recall the three representations for permutations: the *one-line*, *two-line* and *product of disjoint cycles*. For example, we write

$$\sigma\in\mathfrak{S}_9 = 516280374 = \begin{pmatrix} 012345678 \\ 516280374 \end{pmatrix} = (0,5)(1)(2,6,3)(4,8)(7).$$

Consider a permutation σ such that the associated product (27) is non-zero having at least 4 fixed points, i.e., $\mathrm{fix}(\sigma)\geq 4$. It is easy to see that an even number m is in R if and only if $m\equiv 0 \pmod 4$, so that all fixed points are even. Since the number of odd integers in $\{0,1,\dots,k-1\}$ is equal to $\lfloor k/2\rfloor$, there are at least 3 columns of type $\binom{odd}{odd}$ in the two-line representation of the permutation σ. Let $\binom{i_1}{j_1}$, $\binom{i_2}{j_2}$ and $\binom{i_3}{j_3}$ be the first three such columns. By the Pigeonhole Principle, there are at least two numbers among j_1, j_2, j_3 which are congruent modulo 4. Without loss of generality, we assume that j_1 and j_2 are congruent modulo 4. (When all three numbers are congruent, we also choose j_1 and j_2.) We define another permutation τ obtained from σ by exchanging j_1 and j_2 in the bottom line, i.e., $\tau=(j_1,j_2)\circ\sigma$. This procedure is reversible. By comparing the two permutations σ and τ we have the following properties:

(1) $\mathrm{inv}(\sigma)=\mathrm{inv}(\tau)\pm 1$, so that $(-1)^{\mathrm{inv}(\sigma)}=-(-1)^{\mathrm{inv}(\tau)}$.
(2) $i_1+j_2\in R$ and $i_2+j_1\in R$. Since i_1+j_1 and i_2+j_2 are in R and are even, hence must be congruent to 0 modulo 4. Consequently, i_1+j_2 and i_2+j_1 are congruent to 0 modulo 4 and are in R.
(3) $\mathrm{fix}(\sigma)=\mathrm{fix}(\tau)$, i.e., no fixed point has been created. Since i_1+j_2, i_2+j_1 are congruent to 0 modulo 4 (see above item) and i_1, i_2, j_1, j_2 are odd integers, we have $i_1\neq j_2$ and $i_2\neq j_1$.

Thus, the contributions by σ and τ in the summation (26) compensate each other. We can delete the pair $\{\sigma,\tau\}$ from the symmetry group $\mathfrak{S}_k$. The value of the t-Hankel determinant $H_k(\mathbf{r},t)$ defined by (26) does not change. After deleting all the permutations such that $\mathrm{fix}(\sigma)\geq 4$, all remaining permutations have at most 3 fixed points. Hence, the t-Hankel determinant $H_k(\mathbf{r},t)$ is a polynomial in t of degree less than or equal to 3. □

Acknowledgements

The second author would like to thank Zhi-Ying Wen, who invited him to Tsinghua University for a fruitful two-month visit. Part of this research was done during that period.

We are grateful to the two anonymous referees for helpful comments and suggestions.

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