Theoretical Computer Science 412 (2011) 3524–3529

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

An improved lower bound on the sensitivity complexity of graph properties

Xiaoming Sun *

Center for Advanced Study and Institute for Theoretical Computer Science, Tsinghua University, 100084, China

ARTICLE INFO

Article history:
Received 13 December 2010
Received in revised form 13 February 2011
Accepted 22 February 2011
Communicated by D.-Z. Du

Keywords:
Computational complexity
Decision tree complexity
Sensitivity complexity
Graph property

ABSTRACT

Turán (1984) [11] initiated the study of the sensitivity complexity of graph properties. He conjectured that for any non-trivial graph properties on n vertices, the sensitivity complexity is at least $n-1$. He proved an $\lfloor \frac{n}{4} \rfloor$ lower bound for sensitivity in his paper: Turán (1984) [11]. Wegener (1985) [12] proved this conjecture for all *monotone* graph properties. In this paper we improve Turán's lower bound to $\frac{6}{17}n(\approx 0.35n)$. We hope that this will shed some light on the proof of Turán's conjecture.

1. Introduction

Sensitivity complexity $s(f)$ was first introduced by Cook, Dwork and Reischuk [4,5] (under the name critical complexity) for studying the time complexity of CRAW-PRAMs. They showed that $\log_b s(f)$ is a lower bound for the time needed by a PRAM to compute a function f (where $b = (5+\sqrt{21})/2 \approx 4.79$). Simon [10] has shown that the sensitivity complexity of a non-degenerate n-variable Boolean function is at least $\Omega(\log n)$. Turán [11] investigated the sensitivity complexity of *graph properties* (see the definition in Section 2). He proved that for any non-trivial *graph properties* on n vertices, the sensitivity is at least $\lfloor \frac{n}{4} \rfloor$ (the number of variables of graph properties is $\binom{n}{2}$). In [11], Turán also gave an example (the "contained an isolated vertex" property) which has sensitivity complexity $n-1$. He further conjectured that $n-1$ might be the right lower bound. Wegener [12] proved this conjecture for all *monotone* graph properties. In this paper we improve Turán's lower bound for general graph properties. Here is the main theorem of our paper.

Theorem 1. *For any non-trivial graph property f on n vertices, $s(f) \geq \frac{6}{17}n$.*

Wegener's proof relied heavily on the fact that the property is monotone. Our proof strategy is roughly like this: we show that for any two graphs G and H, there always exists a sequence of graphs $G_0, G_1, \ldots, G_t$, where $G_0 = G, G_t = H$, such that for any $0 \leq i \leq t-1$, G_{i+1} is a graph obtained by adding or deleting one edge from graph G_i; more importantly, there are at least αn *isomorphism* ways of adding or deleting this edge. Therefore, if there exists some i with $f(G_{i+1}) \neq f(G_i)$, then $s(f) \geq \alpha n$. So if $s(f) < \alpha n$, then for any two graphs G and H, $f(G) = f(H)$, which contradicts the non-trivial condition of f. We can show that α is at least $\frac{6}{17}$ in this paper.

Related work:

Sensitivity complexity is closely related to *decision tree* complexity and other complexity measures of Boolean functions. Here we only list some results related to the sensitivity complexity. For more results we refer readers to the excellent survey [1] by Buhrman and de Wolf.

* Tel.: +86 10 62792230.
E-mail address: xiaomings@tsinghua.edu.cn.

0304-3975/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2011.02.042

Nisan [8] generalized the concept of sensitivity complexity to *block sensitivity* complexity and demonstrated that the time complexity of CREW-PRAM is actually equal to (up to a constant factor) the logarithm of the block sensitivity. He also showed that block sensitivity, *certificate complexity*, and decision tree complexity are polynomially related. Nisan and Szegedy [9] further showed that the *degree complexity* is also polynomially related to the block sensitivity. But very little is known about sensitivity complexity except the basic fact that it is a lower bound of block sensitivity. It is conjectured that the sensitivity complexity is also polynomially related to all four of the complexity measures mentioned above. Gotsman and Linial [6] have shown that the sensitivity versus degree problem is equivalent to an induced subgraphs problem on the Boolean hypercube studied by Fan Chung et al. [3]. The best known upper bound on the block sensitivity in terms of the sensitivity complexity is still exponential (by Kenyon and Kutin [7]). In his paper [11], Turán also conjectured that for a general class of functions, the *weakly symmetric* functions, the sensitivity complexity has a similar lower bound. Chakraborty [2] disproved this conjecture by giving a cyclically invariant function with sensitivity $O(N^{1/3})$ (N is the number of variables). It is also open whether $\Omega(N^{1/3})$ is a lower bound for the sensitivity complexity of all weakly symmetric functions, or even all cyclically invariant functions.

The rest of the paper is organized as follows: in Section 2 we introduce the definitions and some notation, in Section 3 we prove two structural lemmas which will be used in the proof of the main theorem, and then we prove our main result in Section 4.

2. Preliminaries

Let $f : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function. For an input $x \in \{0, 1\}^n$, x^i denotes the input obtained by flipping the ith bit of x.

Definition 2. The *sensitivity* complexity of f on input x is defined as $s(f, x) = |\{i : f(x) \neq f(x^i)\}|$. The *sensitivity* of the function f is defined as $s(f) = \max_x s(f, x)$.

Definition 3. A Boolean function f is *symmetric* if for every input $x = x_1 \dots x_n$ and every permutation $\pi \in S_n$, $f(x_1, \dots, x_n) = f(\pi(x_1), \dots, \pi(x_n))$.

For a symmetric function, we have the following lower bound on the sensitivity complexity:

Lemma 4 (*Turán [11]*)**.** *For every non-trivial symmetric function $f : \{0, 1\}^n \to \{0, 1\}$, $s(f) > \frac{n}{2}$.*

A generalization of the symmetric function is the *weakly symmetric function*.

Definition 5. A Boolean function f is called *weakly symmetric* (or *transitive-invariant*) if there exists a transitive group[1] $\Gamma \leq S_n$ such that for all $\sigma \in \Gamma$ and every input $x = x_1 \dots x_n$,

$$f(x_1, \dots, x_n) = f(\sigma(x_1), \dots, \sigma(x_n)).$$

In this paper, we are interested in a special class of weakly symmetric functions: *graph properties* — Boolean functions which are independent of the labeling of the vertices of a graph. For example, connectivity, being Hamiltonian, being triangle-free etc are graph properties. Here is the formal definition.

Definition 6. A Boolean function $f : \{0, 1\}^{\binom{n}{2}} \to \{0, 1\}$ is called a *graph property* if for every input $x = (x_{(1,2)}, \dots, x_{(n-1,n)})$ and every permutation $\pi \in S_n$,

$$f(x_{(1,2)}, \dots, x_{(n-1,n)}) = f(x_{(\pi(1),\pi(2))}, \dots, x_{(\pi(n-1),\pi(n))}).$$

For graph $G = (V, E)$, we use $I(G)$ to represent the set of isolated vertices in G. Let $V_d(G) = \{v \in V(G) | \deg(v) = d\}$ and $V_{\geq d}(G) = \{v \in V(G) | \deg(v) \geq d\}$. We also use the notation I, V_d, $V_{\geq d}$ if the graph G referred to is clear from the context.

3. Two structural lemmas

We need the following two lemmas for proving the main theorem.

Lemma 7. *Given graph property f and graph G, if $V_1(G) \neq \emptyset$, then either $s(f) \geq |I(G)| + 1$ or, for any vertex $v \in V_1(G)$ and w adjacent to v, $f(G) = f(G - (v, w))$.*

Proof. Consider graph $G' = G - (w, v)$ and suppose $I(G) = \{u_1, \dots, u_{|I|}\}$; we have

$$G' + (w, u_i) \cong G' + (w, v), \quad i = 1, \dots, |I|,$$

and hence $f(G' + (w, u_i)) = f(G' + (w, v))$. So if $f(G') \neq f(G' + (w, v))$, then $f(G') \neq f(G' + (w, u_i))$ $(i = 1, \dots, |I|)$. Therefore, $s(f, G') \geq |I(G)| + 1$. Otherwise, $f(G') = f(G' + (w, v))$, i.e. $f(G - (w, v)) = f(G)$. □

[1] A group $\Gamma \leq S_n$ is *transitive* if for every $i < j$, there exists $\sigma \in \Gamma$ such that $\sigma(i) = j$.

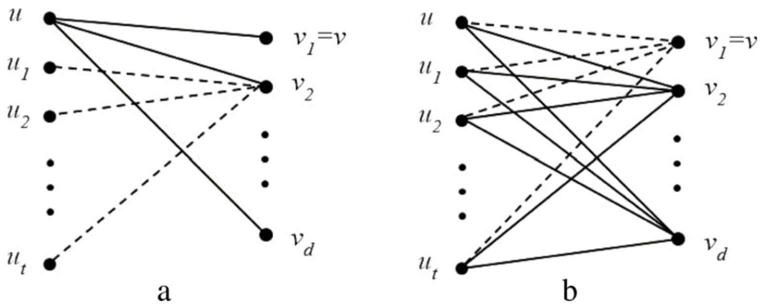

Fig. 1. (a) Function $g_2(x_1, \dots, x_t)$. (b) Function $h(x_0, x_1, \dots, x_t)$.

The following lemma was used implicitly in Turan's proof [11].

Lemma 8. *Given graph property f and graph G, if $E(G) \neq \emptyset$, then either $s(f) \geq |I(G)|/2$ or, for all $e \in E(G)$, $f(G) = f(G - e)$.*

Proof. Suppose $s(f) < |I(G)|/2$; we will deduce that for all $e \in E(G)$, $f(G) = f(G - e)$.

Pick any edge (u, v) from $E(G)$. Suppose that in graph G, $\deg(u) = d$ and vertex u is adjacent to vertices $\{v_1, v_2, \dots, v_d\}$, where $v_1 = v$. Suppose $I(G) = \{u_1, \dots, u_t\}$, where $t = |I(G)|$.

Consider the t-variable Boolean function $g_2 : \{0, 1\}^t \to \{0, 1\}$,

$$g_2(x_1, \dots, x_t) = f(G + x_1(v_2, u_1) + x_2(v_2, u_2) + \cdots + x_t(v_2, u_t)),$$

i.e. we add edge (v_2, u_i) to graph G iff $x_i = 1$ $(i = 1, \dots, t)$; see Fig. 1(a). Since $u_1, \dots, u_t$ are isolated vertices in G, it is easy to see that g_2 is a symmetric function. By Lemma 4, either $s(g_2) > t/2$ or g_2 is a constant function. But g_2 is a restriction of function f, so $s(g_2) \leq s(f)$, and thus $s(g_2) < |I(G)|/2 = t/2$; therefore, g_2 is a constant on every input. In particular, $g_2(1, \dots, 1) = g_2(0, \dots, 0)$, i.e. $f(G + \sum_{i=1}^{t}(v_2, u_i)) = f(G)$. Define $G_2 = G + \sum_{i=1}^{t}(v_2, u_i)$. Consider another Boolean function $g_3 : \{0, 1\}^t \to \{0, 1\}$,

$$g_3(x_1, \dots, x_t) = f(G_2 + x_1(v_3, u_1) + x_2(v_3, u_2) + \cdots + x_t(v_3, u_t)).$$

Similarly, g_3 is a symmetric function, so from Lemma 4 $s(g_3) > t/2$ or g_3 is a constant function. But $s(g_3) \leq s(f) < t/2$, so g_3 is constant, and $f(G_2) = f(G_3)$, where $G_3 = G_2 + \sum_{i=1}^{t}(v_3, u_i)$. Continuing this procedure, we can show that

$$f(G) = f(G_2) = \cdots = f(G_d),$$

where $G_i = G_{i-1} + \sum_{j=1}^{t}(v_i, u_j)$ $(i = 3, \dots, d)$.

Now let us consider the graph $H = G_d - (u, v_1)$. Define the $(t + 1)$-variable function $h : \{0, 1\}^{t+1} \to \{0, 1\}$,

$$h(x_0, x_1, \dots, x_t) = f(H + x_0(v_1, u) + x_1(v_1, u_1) + \cdots + x_t(v_1, u_t)).$$

See Fig. 1(b). Again h is a symmetric function; using Lemma 4, $s(h) > (t + 1)/2$ or h is a constant function. Since $s(h) \leq s(f) < t/2$, h is a constant. In particular, $h(0, 0, \dots, 0) = h(1, 0, \dots, 0)$, i.e. $f(H) = f(H + (v_1, u)) = f(G_d)$.

Next we will delete all the edges between $\{u_1, \dots, u_t\}$ and $\{v_2, \dots, v_d\}$ from H by reversing the adding edge procedure of $G \to G_2 \to \cdots \to G_d$. More precisely, define $H_1 = H$; for $i = 2, \dots, d$, define

$$H_i = H_{i-1} - (v_i, u_1) - (v_i, u_2) - \cdots - (v_i, u_t),$$

and

$$h_i(y_1, \dots, y_t) = f(H_i + y_1(v_i, u_1) + y_2(v_i, u_2) + \cdots + y_t(v_i, u_t)).$$

By Lemma 4 and the fact $s(f) < t/2$ we can show that all the functions $h_2, \dots, h_d$ are constant, which implies $f(H) = f(H_2) = \cdots = f(H_d)$. But if we compare graph G and graph H_d, it is easy to see that $H_d = G - (u, v_1)$. Therefore, $f(G) = f(H_d) = f(G - (u, v))$. □

4. Proof of the main theorem

Without loss of generality we assume that for the empty graph $\bar{K}_n$, $f(\bar{K}_n) = 0$. Since f is a non-trivial property, there must exist a graph G such that $f(G) = 1$. Let us consider graphs in $f^{-1}(1) = \{G | f(G) = 1\}$ with the minimum number of edges. Define $m = \min\{|E(G)| : f(G) = 1\}$.

We claim that if $m \geq \frac{6}{17}n$, then $s(f) \geq \frac{6}{17}n$. Let G be a graph in $f^{-1}(1)$ and $|E(G)| = m \geq \frac{6}{17}n$. Since G has the minimum number of edges, deleting any edges from G will change the value of $f(G)$, i.e. $\forall e \in E(G), f(G - e) = 0$. Therefore, $s(f, G) \geq |E(G)| = m \geq \frac{6}{17}n$. Thus $s(f) \geq \frac{6}{17}n$.

In the following we assume $m < \frac{6}{17}n$. Again let G be a graph in $f^{-1}(1)$ with $|E(G)| = m$. Let us consider the isolated vertices set I; as

$$\sum_{v\in V}\deg(v) = 2|E(G)| = 2m < 2\times\frac{6}{17}n = \frac{12}{17}n,$$

we have

$$|I| = n - |V_{\geq 1}| \geq n - \sum_{v\in V_{\geq 1}}\deg(v) = n - \sum_{v\in V}\deg(v) \geq \frac{5}{17}n.$$

According to whether or not there exists a degree-1 vertex, we separate the proof into two parts:

Case 1: $V_1 \neq \emptyset$, i.e. there exists $v \in V(G)$ with $\deg(v) = 1$. We further consider two subcases here:

(a) There exist v_1 and $v_2 \in V_1$ such that $(v_1, v_2) \in E(G)$.

Let $G' = G - (v_1, v_2)$. Since G has the minimum number of edges, $f(G') = 0$. Suppose $I(G) = \{u_1, \ldots, u_{|I|}\}$; since in graph G, $\deg(v_1) = \deg(v_2) = 1$, then for any $1 \leq i_1 < i_2 \leq |I|$,

$$G' + (u_{i_1}, u_{i_2}) \cong G' + (v_1, v_2) = G.$$

Thus

$$f(G' + (u_{i_1}, u_{i_2})) = f(G) = 1.$$

Similarly, we have

$$f(G' + (u_i, v_1)) = f(G' + (u_i, v_2)) = f(G) = 1, \quad \text{(for all } 1 \leq i \leq |I|)$$

but $f(G') = 0$; therefore,

$$s(f, G') \geq \binom{|I|+2}{2} \geq \binom{\lceil\frac{5}{17}n\rceil + 2}{2} \geq \frac{6}{17}n.$$

(b) Consider any vertices v_1 and $v_2 \in V_1$ with $(v_1, v_2) \notin E(G)$. We will show that $|I(G)| \geq \frac{8}{17}n$ in this case.

$$\sum_{v\in V}\deg(v) = 2|E(G)| = 2m < 2\times\frac{6}{17}n = \frac{12}{17}n,$$

i.e.,

$$\sum_{v\in V_1}\deg(v) + \sum_{v\in V_{\geq 2}}\deg(v) < \frac{12}{17}n. \tag{1}$$

Since no two vertices in V_1 are adjacent, i.e., all the vertices in V_1 are adjacent to vertices in $V_{\geq 2}$, we hence have

$$\sum_{v\in V_1}\deg(v) \leq \sum_{v\in V_{\geq 2}}\deg(v). \tag{2}$$

Combining Eqs. (1) and (2), we have $\sum_{v\in V_1}\deg(v) < \frac{6}{17}n$, i.e. $|V_1| < \frac{6}{17}n$.

If $|V_{\geq 2}| \leq \frac{3}{17}n$, then $|I| = n - |V_1| - |V_{\geq 2}| > n - \frac{6}{17}n - \frac{3}{17}n = \frac{8}{17}n$. Otherwise suppose that $|V_{\geq 2}| > \frac{3}{17}n$; from Eq. (1),

$$|V_1| + 2|V_{\geq 2}| \leq \sum_{v\in V_1}\deg(v) + \sum_{v\in V_{\geq 2}}\deg(v) < \frac{12}{17}n.$$

Hence $|V_1| + |V_{\geq 2}| < \frac{12}{17}n - |V_{\geq 2}| < \frac{9}{17}n$. Therefore, $|I| = n - |V_1| - |V_{\geq 2}| > n - \frac{9}{17}n = \frac{8}{17}n$.

Since $V_1 \neq \emptyset$, by Lemma 7 either $s(f) \geq |I(G)| + 1 > \frac{8}{17}n$ or there exists an edge $e \in G$ with $f(G-e) = f(G)$. But we know that G has the minimum number of edges, so $f(G-e) \neq f(G)$; thus $s(f) \geq \frac{8}{17}n$. This finishes the proof of **Case 1**.

Case 2: $V_1 = \emptyset$, i.e. $\forall v \in V - I$, $\deg(v) \geq 2$. In this case,

$$|I| = n - |V_{\geq 1}| = n - |V_{\geq 2}| \geq n - \frac{1}{2}\sum_{v}\deg(v) = n - m \geq \frac{11}{17}n.$$

(a) There exist v and $w \in V_2$ such that $(v, w) \in E(G)$.

Since $\deg(v) = \deg(w) = 2$, suppose that besides vertex w, vertex v is also adjacent to vertex x; similarly suppose that vertex w is also adjacent to vertex y (x and y could be the same vertex). Pick $2r$ different isolated vertices $\{v_1, \ldots, v_r\}$ and

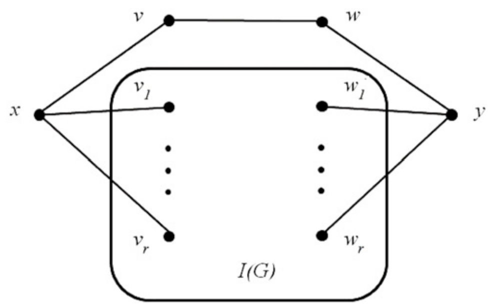

Fig. 2. Graph G_{2r} and $H = G_{2r} - (v, w)$.

$\{w_1, \dots, w_r\}$ from $I(G)$, where $r = \left\lfloor \sqrt{\frac{6}{17}n} \right\rfloor$ ($\frac{11}{17}n \geq 2\left\lfloor \sqrt{\frac{6}{17}n} \right\rfloor$ for $n \geq 4$). Define $G_0 = G, G_i = G_{i-1} + (x, v_i)$ for $i = 1, \dots, r$, and $G_i = G_{i-1} + (y, w_{i-r})$ for $i = r+1, \dots, 2r$. If for some $i \in \{1, 2, \dots, 2r\}, f(G_i) \neq f(G_{i-1})$, then by Lemma 7,

$$s(f) \geq |I(G_i)| + 1 = I(G) - i + 1 \geq \frac{11}{17}n - 2\sqrt{\frac{6}{17}n} + 1 \geq \frac{6}{17}n \quad (\text{for } n \geq 9).$$

So let us assume that $f(G) = f(G_1) = \cdots = f(G_{2r})$.

Now consider the graph $H = G_{2r} - (v, w)$ (see Fig. 2). In graph H, $\deg(v) = \deg(v_1) = \cdots = \deg(v_r) = 1$ and they are both adjacent to x; similarly $\deg(w) = \deg(w_1) = \cdots = \deg(w_r) = 1$ and they are both adjacent to y. Thus

$$H + (v, w) \cong H + (v_i, w) \cong H + (v, w_j) \cong H + (v_i, w_j) \quad (\forall\, i, j = 1, \dots, r).$$

Therefore, if $f(H) \neq f(H + (v, w))$(i.e. $f(G_{2r})$), then

$$s(f, H) \geq (r+1)^2 = \left(\left\lfloor \sqrt{\frac{6}{17}n} \right\rfloor + 1 \right)^2 \geq \frac{6}{17}n.$$

So we can assume that $f(H) = f(G_{2r})$, which implies $f(H) = f(G)$. Now we define another sequence of graphs $H_0 = H$, $H_i = H_{i-1} - (y, w_i)$ for $i = 1, \dots, r$, and $H_i = H_{i-1} - (x, v_{i-r})$ for $i = r+1, \dots, 2r$. By an argument similar to the previous one for $G_0, \dots, G_{2r}$, we can show that either $s(f) \geq \frac{6}{17}n$ or $f(H) = \cdots = f(H_{2r})$. So we have $f(G) = f(H) = f(H_{2r})$. But if we compare graphs G and H_{2r}, we can see that $H_{2r} = G - (v, w)$, which contradicts the minimality of G.

(b) $\forall v_1, v_2 \in V_2, (v_1, v_2) \notin E(G)$. We claim that in this case $|I| \geq \frac{12}{17}n$.

$$2|V_2| + \sum_{v \in V_{\geq 3}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E(G)| = 2m < \frac{12}{17}n. \tag{3}$$

Since no two vertices in V_2 are adjacent, all the vertices in V_2 are adjacent to vertices in $V_{\geq 3}$ ($V_1 = \emptyset$); hence

$$\sum_{v \in V_2} \deg(v) = 2|V_2| \leq \sum_{v \in V_{\geq 3}} \deg(v). \tag{4}$$

From Eq. (4) $+5\times$ (3), we have

$$12|V_2| + 5\sum_{v \in V_{\geq 3}} \deg(v) \leq \sum_{v \in V_{\geq 3}} \deg(v) + \frac{60}{17}n,$$

i.e. $3|V_2| + \sum_{v \in V_{\geq 3}} \deg(v) \leq \frac{15}{17}n$, which implies

$$3|V_2| + 3|V_3| \leq 3|V_2| + \sum_{v \in V_{\geq 3}} \deg(v) \leq \frac{15}{17}n.$$

So $|V_2| + |V_3| \leq \frac{5}{17}n$; therefore, $|I| = n - |V_2| - |V_3| \geq \frac{12}{17}n$.

By Lemma 8, either $s(f) \geq |I(G)|/2 \geq \frac{6}{17}n$ or for all $e \in G, f(G - e) = f(G)$. But we know that G has the minimum number of edges, so $f(G - e) \neq f(G)$; thus $s(f) \geq \frac{6}{17}n$. This ends the whole proof. □

Acknowledgement

The author would like to thank the anonymous referees for their helpful comments and suggestion to improve the presentation of this paper. This work was supported in part by the National Natural Science Foundation of China Grant 60603005, 61033001, 61061130540, the National Basic Research Program of China Grant 2007CB807900, 2007CB807901, and Tsinghua University Initiative Scientific Research Program 2009THZ02120.

References

[1] Harry Buhrman, Ronald de Wolf, Complexity measures and decision tree complexity: a survey, Theor. Comput. Sci. 288 (1) (2002) 21–43.
[2] Sourav Chakraborty, On the sensitivity of cyclically-invariant boolean functions, in: IEEE Conference on Computational Complexity, 2005, pp. 163–167.
[3] Fan R.K. Chung, Zoltán Füredi, Ronald L. Graham, Paul D. Seymour, On induced subgraphs of the cube, J. Comb. Theory, Ser. A 49 (1) (1988) 180–187.
[4] Stephen Cook, Cynthia Dwork, Bounds on the time for parallel RAM's to compute simple functions, in: STOC, 1982, pp. 231–233.
[5] Stephen Cook, Cynthia Dwork, Rüdiger Reischuk, Upper and lower time bounds for parallel random access machines without simultaneous writes, SIAM J. Comput. 15 (1) (1986) 87–97.
[6] Craig Gotsman, Nathan Linial, The equivalence of two problems on the cube, J. Comb. Theory, Ser. A 61 (1) (1992) 142–146.
[7] Claire Kenyon, Samuel Kutin, Sensitivity, block sensitivity, and l-block sensitivity of boolean functions, Inf. Comput. 189 (1) (2004) 43–53.
[8] Noam Nisan, CREW PRAMs and decision trees, SIAM J. Comput. 20 (6) (1991) 999–1007.
[9] Noam Nisan, Mario Szegedy, On the degree of Boolean functions as real polynomials, Comput. Complexity 4 (1994) 301–313.
[10] Hans-Ulrich Simon, A tight $\Omega(\log\log n)$-bound on the time for parallel RAM's to compute nondegenerated Boolean functions, in: FCT, 1983, pp. 439–444.
[11] György Turán, The critical complexity of graph properties, Inf. Process. Lett. 18 (3) (1984) 151–153.
[12] Ingo Wegener, The critical complexity of all (monotone) Boolean functions and monotone graph properties, in: FCT, 1985, pp. 494–502.