Complexity of Connectivity in Cognitive Radio Networks through Spectrum Assignment*

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Abstract. Cognitive Radio Networks (CRNs) are considered as a promising solution to the spectrum shortage problem in wireless communication. In this paper, we address the algorithmic complexity of the connectivity problem in CRNs through spectrum assignment. We model the network of secondary users (SUs) as a potential graph, where if two nodes have an edge between them, they are connected as long as they choose a common available channel. In the general case, where the potential graph is arbitrary and SUs may have different number of antennae, we prove that it is NP-complete to determine whether the network is connectable even if there are only two channels. For the special case when the number of channels is constant and all the SUs have the same number of antennae, which is more than one but less than the number of channels, the problem is also NP-complete. For special cases that the potential graph is complete or a tree, we prove the problem is NP-complete and fixed-parameter tractable (FPT) when parameterized by the number of channels. Furthermore, exact algorithms are derived to determine the connectivity.

1 Introduction

Cognitive Radio is a promising technology to alleviate the spectrum shortage in wireless communication. It allows the unlicensed *secondary users* to utilize the temporarily unused licensed spectrums, referred to as *white spaces*, without interfering with the licensed *primary users*. Cognitive Radio Networks (CRNs) is considered as the next generation of communication networks and attracts numerous research from both academia and industry recently.

In CRNs, each secondary user (SU) can be equipped with one or multiple antennae for communication. With multiple antennae, a SU can communicate on multiple channels simultaneously (in this paper, channel and spectrum are

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* This work was supported in part by the National Basic Research Program of China Grant 2011CBA00300, 2011CBA00302, the National Natural Science Foundation of China Grant 61073174, 61033001, 61061130540, and the Hi-Tech research and Development Program of China Grant 2006AA10Z216.

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A. Bar-Noy and M.M. Halldórsson (Eds.): ALGOSENSORS 2012, LNCS 7718, pp. 108–119, 2013.

used interchangeably.). Through spectrum sensing, each SU has the capacity to measure current available channels at its site, i.e. the channels are not used by the primary users (PUs). Due to the appearance of PUs, the available channels of SUs have the following characteristics [1]: 1)*Spatial Variation*: SUs at different positions may have different available channels; 2)*Spectrum Fragmentation*: the available channels of a SU may not be continuous; and 3) *Temporal Variation*: the available channels of a SU may change over time.

Spectrum assignment is to allocate available channels to SUs to improve system performance such as spectrum utilization, network throughput and fairness. Spectrum assignment is one of the most challenging problems in CRNs and has been extensively studied [12–15].

Connectivity is a fundamental problem in wireless communication. Connection between two nodes in CRNs is not only determined by their distance and their transmission powers, but also related to whether the two nodes has chosen a common channel. Due to the spectrum dynamics, communication in CRNs is more difficult than in the traditional multi-channel radio networks [3]. Authors in [8–10] studied the impact of different parameters on connectivity in large-scale CRNs, such as the number of channels, the activity of PUs, the number of neighbors of SUs and the transmission power.

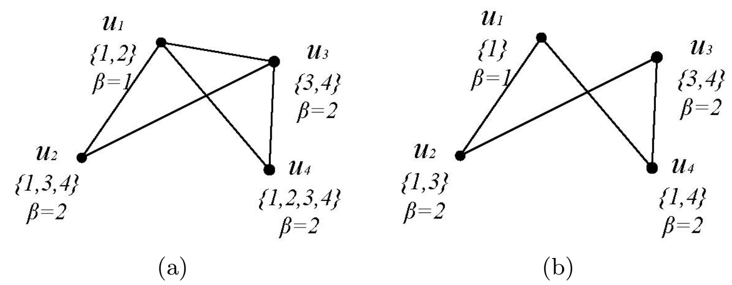

Fig. 1. the general case. a) the potential graph: the set besides each SU is its available channels, and β is its number of antennae. u_2 and u_4 are not connected because they are a pair of heterogenous nodes or their distance exceeds at least one of their transmission ranges. b) the realization graph which is connected: the set beside each SU is the channels assigned to it.

In this paper, we focus on the complexity of connectivity in CRNs through spectrum assignment. We model the network as a potential graph and a realized graph before and after spectrum assignment respectively (refer to Section 2). We start from the most general case, where the network is composed of heterogenous SUs[1], SUs may be equipped with different number of antennae and the potential graph can be arbitrary (Figure 1). Then, we proceed to study the special case when all the SUs have the same number of antennae. If all the SUs are

[1] We assume two heterogenous SUs cannot communicate even when they work on a common channel and their distance is within their transmission ranges.

homogenous with transmission ranges large enough, the potential graph will be a complete graph. For some hierarchically organized networks, e.g. a set of SUs are connected to an access point, the potential graph can be a tree. Therefore, we also study these special cases. Exact algorithms are also derived to determine connectivity for different cases. Our results are listed below. To the best of knowledge, this is the first work that systematically studies the algorithmic complexity of connectivity in CRNs with multiple antennae.

Our Contributions: In this paper we study the algorithmic complexity of the connectivity problem through spectrum assignment under different models. Our main results are as follows.

- When the potential graph is a general graph, we prove that the problem is NP-complete even if there are only two channels. This result is sharp as the problem is polynomial-time solvable when there is only one channel. We also design exact algorithms for the problem. For the special case when all SUs have the same number of antennae, we prove that the problem is NP-complete when $k > \beta \geq 2$, where k and β are the total amount of channels in the white spaces and the number of antennae on an SU respectively.
- When the potential graph is complete[2], the problem is shown to be NP-complete even if each node can open at most two channels. However, in contrast to the general case, the problem is shown to be polynomial-time solvable if the number of channels is fixed. In fact, we prove a stronger result saying that the problem is fixed parameter tractable when parameterized by the number of channels. (See [4] for notations in parameterized complexity.)
- When the potential graph is a tree, we prove that the problem is NP-complete even if the tree has depth one. Similar to the complete graph case, we show that the problem is fixed parameter tractable when parameterized by the number of channels.

Paper Organization: In Section 2 we formally define our model and problems studied in this paper. We study the problem with arbitrary potential graphs in Section 3. The special cases where the potential graph is complete or a tree are investigated in Sections 4 and 5. The paper is concluded in Section 6 with possible future works.

2 System Model and Problem Definition

We first describe the model used throughout this paper. A *cognitive radio network* is comprised of the following ingredients:

- U is a collection of secondary users (SUs) and C is the set of channels in the white spaces.
- Each SU $u \in U$ has a *spectrum map,* denoted by SpecMap(u), which is a subset of C representing the available channels that u can open.

[2] Note that the complete graph is a special case of disk graphs.

- The *potential graph* $\mathcal{PG} = (U, E)$, where each edge of E is also called a *potential edge*. If two nodes are connected by a potential edge, they can communicate as long as they choose a common available channel.
- Each SU $u \in U$ is equipped with a number of antennas, denoted as *antenna budget* $\beta(u)$, which is the maximum number of channels that u can open simultaneously.

For a set S, let 2^S denote the power set of S, i.e., the collection of all subsets of S. A *spectrum assignment* is a function $\mathcal{SA} : U \to 2^C$ satisfying that

$$\mathcal{SA}(u) \subseteq \textsc{SpecMap}(u) \text{ and } |\mathcal{SA}(u)| \leq \beta(u) \text{ for all } u \in U.$$

Equivalently, a spectrum assignment is a way of SUs opening channels such that each SU opens at most β channels and can only open those in its spectrum map.

Given a spectrum assignment $\mathcal{SA}$, a potential edge $\{u, v\} \in E$ is called *realized* if $\mathcal{SA}(u) \cap \mathcal{SA}(v) \neq \emptyset$, i.e., there exists a channel opened by both u and v. The *realization graph* under a spectrum assignment is a graph $\mathcal{RG} = (U, E')$, where E' is the set of realized edges in E. Note that $\mathcal{RG}$ is a spanning subgraph of the potential graph $\mathcal{PG}$. A cognitive radio network is called *connectable* if there exists a spectrum assignment under which the realization graph is connected, in which case we also say that the cognitive radio network is *connected* under this spectrum assignment. Now we can formalize the problems studied in this paper.

The Spectrum Connectivity Problem. The SPECTRUM CONNECTIVITY problem is to decide whether a given cognitive radio network is connectable.

We are also interested in the special case where the number of possible channels is small[3] and SUs have the same antenna budget. Therefore, we define the following subproblem of the SPECTRUM CONNECTIVITY problem:

The Spectrum (k, β)-Connectivity Problem. For two constants $k, \beta \geq 1$, the SPECTRUM (k, β)-CONNECTIVITY problem is to decide whether a given cognitive radio network with k channels in which all SUs have the same budget β is connectable. For convenience we write SPECCON(k, β) to represent this problem.

Finally, we also consider the problem with special kinds of potential graphs, i.e. the potential graph is complete or a tree.

3 The Spectrum Connectivity Problem

In this section, we study the the SPECTRUM CONNECTIVITY problem from both complexity and algorithmic points of view.

3.1 NP-Completeness Results

We show that the SPECTRUM CONNECTIVITY problem is NP-complete even if the number of channels is fixed. In fact we give a complete characterization of the complexity of SPECCON(k, β) by proving the following dichotomy result:

[3] Commonly, the white spaces include spectrums from channel 21 (512Mhz) to 51 (698Mhz) excluding channel 37, which is totally 29 channels [1].

Theorem 1. SPECCON*(k, β) is* NP*-complete for any integers $k > \beta \geq 2$, and is in* P *if $\beta = 1$ or $k \leq \beta$.*

The second part of the statement is easy: When $\beta = 1$, each SU can only open one channel, and thus all SUs should be connected through the same channel. Therefore, the network is connectable if and only if there there exists a channel that belongs to every SU's spectrum map (and of course the potential graph must be connected), which is easy to check. When $k \leq \beta$, each SU can open all channels in its spectrum map, and the problem degenerates to checking the connectivity of the potential graph.

In the sequel we prove the NP-completeness of SPECCON(k, β) when $k > \beta \geq 2$. First consider the case $k = \beta + 1$. We will reduce a special case of the Boolean Satisfiability (SAT) problem, which will be shown to be NP-complete, to SPECCON($\beta + 1, \beta$), thus showing the NP-completeness of the latter.

A clause is called *positive* if it only contains positive literals, and is called *negative* if it only contains negative literals. For example, $x_1 \vee x_3 \vee x_5$ is positive and $\overline{x_2} \vee \overline{x_4}$ is negative. A clause is called *uniform* if it is positive or negative. A *uniform* CNF formula is the conjunction of uniform clauses. Define UNIFORM-SAT as the problem of deciding whether a given uniform CNF formula is satisfiable.

Lemma 1. UNIFORM-SAT *is* NP*-complete.*

Proof. Let F be a CNF formula with variable set $\{x_1, x_2, \ldots, x_n\}$. For each i such that $\overline{x_i}$ appears in F, we create a new variable y_i, and do the following:

- substitute y_i for all occurrences of $\overline{x_i}$;
- add two clauses $x_i \vee y_i$ and $\overline{x_i} \vee \overline{y_i}$ to F. More formally, let $F \leftarrow F \wedge (x_i \vee y_i) \wedge (\overline{x_i} \vee \overline{y_i})$. This ensures $y_i = \overline{x_i}$ in any satisfying assignment of F.

Call the new formula F'. For example, if $F = (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_3)$, then $F' = (x_1 \vee y_2) \wedge (y_1 \vee x_3) \wedge (x_1 \vee y_1) \wedge (\overline{x_1} \vee \overline{y_1}) \wedge (x_2 \vee y_2) \wedge (\overline{x_2} \vee \overline{y_2})$.

It is easy to see that F' is a uniform CNF formula, and that F is satisfiable if and only if F' is satisfiable. This constitutes a reduction from SAT to UNIFORM-SAT, which concludes the proof. ☐

Theorem 2. SPECCON*($\beta + 1, \beta$) is* NP*-complete for any integer $\beta \geq 2$.*

Proof. The membership of SPECCON($\beta + 1, \beta$) in NP is clear. In what follows we reduce UNIFORM-SAT to SPECCON($\beta + 1, \beta$), which by Lemma 1 will prove the NP-completeness of the latter.

Let $c_1 \wedge c_2 \wedge \ldots \wedge c_m$ be an input to UNIFORM-SAT where c_j, $1 \leq j \leq m$, is a uniform clause. Assume the variable set is $\{x_1, x_2, \ldots, x_n\}$. We construct an instance of SPECCON($\beta + 1, \beta$) as follows.

- **Channels:** There are $\beta + 1$ channels $\{0, 1, 2, \ldots, \beta\}$.
- **SUs:** 1) For each variable x_i, there is a corresponding SU X_i with spectrum map SPECMAP(X_i) $= \{0, 1, 2, \ldots, \beta\}$ (which contains all possible channels); 2) for each clause c_j, $1 \leq j \leq m$, there is a corresponding SU C_j with SPECMAP(C_j) $= \{p_j\}$, where $p_j = 1$ if c_j is positive and $p_j = 0$ if c_j is

negative; 3) there is an SU Y_2 with SpecMap$(Y_2) = \{2\}$. For every $1 \le i \le n$ and $2 \le k \le \beta$, there is an SU $Y_{i,k}$ with SpecMap$(Y_{i,k}) = \{k\}$; and 4) all SUs have the same antenna budget β.

- **Potential Graph:** For each clause c_j and each variable x_i that appears in c_j (either as x_i or $\overline{x_i}$), there is a potential edge between X_i and C_j. For each $1 \le i \le n$ and $3 \le k \le \beta$, there is a potential edge between X_i and $Y_{i,k}$. Finally, there is a potential edge between Y_2 and every X_i, $1 \le i \le n$.

Denote the above cognitive radio network by $\mathcal{I}$, which is also an instance of SpecCon$(\beta+1, \beta)$. We now prove that $c_1 \wedge c_2 \wedge \ldots \wedge c_m$ is satisfiable if and only if $\mathcal{I}$ is connectable.

First consider the "only if" direction. Let $A : \{x_1, \ldots, x_n\} \to \{0, 1\}$ be a satisfying assignment of $c_1 \wedge c_2 \wedge \ldots \wedge c_m$, where 0 stands for FALSE and 1 for TRUE. Define a spectrum assignment as follows. For each $1 \le i \le n$, let user X_i open the channels $\{2, 3, \ldots, \beta\} \cup \{A(i)\}$. Every other SU opens the only channel in its spectrum map.

We verify that $\mathcal{I}$ is connected under the above spectrum assignment. For each $1 \le i \le n$, X_i is connected to Y_2 through channel 2. Then, for every $2 \le l \le \beta$, $Y_{i,l}$ is connected to X_i through channel l. Now consider SU C_j where $1 \le j \le m$. Since A satisfies the clause c_j, there exists $1 \le i \le n$ such that: 1) x_i or $\overline{x_i}$ occurs in c_j; and 2) $A(x_i) = 1$ if c_j is positive, and $A(x_i) = 0$ if c_j is negative. Thus X_i and C_j are connected through channel $A(x_i)$. Therefore the realization graph is connected, completing the proof of the "only if" direction.

We next consider the "if" direction. Suppose there is a spectrum assignment that makes $\mathcal{I}$ connected. For every $1 \le i \le n$ and $2 \le l \le \beta$, X_i must open channel l, otherwise $Y_{i,l}$ will become an isolated vertex in the realization graph. Since X_i can open at most β channels in total, it can open at most one of the two remaining channels $\{0, 1\}$. We assume w.l.o.g. that X_i opens exactly one of them, which we denote by a_i.

Now, for the formula $c_1 \wedge c_2 \wedge \ldots \wedge c_m$, we define a truth assignment $A : \{x_1, \ldots, x_n\} \to \{0, 1\}$ as $A(x_i) = a_i$ for all $1 \le i \le n$. We show that A satisfies the formula. Fix $1 \le j \le m$ and assume that c_j is negative (the case where c_j is positive is totally similar). Since the spectrum map of SU C_j only contains channel 0, some of its neighbors must open channel 0. Hence, there exists $1 \le i \le n$ such that $\overline{x_i}$ appears in c_j and the corresponding SU X_i opens channel 0. By our construction of A, we have $A(x_i) = 0$, and thus the clause c_j is satisfied by A. Since j is chosen arbitrarily, the formula $c_1 \wedge c_2 \wedge \ldots \wedge c_m$ is satisfied by A. This completes the reduction from Uniform-SAT to SpecCon$(\beta, \beta + 1)$, and the theorem follows. □

Corollary 1. SpecCon*(k, β) is* NP*-complete for any integers $k > \beta \ge 2$.*

Proof. By a simple reduction from SpecCon$(\beta + 1, \beta)$: Given an instance of SpecCon$(\beta+1, \beta)$, create $k-\beta-1$ new channels and add them to the spectrum map of an (arbitrary) SU. This gives a instance of SpecCon(k, β). Since the new channels are only contained in one SU, they should not be opened, and thus the two instances are equivalent. Hence the theorem follows. □

Theorem 2 indicates that the SPECTRUM CONNECTIVITY problem is NP-complete even if the cognitive radio network only has three channels. We further strengthen this result by proving the following theorem:

Theorem 3. *The* SPECTRUM CONNECTIVITY *problem is* NP-*complete even if there are only two channels.*

Proof. We present a reduction from UNIFORM-SAT similar as in the proof of Theorem 2. Let $c_1 \wedge c_2 \wedge \ldots \wedge c_m$ be a uniform CNF clause with variable set $\{x_1, x_2, \ldots, x_n\}$. Construct a cognitive radio network as follows: There are two channels {0,1}. For each variable x_i there is a corresponding SU X_i with spectrum map $\text{SPECMAP}(X_i) = \{0, 1\}$ and antenna budget $\beta(X_i) = 1$. For each clause c_j there is a corresponding SU C_j with $\text{SPECMAP}(C_j) = \{p_j\}$ and $\beta(C_j) = 1$, where $p_j = 1$ if c_j is positive and $p_j = 0$ if c_j is negative. There is an SU Y with $\text{SPECMAP}(Y) = \{0, 1\}$ and $\beta(Y) = 2$. Note that, unlike in the case of $\text{SPECCON}(k, \beta)$, SUs can have different antenna budgets. Finally, the edges of the potential graph include: $\{X_i, C_j\}$ for all i, j such that x_i or $\overline{x_i}$ appears in c_j, and $\{Y, X_i\}$ for all i. This completes the construction of the cognitive radio network, which is denoted by $\mathcal{I}$. By an analogous argument as in the proof of Theorem 2, $c_1 \wedge c_2 \wedge \ldots \wedge c_m$ is satisfiable if and only if $\mathcal{I}$ is connectable, concluding the proof of Theorem 3. □

Theorem 3 is sharp in that, as noted before, the problem is polynomial-time solvable when there is only one channel.

3.2 Exact Algorithms

In this subsection we design algorithms for deciding whether a given cognitive radio network is connectable. Since the problem is NP-complete, we cannot expect a polynomial time algorithm.

Let n, k, t denote the number of SUs, the number of channels, and the maximum size of any SU's spectrum map, respectively ($t \le k$). The simplest idea is to exhaustively examine all possible spectrum assignments to see if there exists one that makes the network connected. Since each SU can have at most 2^t possible ways of opening channels, the number of assignments is at most 2^{tn}. Checking each assignment takes $\text{poly}(n, k)$ time. Thus the running time of this approach is bounded by $2^{tn}(nk)^{O(1)}$, which is reasonable when t is small. However, since in general t can be as large as k, this only gives a $2^{O(kn)}$ bound, which is unsatisfactory if k is large. In the following we present another algorithm for the problem that runs faster than the above approach when k is large.

Theorem 4. *There is an algorithm that decides whether a given cognitive radio network is connectable in time $2^{O(k+n \log n)}$, where n and k are the number of SUs and channels respectively.*

Proof. Let $\mathcal{I}$ be a given cognitive radio network with potential graph $\mathcal{PG}$. Let n be the number of SUs and k the number of channels. Assume that $\mathcal{I}$ is connected under some spectrum assignment. Clearly the realization graph contains a spanning tree of $\mathcal{PG}$, say T, as a subgraph. If we change the potential graph to T while keeping all other parameters unchanged, the resulting network will still be connected under the same spectrum assignment. Thus, it suffices to check whether there exists a spanning tree T of $\mathcal{G}$ such that $\mathcal{I}$ is connectable when substituting T for $\mathcal{PG}$ as its potential graph. Using the algorithm of [5], we can list all spanning trees of $\mathcal{PG}$ in time $O(Nn)$ where N is the number of spanning trees of $\mathcal{PG}$. By Cayley's formula [2, 11] we have $N \leq n^{n-2}$. Finally, for each spanning tree T, we can use the algorithm in Theorem 9 (which will appear in Section 5) to decide whether the network is connectable in time $2^{O(k)}n^{O(1)}$. The total running time of the algorithm is $O(n^{n-2})2^{O(k)}n^{O(1)} = 2^{O(k+n\log n)}$. □

Combining Theorem 4 with the brute-force approach, we obtain:

Corollary 2. *The* Spectrum Connectivity *problem is solvable can be solved in time $2^{O(\min\{kn,k+n\log n\})}$, with n and k being the number of SUs and channels respectively.*

4 Spectrum Connectivity with Complete Potential Graphs

In this section we consider the special case of the Spectrum Connectivity problem, in which the potential graph of the cognitive radio network is complete. We first show that this restriction does not make the problem tractable in polynomial time.

Theorem 5. *The* Spectrum Connectivity *problem is NP-complete even when the potential graph is complete and all SUs have the same antenna budget $\beta = 2$.*

Proof. The membership in NP is trivial. The hardness proof is by a reduction from the Hamiltonian Path problem, which is to decide whether a given graph contains a Hamiltonian path, i.e., a simple path that passes every vertex exactly once. The Hamiltonian Path problem is well-known to be NP-complete [6]. Let $G = (V, E)$ be an input graph of the Hamiltonian Path problem. Construct an instance of the Spectrum Connectivity problem as follows: The collection of channels is E and the set of SUs is V; that is, we identify a vertex in V as an SU and an edge in E as a channel. For every $v \in V$, the spectrum map of v is the set of edges incident to v. All SUs have antenna budget $\beta = 2$. Denote this cognitive radio network by $\mathcal{I}$. We will prove that G contains a Hamiltonian path if and only if $\mathcal{I}$ is connectable.

First suppose G contains a Hamiltonian path $P = v_1v_2\ldots v_n$, where $n = |V|$. Consider the following spectrum assignment of $\mathcal{I}$: for each $1 \leq i \leq n$, let SU v_i open the channels corresponding to the edges incident to v_i in the path P. Thus all SUs open two channels except for v_1 and v_n each of whom opens only one. For every $1 \leq i \leq n-1$, v_i and v_{i+1} are connected through the channel (edge) $\{v_i, v_{i+1}\}$. Hence the realization graph of $\mathcal{I}$ under this spectrum assignment is connected.

Now we prove the other direction. Assume that $\mathcal{I}$ is connectable. Fix a spectrum assignment under which the realization graph of $\mathcal{I}$ is connected, and consider this particular realization graph $\mathcal{RG} = (V, E')$. Let $\{v_i, v_j\}$ be an arbitrary edge in E'. By the definition of the realization graph, there is a channel opened by both v_i and v_j. Thus there is an edge in E incident to both v_i and v_j, which can only be $\{v_i, v_j\}$. Therefore $\{v_i, v_j\} \in E$. This indicates $E' \subseteq E$, and hence $\mathcal{RG}$ is a connected spanning subgraph of G. Since each SU can open at most two channels, the maximum degree of $\mathcal{RG}$ is at most 2. Therefore $\mathcal{RG}$ is either a Hamiltonian path of G, or a Hamiltonian cycle which contains a Hamiltonian path of G. Thus, G contains a Hamiltonian path.

The reduction is complete and the theorem follows. □

Notice that the reduction used in the proof of Theorem 5 creates a cognitive radio network with an unbounded number of channels. Thus Theorem 5 is not stronger than Theorem 1 or 3. Recall that Theorem 3 says the Spectrum Connectivity problem is NP-complete even if there are only two channels. In contrast we will show that, with complete potential graphs, the problem is polynomial-time tractable when the number of channels is small.

Theorem 6. *The* Spectrum Connectivity *problem with complete potential graphs can be solved in* $2^{2^k+O(k)}n^{O(1)}$ *time, where k is the number of channels and n is the number of SUs.*

Proof. Consider a cognitive radio network $\mathcal{I}$ with SU set U, channel set C and a complete potential graph, i.e., there is a potential edge between every pair of distinct SUs. Let $n = |U|$ and $k = |C|$. For each spectrum assignment $\mathcal{SA}$, we construct a corresponding *spectrum graph* $\mathcal{G}_{\mathcal{SA}} = (V, E)$ where $V = \{C' \subseteq C \mid \exists u \in U \text{ s.t. } \mathcal{SA}(u) = C'\}$ and $E = \{\{C_1, C_2\} \mid C_1, C_2 \in V; C_1 \cap C_2 \neq \emptyset\}$. Thus, V is the collection of subsets of C that is opened by some SU, and E reflexes the connectivity between pairs of SUs that open the corresponding channels. Since each vertex in V is a subset of C, we have $|V| \leq 2^k$, and the number of different spectrum graphs is at most 2^{2^k}.

We now present a relation between $\mathcal{G}_{\mathcal{SA}}$ and the realization graph of $\mathcal{I}$ under $\mathcal{SA}$. If we label each vertex u in the realization graph with $\mathcal{SA}(u)$, and contract all edges between vertices with the same label, then we obtain precisely the spectrum graph $\mathcal{G}_{\mathcal{SA}} = (V, E)$. Therefore, in the language of graph theory, $\mathcal{G}_{\mathcal{SA}} = (V, E)$ is a minor of the realization graph under $\mathcal{SA}$. Since graph minor preserves connectivity, $\mathcal{I}$ is connectable if and only if there exists a connected spectrum graph. Hence we can focus on the problem of deciding whether a connected spectrum graph exists.

Consider all possible graphs $G = (V, E)$ such that $V \subseteq 2^C$, and $E = \{\{C_1, C_2\} \mid C_1, C_2 \in V; C_1 \cap C_2 \neq \emptyset\}$. There are 2^{2^k} such graphs each of which has size $2^{O(k)}$. Thus we can list all such graphs in $2^{2^k+O(k)}$ time. For each graph G, we need to check whether it is the spectrum graph of some spectrum assignment of $\mathcal{I}$. We create a bipartite graph in which nodes on the left side are the SUs in $\mathcal{I}$, and nodes on the right side all the vertices of G. We add an edge between an SU u and a vertex C' of G if and only if $C' \subseteq \text{SpecMap}(u)$ and $|C'| \leq \beta(u)$, that is, u can open C' in a spectrum assignment. The size of H is $\text{poly}(n, 2^k)$ and its construction can be finished in $\text{poly}(n, 2^k)$ time. Now, if G is the spectrum graph of some spectrum assignment $\mathcal{SA}$, then we can identify $\mathcal{SA}$ with a subgraph of H consisting of all edges $(u, \mathcal{SA}(u))$ where u is an SU. In addition, in this subgraph we have

- every SU u has degree exactly one; and
- every node C' on the right side of H has degree at least one.

Conversely, a subgraph of H satisfying the above two conditions clearly induces a spectrum assignment whose spectrum graph is exactly G. Therefore it suffices to examine whether H contains such a subgraph. Furthermore, the above conditions are easily seen to be equivalent to:

- every SU u has degree at least one in G; and
- G contains a *matching* that includes all nodes on the right side.

The first condition can be checked in time linear in the size of H, and the second one can be examined by any polynomial time algorithm for bipartite matching (e.g., [7]). Therefore, we can decide whether such subgraph exists (and find one if so) in time $\text{poly}(n, 2^k)$. By our previous analyses, this solves the Spectrum Connectivity problem with complete potential graphs. The total running time of our algorithm is $2^{2^k+O(k)}\text{poly}(n, 2^k) = 2^{2^k+O(k)}n^{O(1)}$. □

Theorem 7. Spectrum Connectivity *with complete potential graphs is fixed parameter tractable when parameterized by the number of channels.*

5 Trees as Potential Graphs

In this section, we study another special case of the Spectrum Connectivity problem where the potential graph of the cognitive radio network is a tree. Many NP-hard combinatorial problems become easy on trees, e.g., the dominating set problem and the vertex cover problem. Nonetheless, as indicated by the following theorem, the Spectrum Connectivity problem remains hard on trees.

Theorem 8. *The* Spectrum Connectivity *problem is* NP*-complete even if the potential graph is a tree of depth one.*

Proof. We give a reduction from the Vertex Cover problem which is well known to be NP-complete [6]. Given a graph $G = (V, E)$ and an integer r, the Vertex Cover problem is to decide whether there exists r vertices in V that cover all the edges in E. Construct a cognitive radio network $\mathcal{I}$ as follows. The set of channels is $C = \{c_v \mid v \in V\}$. For each edge $e = \{u, v\} \in E$ there is an SU U_e with $\text{SpecMap}(U_e) = \{c_u, c_v\}$ and antenna budget 2. There is another SU M with $\text{SpecMap}(M) = C$ and antenna budget r. The potential graph is a star centered at M, that is, there is a potential edge between M and U_e for every $e \in E$. This finishes the construction of $\mathcal{I}$.

We prove that G has a vertex cover of size r if and only if $\mathcal{I}$ is connectable. First assume G has a vertex cover $S \subseteq V$ with $|S| \leq r$. Define a spectrum assignment $A(S)$ as follows: M opens the channels $\{c_v \mid v \in S\}$, and U_e opens both channels in its spectrum map for all $e \in E$. Since S is a vertex cover, we have $u \in S$ or $v \in S$ for each $e = \{u, v\} \in E$. Thus at least one of c_u and c_v is opened by M, which makes it connected to U_e. Hence the realization graph is connected. On the other hand, assume that the realization graph is connected under some spectrum assignment. For each $e = \{u, v\} \in E$, since the potential edge $\{M, U_e\}$ is realized, M opens at least one of c_u and c_v. Now define $S = \{v \in V \mid c_v \text{ is opened by } M\}$. It is clear that S is a vertex cover of G of size at most $\beta(M) = r$. This completes the reduction, and the theorem follows. □

We next show that, in contrast to Theorems 2 and 3, this special case of the problem is polynomial-time solvable when the number of channels is small.

Theorem 9. *Given a cognitive radio network whose potential graph is a tree, we can check whether it is connectable in $2^{O(t)}(kn)^{O(1)}$ time, where n is the number of SUs, k is the number of channels, and t is the maximum size of any SU's spectrum map. In particular, this running time is at most $2^{O(k)}n^{O(1)}$.*

Proof. Let $\mathcal{I}$ be a given cognitive radio network whose potential graph $\mathcal{PG} = (V, E)$ is a tree. Root $\mathcal{PG}$ at an arbitrary node, say r. For each $v \in V$ let $\mathcal{PG}_v$ denote the subtree rooted at v, and let $\mathcal{I}_v$ denote the cognitive radio network obtained by restricting $\mathcal{I}$ on $\mathcal{PG}_v$. For every subset $S \subseteq \textsc{SpecMap}(v)$, define $f(v, S)$ to be 1 if there exists a spectrum assignment that makes $\mathcal{I}_v$ connected in which the set of channels opened by v is exactly S; let $f(v, S) = 0$ otherwise. For each channel $c \in C$, define $g(v, c)$ to be 1 if there exists S, $\{c\} \subseteq S \subseteq \textsc{SpecMap}(v)$, for which $f(v, S) = 1$; define $g(v, c) = 0$ otherwise. Clearly $\mathcal{I}$ is connectable if and only if there exists $S \subseteq \textsc{SpecMap}(r)$ such that $f(r, S) = 1$.

We compute all $f(v, S)$ and $g(v, c)$ by dynamic programming in a bottom-up manner. Initially all values to set to 0. The values for leaf nodes are easy to obtain. Assume we want to compute $f(v, S)$, given that the values of $f(v', S')$ and $g(v', c)$ are all known if v' is a child of v. Then $f(v, S) = 1$ if and only if for every child v' of v, there exists $c \in S$ such that $g(v', c) = 1$ (in which case v and v' are connected through channel c). If $f(v, S)$ turns out to be 1, we set $g(v, c)$ to 1 for all $c \in S$. It is easy to see that $g(v, c)$ will be correctly computed after the values of $f(v, S)$ are obtained for all possible S. After all values have been computed, we check whether $f(r, S) = 1$ for some $S \subseteq \textsc{SpecMap}(r)$.

Denote $n = |V|$, $k = |C|$, and $t = \max_{v \in V} |\textsc{SpecMap}(v)|$. There are at most $n(2^t + k)$ terms to be computed, each of which takes time $\text{poly}(n, k)$ by our previous analysis. The final checking step takes $2^t \text{poly}(n, k)$ time. Hence the total running time is $2^t \text{poly}(n, k) = 2^t (kn)^{O(1)}$, which is at most $2^{O(k)} n^{O(1)}$ since $t \leq k$. Finally note that it is easy to modify the algorithm so that, given a connectable network it will return a spectrum assignment that makes it connected. □

Corollary 3. Spectrum Connectivity *with trees as potential graphs is fixed parameter tractable when parameterized by the number of channels.*

6 Conclusion

In this paper, we systematically study the algorithmic complexity of connectivity problem in cognitive radio networks through spectrum assignment. The hardness of the problem in the general case and several special cases are addressed. Our work gives a better understanding of the complexity of the problem. Exact algorithms are also derived to check whether the network is connectable. Due to interference, the connected nodes can not communicate simultaneously. One meaningful extension of this work is how to schedule the links such that the network throughput is optimized under realistic interference models. Another future work is to design efficient distributed channel assignment algorithms to achieve network connectivity.

Acknowledgements. The authors would like to give thanks to Dr. Thomas Moscibroda at Microsoft Research Asia for his introduction of the original problem.

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