Multi-rate Sequential Data Transmission

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***Abstract*—We investigate the data transmission problem in which a sequence of data is broadcast to a number of receivers via erasure channels with different erasure probabilities. Accordingly, the receivers wish to decode the data sequentially at different rates. We present a formulation of the problem and propose an optimal coding scheme. Our results can be employed in the streaming of a video clip by broadcasting, so that receivers with different bandwidths can play the video at different speeds. Specifically, receivers with sufficiently large bandwidth can play the video at normal speed, while others can play the video with pauses, or at a slower speed using time-scale modification.**

I. Problem Formulation

We consider transmitting data sequentially from a sender to a receiver through a memoryless erasure channel. We assume that the erasure channel is binary, and one bit is transmitted on the channel every unit time. Our results can readily be extended if the channel is instead a packet erasure channel.

Let the message sequence be a sequence of bits $M_1, M_2, ...$, where $M_i \overset{i.i.d.}{\sim} \mathrm{Bern}(1/2)$. The sender encodes the message sequence into a sequence of coded bits $X_1, X_2, ...$ and transmits them on the erasure channel. The coded bits are received at the receiver as $Y_1, Y_2, ...$, where $Y_i = X_i$ if no erasure occurs, otherwise $Y_i = e$. Based on these received symbols, the receiver tries to decode the message sequence as $\widetilde{M}_1, \widetilde{M}_2, \ldots$ Let $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} \{0, 1, e\}^n$ be the set of all possible sequences of received symbols.

In our setting, the erasure probability of the channel, or equivalently the channel capacity, is unspecified. We want to design a coding scheme that can guarantee different decoding rates for different channel capacities. This formulation also applies when the sender broadcasts a sequence of data to a number of receivers through erasure channels with a common input but different capacities.

In the sequel, we use the notation $Y_a^b = (Y_a, Y_{a+1}, ..., Y_b)$.

Definition 1 (MRS code)**.** A *multi-rate streaming code* (MRS code) is specified by a pair of encoding and decoding functions. The encoding function

$$\mathrm{Enc} : \{0, 1\}^{\mathbb{N}} \times \mathbb{N} \to \{0, 1\},\ (\{M_i\}, j) \mapsto X_j$$

maps the message bits $\{M_i\}$ to the coded bits $\{X_j\}$. The decoding function

$$\mathrm{Dec} : \mathcal{Y} \times \mathbb{N} \to \{0, 1\},\ (Y_1^n, i) \mapsto \widetilde{M}_i$$

maps the received symbols $\{Y_j\}$ to the decoded bits $\left\{\widetilde{M}_i\right\}$.

Note that the MRS code does not admit a fixed decoding rate like a block code. Instead its rate depends on the capacity of the erasure channel. Also note that in the above definition, the encoder can use the infinitely long message sequence for the encoding of every bit in the coded sequence. We will see in the next section that this assumption is actually superfluous, because there exist optimal codes for which the encoder uses only a finite segment of the message sequence to generate each bit, where the length of the segment increases with time.

Definition 2 (admissible pair)**.** For an MRS code, a capacity-rate pair (c, r) is *ϵ-admissible* if there exist N_0 such that when the receiver receives symbols from the sender through an erasure channel with capacity c (i.e., with erasure probability $1 - c$),

$$\mathbb{P}\left\{M_i \neq \widetilde{M}_i\left(Y_1^N\right)\right\} \leq \epsilon$$

for any $N \geq N_0$ and $i \leq N(r - \epsilon)$.

In other words, if (c, r) is ϵ-admissible, any receiver with channel capacity c can decode the first $N(r - \epsilon)$ message bits $M_1^{N(r-\epsilon)}$ with bit error probability less than ϵ when the first N symbols Y_1^N are received, for sufficiently large N, i.e., the receiver can decode the message causally at rate $r - \epsilon$ bits per channel use. Note that if $r \leq \epsilon$, then the capacity-rate pair (c, r) is always ϵ-admissible by any MRS code.

It is clear that for an MRS code, if (c, r) is ϵ-admissible, then all the pairs in $\{(c', r')|r' \leq r, c \leq c' \leq 1\}$ are ϵ-admissible. Therefore we can use a function to characterize all ϵ-admissible capacity-rate pairs of the code. We call $r : [0, 1] \to [0, \infty)$ a *rate-capacity function* if it is monotonically increasing, right continuous, and $r(0) = 0$.

Definition 3 (rate of MRS code)**.** A rate-capacity function $r(c)$ is called *ϵ-admissible* by a code if all of the pairs $(c, r(c))$ are ϵ-admissible by the code.

Definition 4 (achievable rate-capacity functions)**.** A rate-capacity function $r(c)$ is *achievable* if for any $\epsilon > 0$, there exist a code where $r(c)$ is ϵ-admissible by that code.

This work was done when C. T. Li was with The Chinese University of Hong Kong.

II. MAIN RESULT

The following theorem gives the achievable region for the rate-capacity function.

Theorem 5. *A rate-capacity function $r(c)$ is achievable if and only if*

$$\int_0^1 \frac{1}{c} dr(c) = \lim_{t \to 0^+} \int_t^1 \frac{1}{c} dr(c) \leq 1.$$

This theorem completely characterizes the fundamental tradeoff between the available bandwidth and the decoding rate of the receiver.