Dynamic Price Sequence and Incentive Compatibility*
(Extended Abstract)

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Abstract. We introduce and study a new auction model in which a certain type of goods is offered over a period of time, and buyers arrive at different times and stay until a common deadline (unless their purchase requests have been fulfilled). We examine in this model incentive compatible auction protocols (*i.e.*, those that induce participants to bid their true valuations).

We establish an interesting connection between incentive compatibility and price sequence: incentive compatibility forces a non-decreasing price sequence under some assumptions on market pricing schemes. We should point out that negation of our assumptions would require market distortions to some extent.

Our protocol may not ensure that one item must be sold everyday. Imposing such a market intervention, we show an impossibility result that deterministic incentive compatible auction protocols do not exist. With randomized relaxation, we give such an incentive compatible auction protocol. We also discuss incentive compatible protocols under other market conditions.

1 Introduction

The interplay of Computer Science and Economics has for quite a long time leaned towards the application of computer science concepts to those of economics [15,6,17,7,8]. Recently, many interesting ideas in economics, including the concept of incentive compatibility [19,5,11], which has played a central role in the studies of auction and related economic issues, started to make their ways into the studies of Computer Science and the Internet [16].

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The new economic platform of the Internet and electronic goods has brought renewed interests and new insight into the age-old problem. In recent work of digital-goods, where items can be sold in unlimited number of copies [10, 9], the main concerns have been incentive-compatibility and profit-maximizing for the auctioneer. One interesting result states that a digital-goods auction is incentive-compatible if and only if it is bid-independent [10,9]. As an example of bid-independent auction, the auctioneer can choose an arbitrary price at every instance of sales. Lavi and Nisan [12] studied the online auction where the auctioneer is required to respond to each bid as it arrives at different times, and characterized the incentive compatible protocols. Blum et al. [4] studied a general model in which both buyers and sellers stay for some periods after arriving, and discussed competitive strategies. For similar discussions on online auction and incentive compatibility see, *e.g.*, [1,2,3].

In the markets for many products, however, the price sequence exhibits certain patterns. For example, air-ticket price tends to rise towards the take-off date (of course, there are exceptions such as the last-minute price). In this paper we establish an interesting connection between incentive compatibility and price sequence towards a deadline in a semi-dynamic time auction setting. Our model is different from the standard online auction model in the following way: In our model, buyers may arrive at different times for a certain type of goods and they stay until a deadline or be offered to get goods, while in the ordinary online model, buyers arrive and leave at the same time with or without the goods. Both the standard model and ours may be practical models, relevant for modelling reality in different situations.

The price is shown going up if we want to have an incentive compatible auction protocol over days some items are sold, under mild assumptions. Our assumptions require that the market allows anyone to win if he bids sufficiently higher than all others, that the price does not depend on the particular buyers but on the bids submitted, and that the price may not go down if all bids are higher or equal. It is clear that lifting of those restrictions may be viewed as market interventions. In this case, social harmony relies on an inflationary economy. It is interesting to see such a phenomenon even under this limited constraint, in particular without introduction of interest rate or discounted future utility.

Our work may reveal an interesting direction in the study of price dynamics. The dynamics of goods prices is a difficult problem and is proposed, under the general equilibrium pricing model, by Smale as one of the most important mathematical problems for the 21st century [18]. Our study is based on an alternative economic pricing model.

We introduce some properties of incentive compatible auction protocols in Section 2, together with notations. Central to the concepts discussed here is that of critical value. The main idea is that the winning buyer pays a price that is dependent on the submitted bids of the other buyers. This idea is well known, and has been used in [13,14] to study combinatorial auctions, where interesting characterizations are obtained for incentive compatible protocols. Our presentation is motivated by, but slightly different from, their works in that the payment scheme is a little bit different and the starting axioms are also somewhat different. Therefore, we supply the lemmas in our model for completeness of the presentation, which by no means claims the results in Subsection 2.1 are our own.

In Section 3, we propose a deterministic incentive compatible protocol for the semi-dynamic auction model. Noticeably, the protocol forces a non-decreasing price sequence. In Section 4, we give strong evidence that this is to some extent unavoidable, by proving that the price sequence is non-decreasing for any deterministic incentive compatible protocol.

In Section 5, we discuss the necessity of those assumptions and present various cases of price sequences under other market conditions. Note that our incentive compatible auction protocols may not sell one item every day. We show that introducing such a market restriction will result in an impossibility result for deterministic incentive compatible protocols. Whereas for randomized relaxation, we give such an auction protocol to reach incentive compatibility in expectation. We also discuss auction protocols that utilize customer discriminating strategies to obtain incentive compatibility. Finally, we conclude our work with remarks and future studies in Section 6.

2 Preliminaries

We consider a price-based auction model in which an auctioneer sells a set of homogeneous goods to potential buyers. Each buyer desires exactly one item of the goods (buyers with multiple units requests can be reduced to this one). We denote buyers by i, $i = 1, 2, \ldots, n$. Each buyer i has a privately known *valuation* $v_i \in \mathbb{N}$, representing the maximal value that i would like to pay for the goods.

Each buyer i submits a *bid* $b_i \in \mathbb{N} \cup \{0\}$ to the auctioneer. When receiving all submitted bids from buyers, the auctioneer specifies the *winners* and the *price* $p \in \mathbb{R}^+ \cup \{0\}$ of the goods. If buyer i wins the goods, *i.e.*, i is a winner, his *utility* is $u_i = v_i - p$. If i does not win the goods, his *utility* is zero.

Here we assume all buyers are rational and aim to maximize their utilities. Note that to maximize the utility value, buyers might not submit their valuations truthfully according to different auction protocols. We say an auction is *incentive compatible* (or *truthful*) if for any buyer i and the submitted bids of other buyers, buyer i's utility is maximized by submitting his true valuation, *i.e.*, $b_i = v_i$.

We shall discuss some properties of incentive compatible auction protocols and then introduce notations for our semi-dynamic model.

2.1 Critical Values for Buyers Under Incentive Compatible Auctions

In this paper, we consider auctions with the *non-trivial property*: Any buyer with $b_i = 0$ will not win the goods; whereas if a buyer bids sufficiently large (*e.g.*, $b_i = +\infty$), he must win the goods.

Lemma 1 *For any incentive compatible auction, the non-trivial property implies the* participation constraints*: If buyer i with bid b_i wins, then we must have $b_i \geq p$.*

We establish the following observations to winners and losers, respectively. Most of the similar properties are previously known (see, *e.g.*, for single-minded auction [14]). We present them for the completeness of our discussion.

Lemma 2 *In incentive compatible auction ψ with non-trivial property, assume buyer i with bid b_i wins the goods at price p. If i bids $b_i' > p$, $b_i' \neq b_i$, rather than b_i, as long as the submitted bids of other buyers do not change, he still wins the goods at the same price p. In addition, if i bids $b_i' < p$, he will not win the goods.*

Lemma 3 *In incentive compatible auction ψ with non-trivial property, assume buyer i with bid b_i does not win the goods. Then there exists a unique minimal integer (critical value) $r_i(\psi, b_{-i}) > b_i$ such that i always wins the goods when he bids $b_i' \geq r_i(\psi, b_{-i})$, where b_{-i} is the collection of submitted bids of buyers except i.*

The above two lemmas define the concept of *critical value*: the one for all the winners is the same: the price; and the one for the losers may not be the same and be the price. We will make use of the concept in the following discussions.

2.2 Semi-dynamic Auction Model

We consider a special type of auction model, *semi-dynamic auction*. An auctioneer sells a type of goods to potential buyers. The process of auction will last for several consecutive (and discrete) time units. For convenience, we shall use *day* as the time unit, denoted by t. Some units of the goods (determined by the auction protocol) will be sold each day.

Let $b_{i,t} \in \mathbb{N} \cup \{0\}$ be the submitted *bid* of buyer i on the t-th day, and $p_t \in \mathbb{R}^+ \cup \{0\}$ be the *price* of the goods on the t-th day. Note that for any buyer, we allow he submits different bids on different days. If buyer i wins the goods on the t-th day, his *utility* is $u_i = v_i - p_t$, where v_i is the true valuation of i. Otherwise, his *utility* is zero. We will use the following notations:

- D: The time span, *i.e.*, the number of days.
- $d_i \in \{1, \ldots, D\}$: The first day that i can appear as a buyer. It may choose to arrive later as an adversary action but not earlier than d_i. We assume that i appears in the continuous days of the domain $\{d_i, \ldots, D\}$, unless he wins the goods (and consequently, quit).
- $r_{i,t}$: The *critical value* of buyer i at time t. Let A_t be the collection of buyers that appear on the t-th day, $1 \leq t \leq D$. For any time t and $i \in A_t$, if i is a loser, define $r_{i,t} = r_i(\psi, b_{-i})$ (the value defined in Lemma 3). If i is a winner, define $r_{i,t} = p_t$. Let $R_t = \max_{i \in A_t} r_{i,t}$.

An auction protocol is called *incentive compatible* if for any time t and any set of submitted bids of other buyers, the utility of buyer i is maximized by submitting his true valuation, *i.e.*, $b_{i,t} = v_i$, for all $d_i \leq t \leq D$.

Here, we should get the meaning of price p_t: It is possible that not every buyer i with bid $b_{i,t} > p_t$ would win the goods. In this case, there may be a fixed quantity, say δ_t, of the goods for sale on each day. There might be more than δ_t buyers bidding higher than p_t, some buyers would still lose while others are selected winners according to the auction protocol.

3 An Incentive Compatible Semi-dynamic Auction Protocol

Let Ψ be the collection of all incentive compatible auction protocols for the ordinary one period case (*i.e.*, $D = 1$) satisfying all buyers with bids higher than the price win the goods (for example, Vickrey auction [19]). For any $\psi \in \Psi$, let $p(\psi, Z)$ be the price of the goods when the auctioneer selects auction protocol ψ, upon receiving submitted bids vector Z.

Deterministic Auction Scheme:

1. The auctioneer selects $\psi \in \Psi$ arbitrarily, and sets $R_0 = 0$.
2. For $t = 1, \ldots, D$
   (i) let $p_t = \max\{R_{t-1}, p(\psi, Z_t)\}$ be the price of the goods on the t-th day, where Z_t is the submitted bids vector this day,
   (ii) all buyers with bids higher than p_t win the goods,
   (iii) compute the critical value $r_{i,t}$ for each buyer in A_t, and let $R_t = \max_{i \in A_t} r_{i,t}$.

Example 1 We assume that for each day, the auctioneer always selects 1-item Vickrey (second-price) auction [19]. On the first day, for instance, buyers $A_1 = \{1, \ldots, k_1\}$ appear to the auction with submitted bids $b_{1,1} \geq b_{2,1} \geq \cdots \geq b_{k_1,1}$ (ties are broken arbitrarily), respectively. Therefore, according to the above Deterministic Auction Protocol, the price of the goods is the second highest bid $b_{2,1}$ (*i.e.*, $p_1 = b_{2,1}$). If $b_{1,1} > b_{2,1}$, then buyer 1 wins the goods; otherwise, no buyer wins the goods. In this case, the critical value for every loser is $b_{1,1}$. Hence, $R_1 = b_{1,1}$. On the next day, if the second highest bid is not less than R_1, then price p_2 is set to be that bid; otherwise, $p_2 = R_1$.

Theorem 1 *The above Deterministic Auction Protocol is incentive compatible.*

Intuitively and informally, since the price goes up, the best chance of the buyers is at the first day of entry to the market. They would not lie by the incentive compatible requirement for the single period. The detailed proof is omitted here and will be presented in the journal version.

We comment that if we change R_{t-1} in determining p_t to anything smaller, say $R_{t-1} - \epsilon$, the protocol is no longer incentive compatible. In particular we cannot replace R_{t-1} by p_{t-1} in the protocol, as the following example shows.

Example 2 We still consider 1-item Vickrey auction. On the first day, three buyers come to auction with submitted bids 20,15,10, respectively. Specifically, we consider the behavior of buyer 2, let his valuation be 15 (*i.e.*, $v_2 = 15$). If he bids 15 truthfully on the first day, then we know that (i) buyer 1 (with submitted bid 20) wins, (ii) $p_1 = 15$, and (iii) on the second day, $p_2 = \max\{p_1, 10\} = 15$, which implies that the utility of buyer 2 is always zero. If buyer 2 bids 11 untruthfully, however, then $p_1 = 11$ and $p_2 = \max\{p_1, 10\} = 11$. Thus, he wins the goods on the second day with utility $15 - 11 > 0$.

4 Non-decreasing Property of Price Sequence

We prove here that the price sequence is non-decreasing in general if we assume the auction protocol is required to be incentive compatible. We make the following mild assumptions on the pricing protocols:

- *Non-trivial:* As defined in Section 2.
- *Non-discriminating*: The price p_t only depends on the sets of submitted bids in the previous t rounds: $p_t(B_1, B_2, \ldots, B_t)$, where B_j is the (multi) set of submitted bids on the j-th day, $1 \leq j \leq t$. That is, the bids are detached from buyers when determining prices. As a special case, if two buyers exchange their bids at a given time, the price does not change: $p(i : \alpha; j : \beta) = p(i : \beta; j : \alpha)$. where $p(i : \alpha; j : \beta)$ denotes the price of the goods when i bids α and j bids β.
- *Monotone*: For any time t, t_1, $1 \leq t_1 \leq t$, $p_t(B_1, B_2, \ldots, B_{t_1} \cup \{\alpha\}, \ldots, B_t) \geq p_t(B_1, B_2, \ldots, B_{t_1} \cup \{\alpha'\}, \ldots, B_t)$, for any $\alpha > \alpha'$.

Note that, non-trivial property and non-discriminating property are related but the former statement is about the winners of the goods and the latter one is about the winning price. Both are axioms describing the anonymity of the buyers.

Lemma 4 *Let $t > 0$ and $p_t = p(i : b_{i,t}; j : b_{j,t})$. If $b_{i,t} \geq b_{j,t} > p_t$ or $b_{i,t} > b_{j,t} \geq p_t$, and if buyer j wins, then buyer i also wins the goods at time t.*

Sketch of the Proof. Assume, to the contrary, that buyer i does not win the goods. Let $r_{i,t}$ be the critical value of i at time t. Due to Lemma 3, we know that $r_{i,t} > b_{i,t}$. Since $r_{i,t}, b_{i,t} \in \mathbb{N}$, we have $r_{i,t} - 1 \geq b_{i,t}$. Note that $b_{i,t} > p_t$, it follows that

$$p_t \; < \; r_{i,t} - 1. \tag{1}$$

If buyer i bids $r_{i,t}$, he wins the goods at price $p(i : r_{i,t}; j : b_{j,t})$. We claim that

$$r_{i,t} - 1 \leq p(i : r_{i,t}; j : b_{j,t}) \tag{2}$$

Otherwise, $p(i : r_{i,t}; j : b_{j,t}) < r_{i,t} - 1$. By Lemma 2, if buyer i bids $r_{i,t} - 1$, he also wins the goods. This contradicts to $r_{i,t}$'s definition.

Since $b_{j,t} \leq b_{i,t}$, we have

$$p(i : r_{i,t}; j : b_{j,t}) \leq p(i : r_{i,t}; j : b_{i,t}), \tag{3}$$

due to the monotone property. By Lemma 2 and $p(i : b_{i,t}; j : b_{j,t}) = p_t < r_{i,t}$, we have

$$p(i : b_{i,t}; j : r_{i,t}) = p(i : b_{i,t}; j : b_{j,t}). \tag{4}$$

Combining (1), (2), (3), (4), we have $p(i : b_{i,t}; j : r_{i,t}) = p(i : b_{i,t}; j : b_{j,t}) = p_t < r_{i,t} - 1 \leq p(i : r_{i,t}; j : b_{j,t}) \leq p(i : r_{i,t}; j : b_{i,t})$, which contracts to the non-discriminating property. □

Lemma 5 *For any time $t > 0$, assume the price set by the auctioneer is p_t. Then any buyer with bid $b_{i,t} > p_t$ must win the goods at time t.*

Sketch of the Proof. Assume to the contrary, that there exists a loser i with bid $b_{i,t} > p_t$. Suppose buyer $1', \ldots, \delta'_t$ win the goods with submitted bids $b_{1',t} \geq b_{2',t} \geq \cdots \geq b_{\delta'_t,t}$, respectively, where δ_t is the fixed quantity of the goods for sale at time t. For buyer $1'$, due to Lemma 2, we know that if he bids $b'_{1',t} = b_{i,t} > p_t$, he still wins the goods at price p_t. Note that $b'_{1',t} = b_{i,t} < b_{\delta'_t,t} \leq \cdots \leq b_{2',t}$. From Lemma 4, we know that buyer i, along with $1', 2', \ldots, \delta'_t$, should also win the goods when $1'$ bids $b'_{1',t}$. That is, there are at least $\delta_t + 1$ items to be sold at time t, which contradicts to our assumption. □

Lemma 6 *For any two days $t, t+1$, $1 \leq t < D$, if at least one item of the goods is sold, the price must satisfy $p_t \leq p_{t+1}$.*

Sketch of the Proof. Assume buyers $A_t = \{1, \ldots, k_t\}$ appear on the t-th day, with submitted bids $b_{1,t} \geq b_{2,t} \geq \cdots \geq b_{k_t,t}$, and buyers $A_{t+1} = \{1', \ldots, k'_{t+1}\}$ appear on the $(t+1)$st day, with submitted bids $b_{1',t+1} \geq b_{2',t+1} \geq \cdots \geq b_{k'_{t+1},t+1}$, respectively.

Due to Lemma 4 and our assumptions, we know that buyer 1 (and $1'$) win the goods at price p_t (and p_{t+1}) on the t-th (and $(t+1)$st) day, respectively. Assume to the contrary that $p_t > p_{t+1}$. We following consider two cases.

Case 1. There exists a winner j' on the $(t+1)$st day such that $d_{j'} < t+1$, *i.e.*, he loses on the t-th day. Let $r_{j',t} > b_{j',t}$ be his critical value on that day. Then $p(j' : r_{j',t}) \geq p(j' : b_{j',t}) = p_t > p_{t+1}$, where the first inequality is due to the monotone property of the price function. Consider the state that $r_{j',t}$ be the true valuation of buyer j', if he bids truthfully on the t-th day, his utility is $r_{j',t} - p(j' : r_{j',t})$. Whereas if he bids $b_{j',t}$ on the t-th day, and bids $b_{j',t+1}$ on the $(t+1)$st day, he will win the goods with utility $r_{j',t} - p_{t+1} > r_{j',t} - p(j' : r_{j',t})$. A contradiction to incentive compatibility.

Case 2. For all winner j' on the $(t+1)$st day, $d_{j'} = t+1$. Specially, $d_{1'} = t+1$. For the buyer 1 on the t-th day, let $b_{1,t}$ be his true valuation, *i.e.*, $v_1 = b_{1,t}$. Hence, if he bids truthfully on the t-th day, his utility is $v_1 - p_t$. Following we consider the case that buyer 1 bids zero on the t-th day (which implies that he loses on the t-th day), and bids $b_{1',t+1}$ on the $(t+1)$st day. We remove buyer $1'$ from the auction on the $(t+1)$st day (note that $d_{1'} = t+1$). Assume the new price is p'_{t+1}. Note that the set of bids on $(t+1)$st day is the same now, due to the monotone property of the price function, we have $p'_{t+1} \le p_{t+1}$. Now the utility of buyer 1 is $v_1 - p'_{t+1} \ge v_1 - p_{t+1} > v_1 - p_t$. A contradiction.

Therefore, the lemma follows. □

Theorem 2 *Let $p_1, \ldots, p_D$ be a price sequence of consecutive transactions, then $p_1 \le p_2 \le \cdots \le p_D$.*

Sketch of the Proof. We may skip all the days with no transactions, and the protocol and the transactions will not change. Then the theorem follows by Lemma 6. □

5 Incentive Compatibility Under Other Market Conditions

In this section, we discuss incentive compatible protocols under various market conditions.

5.1 An Impossibility Result

Theorem 3 *For any buyers with arbitrary bids, if $D > 1$ and at least one item of the goods is sold each day, then the deterministic incentive compatible auction protocol satisfying non-trivial, non-discriminating and monotone properties does not exist.*

The key point of the theorem is the non-decreasing property showed by Lemma 6 and the fact that at least one item of the goods is sold each day.

Sketch of the Proof of Theorem 3. Note that on the t-th day, all buyers with bids higher than p_t win the goods. According to our requirement, however, at least one buyer will win the goods on the $(t+1)$st day, no matter what bids of buyers are submitted. Hence, we may consider a special case that the submitted bid of each buyer on the $(t+1)$st day is strictly smaller than p_t. Therefore, we must have $p_t > p_{t+1}$, where p_{t+1} is the price of the goods on the $(t+1)$st day, which contradicts to Lemma 6. □

5.2 A Randomized Incentive Compatible Auction Protocol

The impossibility result leaves open the question whether we can ensure incentive compatibility when a fixed number of items are required to be sold each time. In this subsection, we introduce one randomized solution under the following restrictions:

– For convenience, we assume that one item is sold each time. That is, there are totally m items to be sold in $D = m$ continuous days, one item each day. The general case is similar.
– As in a ration system of war time, we assume that each buyer bids for the goods only once, and his following bid is the same to his first commitment. That is, $b_{i,t} = b_{i,d_i}$, for $d_i \leq t \leq D$.

Note that for randomized protocols, the meaning of incentive compatibility is to guarantee that truthful bid always maximizes a buyer's expected utility, *i.e.*, the auction is *incentive compatible in expectation.*

Randomized Auction Protocol:

1. For $t = 1, \ldots, D$
   (i) For each buyer, its entry bid is taken as its bids at subsequent time,
   (ii) let the price p_t be the $(m + 2 - t)$st highest submitted bid at time t,
   (iii) sell one item to one of the first $(m+1-t)$ buyers whose bids are not less than p_t, with probability $\frac{1}{m+1-t}$ each (*i.e.*, exactly one buyer wins).

For example, if $m = 2$, the auctioneer sells two items in two continuous days. On the first day, define the price to be the third highest submitted bid, and sell one item to the first two buyers with probability 1/2 each. On the second day, sell the remaining item in terms of 1-item Vickrey auction [19].

Lemma 7 *The price of the goods is non-decreasing, i.e.,* $p_1 \leq p_2 \leq \ldots \leq p_m$.

Therefore if the buyer wins the goods with zero probability on the t-th day, he still can not win in the following days.

Theorem 4 *The above Randomized Auction Protocol sells exactly one item of the goods and is incentive compatible in expectation.*

Sketch of the Proof. For convenience, we denote the submitted bid of buyer i by b_i. For arbitrary fixed submitted bids of other buyers, we only need to prove that for any b_i, we have $E[u_i(v_i)] \geq E[u_i(b_i)]$, where $E[u_i(b_i)]$ is the expected utility of i when submitting b_i. Without loss of generality, assume that $d_i = 1$, *i.e.*, buyer i appears on the first day.

Let $S_t = \{j \mid d_j = t, j \neq i\}$, $t = 1, \ldots, m$. Let S_0 denote the losers before buyer i appears. Next, we prove that for any $S_0, S_1, \ldots, S_m$, and b_i,

$$E[u_i(v_i; S_0, S_1, \ldots, S_m)] \;\geq\; E[u_i(b_i; S_0, S_1, \ldots, S_m)], \tag{5}$$

by mathematical induction on the number of items m.

If $m = 1$, it is equivalent to the deterministic 1-item Vickrey auction, so we always have $u_i(v_i; S_0, S_1) \geq u_i(b_i; S_0, S_1)$. Assume (5) holds for the case of $(m - 1)$. Following we consider there are m items to be sold.

Case 1. $v_i \le p_1(i:v_i)$, where $p_1(i:v_i)$ is the price of the goods of the first day when i bids v_i. Therefore, $u_i(v_i; S_0, S_1, \ldots, S_m) = 0$. If $b_i \le p_1(i:b_i)$, then $u_i(b_i; S_0, S_1, \ldots, S_m) = 0$, and (5) holds. Otherwise, $b_i > p_1(i:b_i)$. It is easy to see that $p_1(i:b_i) \ge v_i$. By Lemma 7 we know that the price is non-decreasing. Thus $E[u_i(b_i; S_0, S_1, \ldots, S_m)] \le 0$.

Case 2. $v_i > p_1(i:v_i)$, then

$$E[u_i(v_i; S_0, S_1, \ldots, S_m)]$$
$$= \frac{1}{m}(v_i - p_1(i:v_i)) + \frac{1}{m} \sum_{\substack{j:j\neq i \\ b_{j,1} > p_1(i:v_i)}} E\left[u_i(v_i; S_0 \cup S_1 - \{j\}, S_2, \ldots, S_m)\right]. \quad (6)$$

We may assume that $b_i > p_1(i:b_i)$, otherwise

$$E[u_i(b_i; S_0, \ldots, S_m)] \le 0 \le E[u_i(v_i; S_0, \ldots, S_m)].$$

It is easy to see that $p_1(i:b_i) = p_1(i:v_i)$, and

$$E[u_i(b_i; S_0, S_1, \ldots, S_m)]$$
$$= \frac{1}{m}(v_i - p_1(i:b_i)) + \frac{1}{m} \sum_{\substack{j:j\neq i \\ b_{j,1} > p_1(i:b_i)}} E[u_i(b_i; S_0 \cup S_1 - \{j\}, S_2, \ldots, S_m)]. \quad (7)$$

By the induction hypothesis, we have

$$E[u_i(v_i; S_0 \cup S_1 - \{j\}, S_2, \ldots, S_m)] \ge E[u_i(b_i; S_0 \cup S_1 - \{j\}, S_2, \ldots, S_m)]. \quad (8)$$

Combining (6), (7), (8), we have

$$E\left[u_i(v_i; S_0, S_1, \ldots, S_m)\right]$$
$$= \frac{1}{m}(v_i - p_1(i:v_i)) + \frac{1}{m} \sum_{\substack{j:j\neq i \\ b_{j,1} > p_1(i:v_i)}} E[u_i(v_i; S_0 \cup S_1 - \{j\}, S_2, \ldots, S_m)]$$
$$\ge \frac{1}{m}(v_i - p_1(i:b_i)) + \frac{1}{m} \sum_{\substack{j:j\neq i \\ b_{j,1} > p_1(i:b_i)}} E[u_i(b_i; S_0 \cup S_1 - \{j\}, S_2, \ldots, S_m)]$$
$$= E[u_i(b_i; S_0, S_1, \ldots, S_m)].$$

Hence (5) holds for any m, and the theorem follows. □

5.3 Discriminative Incentive Compatible Auction Protocols

If discriminative pricing scheme is allowed, such as the case in many information products, software systems, for example, the price sequence over time may not be decreasing.

As an example, we may sort the customers according to their names. At time t, we consider the first buyer in the ordered list. If his submitted bid is not less than that of the second buyer, he wins the goods at the price of the second buyer's bid. Otherwise, we remove this buyer from the list and consider the next one. In this protocol, we exactly sell one item every day. It is not hard to see this is a bid-independent protocol. And it is not difficult to verify it is incentive compatible.

Other interesting incentive compatible auction protocols exist when discriminative pricing protocols are used.

6 Conclusion and Discussions

In this paper, we discuss the connections between incentive compatibility and price sequence for the semi-dynamic auction model, where the auction lasts for several consecutive time units and buyers appear to the auction in the continuous time units until he wins the goods. The problem deserves further investigation into other different models.

– As an example, suppose that all buyers come to auction on the first day with different maximum departure dates, what is the characterization on price dynamics for incentive compatible protocols?

Note that there is a symmetry with respect to time in comparison with the model discussed here. However, it is not very clear how would the approach be carried over for it. More generally, it would be interesting to understand the full dynamics of price system in response to the dynamics of participating agents of the market.

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