Testing multipartite entanglement with Hardy's nonlocality

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Multipartite quantum states may exhibit different types of quantum entanglement in that they cannot be converted into each other by local quantum operations only, and fully understanding mathematical structures of different types of multipartite entanglement is a very challenging task. In this paper, from the viewpoint of Hardy's nonlocality, we compare W and GHZ (Greenberger-Horne-Zeilinger) states and show a couple of crucial different behaviors between them. Particularly, by developing a geometric model for Hardy's nonlocality problem of W states, we derive an upper bound for its maximal violation probability, which turns out to be strictly smaller than the corresponding probability of GHZ states. This gives us a new comparison between these two quantum states, and the result is also consistent with our intuition that GHZ states are more entangled. Furthermore, we generalize our approach to obtain an asymptotic characterization for general N-qubit W states, revealing that when N goes up, the speed that the maximum violation probabilities decay is exponentially slower than that of general N-qubit GHZ states. We provide some numerical simulations to verify our theoretical results.

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I. INTRODUCTION

Entanglement plays a central role in quantum information processing tasks, and it is often entanglement that makes quantum schemes enjoy remarkable advantage over their classical counterparts. Therefore, studying and characterizing the properties of quantum entanglement is naturally an important and fundamental problem. At present, the structure of quantum entanglement for bipartite quantum states has been relatively clear, especially the case of pure states. However, the situation of multipartite entanglement is much more complicated, and it is still far from being understood very well. Nevertheless, a remarkable fact on multipartite entanglement has been well known: that is, multipartite quantum states can be entangled in different ways, in that different kinds of multipartite entanglement cannot be converted into each other by local operations only [1]. A most famous example that demonstrates this fact is Greenberger-Horne-Zeilinger (GHZ) and W states, as they are two different forms of entanglement in three-qubit quantum states [1].

Different entanglement forms exhibit different properties. In the example of GHZ and W states, it has been well known that the GHZ state is more entangled, but the W state is more robust against qubit loss. For the general case of multipartite entanglement, however, very little like this is known. In order to gain a deep understanding of this problem, characterizing different entanglement forms from more viewpoints is highly demanded.

One attempt of this kind is comparing the underlying quantum nonlocality produced by different forms of multipartite entanglement, which is usually indicated by the fact that involved quantum states violate certain Bell inequalities, which is called Bell nonlocality. Particularly, since Bell nonlocality is revealed by outcome statistics of quantum measurements, this approach allows us to observe the differences between multipartite entangled quantum states directly in quantum laboratories [2]. What is more, the task can be fulfilled in a device-independent manner, making it possible to compare different kinds of multipartite entanglement *reliably* by using unreliable quantum devices, as one does not have to care about the internal workings of involved quantum devices. In fact, this approach has been studied extensively for years, and a lot of interesting results that certify the existence of multipartite entanglement have been reported [2–6]. As an example, it has been shown that Bell inequalities exist such that they can be violated by W states but not by GHZ states, and vice versa [2]. Furthermore, Bell nonlocality has even been utilized to quantify quantum entanglement in a device-independent manner, though for now most results are focusing on bipartite cases [7–10].

Therefore, it can be said that quantum nonlocality gives us a powerful tool to study quantum entanglement. Interestingly, apart from Bell inequalities, there exist other ways to exhibit nonlocality without resorting to inequalities [11–14], and Hardy's paradox is a famous framework of this kind [14]. For convenience, in later discussions we use Hardy's nonlocality to address the nonlocal property revealed by Hardy's paradox.

The original Hardy's nonlocality problem was a proof of entanglement for almost all two qubit states [14], and later was generalized to scenarios of multiple qubits, multiple settings, and qudit states [15–20]. Furthermore, a lot of experiments have been performed to confirm the paradox [21–23]. In this paper, our comparisons will be based on Hardy's nonlocality problem for multiqubit states proposed in [16], which can be formulated as below. Consider an N-qubit quantum state $|\psi\rangle$ and two sets of observables U_i and D_i ($i \in [N]$, where

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$[N] \equiv \{1, 2, \ldots, N\}$), where the subscript i represents that the observable measures the ith qubit alone. The observables are set up so that

$$\begin{aligned}
&P(D_1U_2\cdots U_N|{+}{+}\cdots{+}) = 0,\\
&P(U_1D_2\cdots U_N|{+}{+}\cdots{+}) = 0,\\
&\qquad\vdots\\
&P(U_1U_2\cdots D_N|{+}{+}\cdots{+}) = 0,\\
&P(D_1D_2\cdots D_N|{-}{-}\cdots{-}) = 0,\\
&P(U_1U_2\cdots U_N|{+}{+}\cdots{+}) > 0,
\end{aligned}$$

where $P(A_1A_2\cdots A_N|+++)$ denotes the joint probability when one measures the ith qubit with the measurement setting A_i and gets the outcome $+$, and the other expressions are similar. And for convenience, we call the first $N+1$ relations *equation constraints*.

It turns out that quantum entanglement is necessary to manifest Hardy's nonlocality, i.e., satisfy all the constraints above [14,16]. Indeed, in any classical scenario where each local measurement is independent of the others, the last inequality gives $P(U_i|+) > 0$ for all $i \in [N]$, implying that $P(D_i|+) = 0$ for all $i \in [N]$, which is a contradiction to $P(D_1D_2\cdots D_N|{-}{-}\cdots{-}) = 0$. Therefore, if a classical system satisfies the first $N+1$ constraints, we must have that $P(U_1U_2\cdots U_N|{+}{+}\cdots{+}) = 0$, and the violation to this relation means that the system must be quantum. For convenience, when the first $N+1$ constraints are satisfied, we call the maximal value of $P(U_1U_2\cdots U_N|{+}{+}\cdots{+})$ the maximal violation probability.

Since Hardy's nonlocality reveals quantumness of entangled quantum states, a nature question is, can we utilize it to look into the essential properties of multipartite entanglement, like describing the differences between multipartite entanglement forms? In fact, to our knowledge Hardy's nonlocality has not been utilized to compare different entanglement structures of multipartite quantum states. In this paper, we show that this is indeed possible. Particularly, complementing a previous work that investigated Hardy's nonlocality for multipartite GHZ states [16], we analyze Hardy's nonlocality for multipartite W states, which exhibits some crucial differences between them, confirming the above possibility.

To achieve this, by developing a new geometric model for W states in Hardy's nonlocality problem, we derive an upper bound of the maximal violation probability for the perfect three-qubit W state, which is 1/9 and strictly smaller than the corresponding probability of the perfect three-qubit GHZ state, 0.125. Note that this comparison is consistent with our intuition that the GHZ state is more entangled, though we have known that entanglement and nonlocality are two different computational resources. Furthermore, we also obtain an asymptotic lower bound of maximum violation probabilities for multipartite W states as well, which is roughly $\Omega(1/N)$. And this means that when N goes up, the speed that maximum violation probabilities for multipartite W states decay is exponentially slower than that of multipartite GHZ states. We also provide some numerical simulation results to verify our theoretical results. Therefore, our results indicate a couple of crucial different behaviors of W states and GHZ states from the viewpoint of Hardy's nonlocality.

As stressed above, characterizing the structures of multipartite quantum entanglement is a fundamental problem in quantum information. Our main results imply that Hardy's nonlocality provides us an additional viewpoint from which to look into this problem. Since Hardy's nonlocality is described by quantum measurement outcome statistics only, this new approach is also of a device-independent nature, thus bringing us the convenience of reliable physical implementations by unreliable quantum devices.

II. GEOMETRIC MODEL FOR GENERALIZED THREE-QUBIT W STATES

In this section, we first consider the *generalized W state*, which can be expressed as

$$|\psi\rangle = a_1|100\rangle + a_2|010\rangle + a_3|001\rangle, \tag{1}$$

where $a_i \neq 0$ and $\sum_i |a_i|^2 = 1$. By applying local phases on the basis state $|1\rangle$ of each qubit, we may assume without loss of generality that $a_i > 0$.

Note that the constraints and the objective in Hardy's nonlocality problem are given by relations on joint probability distributions of measurement outcomes of observables. We now develop a geometric model to represent local observables for generalized W states, which allows us to formulate these joint probability distributions in the language of vectors. Later we will see that the geometric model can be generalized to N-qubit generalized W states.

Given an observable A, let λ be one of its eigenvalues with one-dimensional eigenspace; its corresponding eigenstate can be written as

$$|\phi\rangle = \cos\varphi\,|0\rangle + e^{i\theta}\sin\varphi\,|1\rangle, \tag{2}$$

where $\varphi \in [0, \pi/2]$ and $\theta \in [0, 2\pi)$.

To build our geometric model, when $\varphi \neq \pi/2$ we make the following definition.

Definition 1. The *representation vector* of the observable/eigenvalue pair (A, λ) is defined as

$$v(A,\lambda) \equiv (\tan\varphi\cos\theta, \tan\varphi\sin\theta)^T \in \mathbb{R}^2, \tag{3}$$

and, for convenience, when $\varphi \neq \pi/2$ we say $v(A,\lambda)$ is *well defined*.

Recall that Hardy's nonlocality problem is a maximization problem among local observables U_i and D_i with eigenvalues ± 1, where the subscript $i \in [3]$ indicates the observable measuring the ith qubit:

$$\begin{aligned}
\text{maximize}\quad & P(U_1U_2U_3|+++), && (4)\\
\text{subject to}\quad & P(D_1U_2U_3|+++) = 0, && (5)\\
& P(U_1D_2U_3|+++) = 0, && (6)\\
& P(U_1U_2D_3|+++) = 0, && (7)\\
& P(D_1D_2D_3|---) = 0. && (8)
\end{aligned}$$

By our geometric model, the above conditions can be restated, as shown in the following proposition.

Proposition 2. Let A_1, A_2, A_3 be observables with all their eigenspaces being one-dimensional. Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues corresponding to A_1, A_2, A_3, respectively. Suppose $v(A_i, \lambda_i)$ is well defined for $i \in [3]$. Let $t_i = a_i \cdot v(A_i, \lambda_i)$. Then

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = \frac{\|t_1 + t_2 + t_3\|^2}{\left(1 + \frac{1}{a_1^2}\|t_1\|^2\right)\left(1 + \frac{1}{a_2^2}\|t_2\|^2\right)\left(1 + \frac{1}{a_3^2}\|t_3\|^2\right)}. \quad (9)$$

Proof. Suppose the eigenstate for (A_i, λ_i) pair is

$$|\phi_i\rangle = \cos\varphi_i |0\rangle + e^{i\theta_i}\sin\varphi_i |1\rangle,$$

where $\varphi_i \in [0, \pi/2)$ and $\theta_i \in [0, 2\pi)$.

By the postulate of quantum measurement, we have that

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = \langle\psi| (|\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2| \otimes |\phi_3\rangle\langle\phi_3|) |\psi\rangle = a^T Q a,$$

where Q is a positive semidefinite matrix defined as

$$Q = \begin{pmatrix} 1 & \cos(\theta_1 - \theta_2) & \cos(\theta_1 - \theta_3) \\ \cos(\theta_1 - \theta_2) & 1 & \cos(\theta_2 - \theta_3) \\ \cos(\theta_1 - \theta_3) & \cos(\theta_2 - \theta_3) & 1 \end{pmatrix}, \quad (10)$$

and a is a vector defined as

$$a = \begin{pmatrix} a_1 \sin\varphi_1 \cos\varphi_2 \cos\varphi_3 \\ a_2 \cos\varphi_1 \sin\varphi_2 \cos\varphi_3 \\ a_3 \cos\varphi_1 \cos\varphi_2 \sin\varphi_3 \end{pmatrix}. \quad (11)$$

The matrix Q admits a factorization $Q = B^T B$, where

$$B = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 \end{pmatrix}.$$

Therefore, the factorization gives

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = \|Ba\|^2. \quad (12)$$

Extracting the factor $\prod_i \cos^2\varphi_i$ from the outcome probability, we have that

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = \left(\prod_i \cos^2\varphi_i\right) \left\| B \begin{pmatrix} a_1 \tan\varphi_1 \\ a_2 \tan\varphi_2 \\ a_3 \tan\varphi_3 \end{pmatrix} \right\|^2.$$

According to the definition of $v(A_i, \lambda_i)$, it holds that

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = \left(\prod_i \cos^2\varphi_i\right) \|t_1 + t_2 + t_3\|^2.$$

In the meanwhile, the cosine factors can be rewritten as

$$\cos^2\varphi_i = \frac{1}{1 + \tan^2\varphi_i} = \frac{1}{1 + \frac{1}{a_i^2}\|t_i\|^2},$$

which means that

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = \frac{\|t_1 + t_2 + t_3\|^2}{\left(1 + \frac{1}{a_1^2}\|t_1\|^2\right)\left(1 + \frac{1}{a_2^2}\|t_2\|^2\right)\left(1 + \frac{1}{a_3^2}\|t_3\|^2\right)}.$$

This concludes the proof. ■

We immediately have the following corollary:

Corollary 3. Let A_1, A_2, A_3 be observables with all their eigenspaces being one-dimensional, and let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues corresponding to A_1, A_2, A_3, respectively. Suppose $v(A_i, \lambda_i)$ is well defined for $i \in [3]$. Let $t_i = a_i \cdot v(A_i, \lambda_i)$. Then

$$P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3) = 0$$

if and only if

$$t_1 + t_2 + t_3 = 0. \quad (13)$$

In order to formulate all constraints in Hardy's nonlocality problem, we make the following further definitions.

Definition 4. For $i \in [3]$, if $v(U_i, +1)$, $v(D_i, +1)$, and $v(D_i, -1)$ are well defined, let

$$u_i = a_i v(U_i, +1), \quad (14)$$

$$v_i = a_i v(D_i, +1), \quad (15)$$

$$w_i = a_i v(D_i, -1). \quad (16)$$

With the new notations, we now translate the constraints in Hardy's nonlocality problem in the language of vectors defined above. First, by Corollary 3, we have

$$v_i = -\sum_{j \neq i} u_j \quad (17)$$

for $i \in [3]$, and

$$\sum_j w_j = 0. \quad (18)$$

Second, by Definition 1, we have $v_i = -\frac{a_i^2 w_i}{\|w_i\|^2}$. Indeed, suppose the eigenstate for $(D_i, +1)$ is

$$|\phi_+\rangle = \cos\alpha_i |0\rangle + e^{i\beta_i}\sin\alpha_i |1\rangle.$$

Then the eigenstate for $(D_i, -1)$ is

$$|\phi_-\rangle = \sin\alpha_i |0\rangle - e^{i\beta_i}\cos\alpha_i |1\rangle.$$

Now, by Definition 1, we have that

$$v_i = a_i(\tan\alpha_i \cos\beta_i, \tan\alpha_i \sin\beta_i)^T,$$

$$w_i = -a_i(\cot\alpha_i \cos\beta_i, \cot\alpha_i \sin\beta_i)^T,$$

hence $v_i = -\frac{a_i^2 w_i}{\|w_i\|^2}$.

With the above observations, when all representation vectors are well defined, the probability maximization problem can be rewritten as

$$\text{maximize} \quad P(U_1U_2U_3|+++) = \frac{\|u_1 + u_2 + u_3\|^2}{\left(1 + \frac{1}{a_1^2}\|u_1\|^2\right)\left(1 + \frac{1}{a_2^2}\|u_2\|^2\right)\left(1 + \frac{1}{a_3^2}\|u_3\|^2\right)}$$

$$= \frac{\|v_1+v_2+v_3\|^2/4}{\left(1+\frac{1}{4a_1^2}\|v_2+v_3-v_1\|^2\right)\left(1+\frac{1}{4a_2^2}\|v_3+v_1-v_2\|^2\right)\left(1+\frac{1}{4a_3^2}\|v_1+v_2-v_3\|^2\right)},$$

$$\text{subject to} \qquad w_1+w_2+w_3=0,$$

where $w_i (i \in [3])$ are the variables and

$$v_i = -\frac{a_i^2 w_i}{\|w_i\|^2}.$$

III. BOUNDING THE VIOLATION PROBABILITY FOR THE W STATE

Based on the geometric model introduced above, we now prove our first main result, which shows that for the perfect W state the maximal violation probability in Hardy's nonlocality problem is upper bound for 1/9. Since the geometric model supposes all representation vectors are well defined, we first consider this case, then we show that the conclusion can be generalized to arbitrary case.

Lemma 5. Let U_i and D_i be observables in Hardy's nonlocality problem for the W state $|\psi\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$. If $v(U_i,+1)$, $v(D_i,+1)$, and $v(D_i,-1)$ are well defined for $i \in [3]$, then all equation constraints being satisfied implies that

$$P(U_1U_2U_3|+++) \leqslant 1/9.$$

Proof. We have known that the target probability can be expressed as

$$P(U_1U_2U_3|+++) = \frac{\frac{1}{4}\|v_1+v_2+v_3\|^2}{\left(1+\frac{3}{4}\|-v_1+v_2+v_3\|^2\right)\left(1+\frac{3}{4}\|v_1-v_2+v_3\|^2\right)\left(1+\frac{3}{4}\|v_1+v_2-v_3\|^2\right)}.$$

Expanding the denominator gives

$$\begin{aligned} P(U_1U_2U_3|+++) &\leqslant \frac{\frac{1}{4}\|v_1+v_2+v_3\|^2}{1+\frac{3}{4}(\|-v_1+v_2+v_3\|^2+\|v_1-v_2+v_3\|^2+\|v_1+v_2-v_3\|^2)} \\ &= \frac{\frac{1}{4}\|v_1+v_2+v_3\|^2}{1+\frac{9}{4}\|v_1+v_2+v_3\|^2-6(v_1\cdot v_2+v_2\cdot v_3+v_3\cdot v_1)}, \end{aligned}$$

where $v_1 \cdot v_2$ is the inner product of v_1 and v_2.

Since isometries preserve inner products, we may assume that

$$\begin{aligned} w_1 &= -(M,0)^T, \\ w_2 &= -(x,y)^T, \\ w_3 &= -(-x-M,-y)^T \end{aligned}$$

without loss of generality, where we have utilized the relation $w_1+w_2+w_3=0$.

By the assumption that all $v(D_i,+1)$ and $v(D_i,-1)$ are well defined, we have $w_1, w_2, w_3 \neq 0$; that is, $M \neq 0$, $x^2+y^2 \neq 0$ and $(x+M)^2+y^2 \neq 0$.

By relation $v_i = -w_i/(3\|w_i\|^2)$, we have

$$\begin{aligned} v_1 &= (1/M,0)^T/3, \\ v_2 &= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)^T \Big/ 3, \\ v_3 &= \left(\frac{-x-M}{(x+M)^2+y^2}, \frac{-y}{(x+M)^2+y^2}\right)^T \Big/ 3. \end{aligned}$$

Then

$$\begin{aligned} v_1\cdot v_2+v_2\cdot v_3+v_3\cdot v_1 &= \frac{\frac{x}{M}[(x+M)^2+y^2]-x(x+M)-y^2-\frac{x+M}{M}(x^2+y^2)}{9(x^2+y^2)[(x+M)^2+y^2]} \\ &= \frac{-2y^2}{9(x^2+y^2)[(x+M)^2+y^2)]} \leqslant 0. \end{aligned}$$

Therefore,

$$P(U_1U_2U_3|+++) \leqslant \frac{\frac{1}{4}\|v_1+v_2+v_3\|^2}{1+\frac{9}{4}\|v_1+v_2+v_3\|^2} \leqslant 1/9.$$

We now show that the assumption in Lemma 5, that all representation vectors involved in the Hardy's nonlocality problem are well defined, can be removed, which means that in this case the upper bound in Lemma 5 is still correct.

For this, we first suppose two of $v(U_i,+1)$ are not well defined; then it can be seen that the vector a in Eq. (11) for $P(U_1U_2U_3|+++)$ is zero, thus Eq. (12) indicates that $P(U_1U_2U_3|+++)=0$, and it does not hurt the upper bound. Second, a similar argument shows that if one of $v(A_i|\lambda_i)$ is not well defined, then $P(A_1A_2A_3|\lambda_1\lambda_2\lambda_3)=0$ implies that there must be another $i'\neq i$ such that $v(A_{i'}|\lambda_{i'})$ is not well defined either.

Then combining the above two observations, we can rule out the possibility that only one of $v(U_i,+1)$, say $v(U_1,+1)$, is not well defined. If this is the case, then Eqs. (6) and (7) mean that $v(D_2,+1)$ and $v(D_3,+1)$ are not well defined, i.e., $v(D_2,-1)=0$ and $v(D_3,-1)=0$. By applying Corollary 3 on $P(D_1D_2D_3|---)=0$, we have that $v(D_1,-1)=0$, and this indicates that $v(D_1,+1)$ is not well defined either. However, we know that $P(D_1U_2U_3|+++)=0$, and this requires that at least one of $v(U_2,+1)$ and $v(U_3,+1)$ is not well defined, which is a contradiction. In summary, if any vector in $v(U_i,+1)$, $v(D_i,+1)$, and $v(D_i,-1)$ is not well defined, satisfying all equation constraints means that $P(U_1U_2U_3|+++)=0$. Therefore, we have the following theorem.

Theorem 6. Let U_i and D_i be observables in Hardy's nonlocality problem for the W state $|\psi\rangle=1/\sqrt{3}(|001\rangle+|010\rangle+|100\rangle)$. Then all equation constraints being satisfied implies that

$$P(U_1U_2U_3|+++)\leqslant 1/9.$$

Theorem 6 essentially states that the violation probability of the perfect three-qubit W state is upper bounded by 1/9. For comparison, it has been shown that obtaining a violation probability of 0.125 is possible from the perfect three-qubit GHZ state [16].

IV. GENERALIZATION TO *N* QUBIT W STATES

In this section, our first task is to show that the geometric model introduced above can be generalized to N-qubit W states with $N>3$.

The N-qubit generalized W state is defined as

$$|\psi\rangle=a_1|10\cdots0\rangle+a_2|01\cdots0\rangle+\cdots+a_N|00\cdots1\rangle,$$

where $a_i>0$ for all $i\in[N]$ and $\sum_i a_i^2=1$. When $a_i=1/\sqrt{N}$ for all $i\in[N]$, we call it the N-qubit perfect W state, denoted by $|W_N\rangle$. For convenience, we denote the N-qubit perfect GHZ state as

$$|\mathrm{GHZ}_N\rangle=\frac{1}{\sqrt{2}}(|00\cdots0\rangle+|11\cdots1\rangle). \tag{19}$$

Following Definition 1, the joint measurement outcome probability formula is readily generalized as Proposition 7.

Proposition 7. Let $A_i(i\in[N])$ be observables with all their eigenspaces being one-dimensional. Let λ_i be an eigenvalue corresponding to A_i. Suppose $v(A_i,\lambda_i)$ is well defined for $i\in[N]$. Let $v_i=a_i\cdot v(A_i,\lambda_i)$. Then

$$P(A_1\cdots A_N|\lambda_1\cdots\lambda_N)=\frac{\left\|\sum_{i=1}^N v_i\right\|^2}{\prod_{i=1}^N\left(1+\frac{1}{a_i^2}\|v_i\|^2\right)}.$$

■

Corollary 8. Let $A_i(i\in[N])$ be observables with all their eigenspaces being one-dimensional. Let λ_i be an eigenvalue corresponding to A_i. Suppose $v(A_i,\lambda_i)$ is well defined for $i\in[N]$. Let $v_i=a_i\cdot v(A_i,\lambda_i)$. Then

$$P(A_1\cdots A_N|\lambda_1\cdots\lambda_N)=0$$

if and only if

$$\sum_{i=1}^N v_i=0.$$

The N-qubit Hardy's nonlocality can be restated as the following maximization problem among the observables U_i and D_i:

$$\begin{aligned}
\text{maximize}\quad & P(U_1U_2\cdots U_N|++\cdots+),\\
\text{subject to}\quad & P(D_1U_2\cdots U_N|++\cdots+)=0,\\
& P(U_1D_2\cdots U_N|++\cdots+)=0,\\
& \vdots\\
& P(U_1U_2\cdots D_N|++\cdots+)=0,\\
& P(D_1D_2\cdots D_N|--\cdots-)=0.
\end{aligned}$$

Definition 9. For $i\in[N]$, if $v(U_i,+1)$, $v(D_i,+1)$, and $v(D_i,-1)$ are well defined, let

$$\begin{aligned}
u_i&=a_iv(U_i,+1),\\
v_i&=a_iv(D_i,+1),\\
w_i&=a_iv(D_i,-1).
\end{aligned}$$

Additionally, let $u=\sum_{i\in[N]}u_i$, $v=\sum_{i\in[N]}v_i$, and $w=\sum_{i\in[N]}w_i$.

By the constraints in Hardy's nonlocality, we have the following relations:

$$\begin{aligned}
\forall i\in[N],\quad & v_i=-a_i^2w_i/\|w_i\|^2,\\
\forall i\in[N],\quad & v_i=-(u-u_i),\\
\forall i\in[N],\quad & u_i=v_i+u,\\
& u=-\frac{v}{N-1}.
\end{aligned}$$

Under the relations above, the probability maximization problem becomes:

$$\begin{aligned}
\text{maximize}\quad & P(U_1U_2\cdots U_N|++\cdots+)\\
&=\frac{\|u\|^2}{\prod_{i=1}^N\left(1+\frac{1}{a_i^2}\|u_i\|^2\right)}\\
&=\frac{\|v\|^2/(N-1)^2}{\prod_{i=1}^N\left(1+\|v-(N-1)v_i\|^2/\left(\left((N-1)^2a_i^2\right)\right)\right)}\\
\text{subject to}\quad & w=0,
\end{aligned}$$

where $w_i (i \in [N])$ are the variables and

$$v_i = -\frac{a_i^2 w_i}{\|w_i\|^2}.$$

We now turn to the second task of this section. Differently from the three-qubit case, we consider lower bounding the maximum violation probability of the Hardy nonlocality problem when N is large. The following theorem gives such an asymptotic lower bound. Since we are focusing on a lower bound, we can suppose that all the involved representation vectors are well defined.

Theorem 10. Let $P(N)$ denote the maximum violation probability in Hardy's nonlocality problem for the perfect N-qubit W states. Then $P(N) = \Omega(N^{-1})$.

Proof. For simplicity, we represent the vectors u_i, v_i, w_i with one real number each, in the sense that their second component is equal to zero.

Let $w_i = 1/(N-1)$ for $i \in [N-1]$ and $w_N = -1$. Then, $\forall i \in [N-1]$,

$$v_i = -\frac{N-1}{N}, \quad v_N = \frac{1}{N},$$
$$u = \frac{N-2}{N-1}, \quad u_i = -\frac{1}{N(N-1)},$$
$$u_N = \frac{N^2-N-1}{N(N-1)}.$$

Now let N tend to $+\infty$. By Proposition 7, the violation probability under this settings is

$$\frac{\|u\|^2}{\prod_{i=1}^{N}\left(1+\frac{1}{a_i^2}\|u_i\|^2\right)} \approx \frac{1}{\left(1+\frac{1}{N(N-1)^2}\right)^{N-1}\left(1+\frac{(N^2-N-1)^2}{N(N-1)^2}\right)}$$
$$\approx \frac{1/N}{\left(1+\frac{1}{N(N-1)^2}\right)^{N-1}}$$
$$\approx 1/N.$$

Therefore, we have $P(N) = \Omega(N^{-1})$. ■

As a comparison, it has been known that the maximum violation probabilities for the N-qubit perfect GHZ states diminish exponentially with N [16], thus we witness another sharp difference between asymptotic behaviors of $|W_N\rangle$ and $|\mathrm{GHZ}_N\rangle$ when N tends to infinity. Therefore, on one hand $|\mathrm{GHZ}_N\rangle$ enjoys stronger nonlocality than $|W_N\rangle$ if $N = 3$, but on the other hand when N becomes larger the speed at which Hardy's nonlocality of $|W_N\rangle$ decays is much slower.

V. NUMERICAL SIMULATION RESULTS

We made the following numerical simulations to verify our theoretical results, where very good consistency can be seen. In the meantime, phenomena that deserve further study can also be observed. Our numerical results are obtained via the optimization package SciPy.

A. The three-qubit perfect W state

We first consider the case of the three-qubit perfect W state. In order to parametrize the involved measurements, let

$$v(U_i, +1) \equiv (\tan\varphi_{1,i}\cos\theta_{1,i}, \tan\varphi_{1,i}\sin\theta_{1,i})^T \in \mathbb{R}^2 \quad (20)$$

and

$$v(D_i, +1) \equiv (\tan\varphi_{2,i}\cos\theta_{2,i}, \tan\varphi_{2,i}\sin\theta_{2,i})^T \in \mathbb{R}^2. \quad (21)$$

For this, we consider the following construction, which is guided by the geometric model:

$$\tan\varphi_{11} = M/4, \quad \tan\varphi_{12} = 5M/4, \quad \tan\varphi_{13} = M/4, \quad (22)$$

$$\theta_{11} = 0, \quad \theta_{12} = \pi, \quad \theta_{13} = 0, \quad (23)$$

$$\tan\varphi_{21} = M, \quad \tan\varphi_{22} = M/2, \quad \tan\varphi_{23} = M, \quad (24)$$

$$\theta_{21} = 0, \quad \theta_{22} = \pi, \quad \theta_{23} = 0, \quad (25)$$

where $M > 0$ is a parameter. In this setting, our numerical simulations show that when M is picked to maximize the violation probability, the outcome probability is about 0.071 868, which is indeed below the theoretical upper bound 1/9 that our theoretical result proves, and the maximum violation probability 0.125 that the perfect three-qubit GHZ state can achieve [16].

B. The three-qubit generalized W state

We now turn to generalized three-qubit W states, defined as $|\psi\rangle = a_1|100\rangle + a_2|010\rangle + a_3|001\rangle$. Our numerical simulations show that, when $a_1 = 0.448\,473$, $a_2 = 0.632\,011$, and $a_3 = 0.632\,008$, the configuration

$$\tan\varphi_{11} = 1.320\,219, \quad \tan\varphi_{12} = 0.147\,611,$$
$$\tan\varphi_{13} = 0.147\,611,$$
$$\theta_{11} = \pi, \quad \theta_{12} = 0, \quad \theta_{13} = 0,$$
$$\tan\varphi_{21} = 0.295\,222, \quad \tan\varphi_{22} = 1.172\,608,$$
$$\tan\varphi_{23} = 1.172\,607,$$
$$\theta_{21} = \pi, \quad \theta_{22} = 0, \quad \theta_{23} = 0,$$

achieves violation probability of 0.097 738 1, which is higher than that of the perfect three-qubit W state. Notice that both resulting sequences $(\varphi_{1,i})$ and (a_i) exhibit $S_{N-1} = S_2$ symmetry, which matches the W state when the amplitudes are ignored. Therefore, we conjecture that the maximum violation probability can be achieved when $a_1 = a_2 = \cdots = a_{N-1}$ and $\varphi_{1,i} = \varphi_{2,i} = \cdots = \varphi_{N-1,i}$. If the conjecture is proved, then the maximization problem would be simplified in the sense that at most two real parameters would be free regardless of N.

Figure 1 is a color plot of the violation probability maximized using the optimization package for different amplitudes. The horizontal and vertical axes represents α and β from 0 to $\pi/2$, respectively, which are used in the definitions

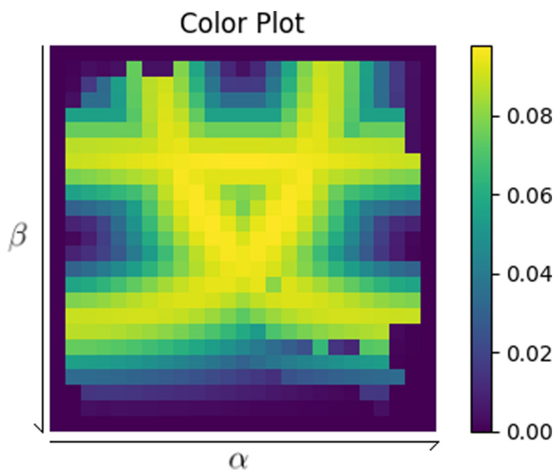

FIG. 1. The color plot of the maximum violation probability for different amplitude settings (a_1, a_2, a_3). The amplitudes are parametrized using spherical coordinates, with angles α and β ranging from 0 to $\pi/2$. The angle α increases from left to right (horizontally), while β increases from top to bottom (vertically).

of amplitudes as

$$a_1 = \cos\beta\cos\alpha, \quad a_2 = \cos\beta\sin\alpha, \quad a_3 = \sin\beta.$$

It is interesting to see that the area with larger violation probability forms three yellow bands.

C. The N-qubit perfect W state

For the N-qubit perfect W state, our numerical experiment gives the maximum violation probabilities $P(N)$ shown in Fig. 2. It is evident from the figure that $P(N)$ is unimodal in the range $3 \leqslant N \leqslant 10$ and is maximized at $N = 5$. For now no theoretical analysis can explain this fact, and this is an intriguing phenomenon worth further study. Additionally, it can be observed that the maximum violation probabilities assumed by the perfect N-qubit GHZ states are all below $P(N)$ except when $N = 3$.

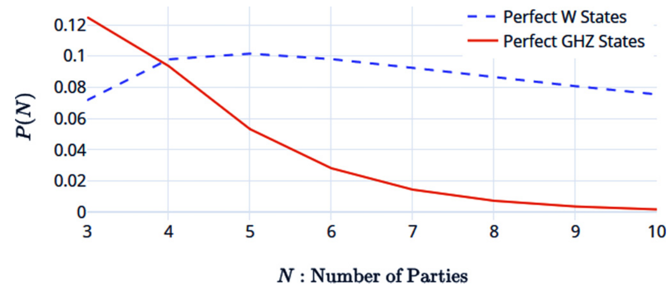

FIG. 2. The maximal violation probabilities of N-qubit perfect W states and GHZ states.

VI. CONCLUSION

In this paper, we have analyzed Hardy's nonlocality for W states. For this purpose, we develop a geometric model for general W states, and this model allows us to describe the constraints in Hardy's nonlocality problem as relations on vectors, which in turn makes it convenient to characterize the target violation probability.

Concretely, for the perfect three-qubit W state, we have shown that its violation probability is upper-bounded by 1/9.

As a comparison, the perfect three-qubit GHZ state has maximum violation probability 0.125, and the stronger correlation provided by the GHZ state is also consistent with our intuition that it is more entangled than the W state, though we know that entanglement and nonlocality are two different computational resources.

For the perfect N-qubit W states where $N \geqslant 4$, we have shown that their maximum violation probabilities are at least $\Omega(N^{-1})$, making another sharp comparison with the perfect N-qubit GHZ states, since the maximum violation probabilities of the latter decay exponentially when N goes up.

Therefore, it can be seen that Hardy's nonlocality indeed provides an additional viewpoint from which to distinguish the different entanglement structures of GHZ and W states. We hope this approach can be generalized to more complicated and more general multipartite quantum states, and eventually give us a new approach to characterize multipartite entanglement in a device-independent manner.

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[1] W. Dur, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).
[2] N. Brunner, J. Sharam, and T. Vértesi, Testing the Structure of Multipartite Entanglement with Bell Inequalities, Phys. Rev. Lett. **108**, 110501 (2012).
[3] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Bell-Type Inequalities to Detect True n-Body Nonseparability, Phys. Rev. Lett. **88**, 170405 (2002).
[4] J. D. Bancal, N. Gisin, Y. C. Liang, and S. Pironio, Device-Independent Witnesses of Genuine Multipartite Entanglement, Phys. Rev. Lett. **106**, 250404 (2011).
[5] M. Zwerger, W. Dür, J. D. Bancal, and P. Sekatski, Device-Independent Detection of Genuine Multipartite Entanglement for All Pure States, Phys. Rev. Lett. **122**, 060502 (2019).
[6] A. Aloy, J. Tura, F. Baccari, A. Acín, M. Lewenstein, and R. Augusiak, Device-Independent Witnesses of Entanglement Depth from Two-Body Correlators, Phys. Rev. Lett. **123**, 100507 (2019).
[7] F. Verstraete and M. M. Wolf, Entanglement versus Bell Violations and Their Behavior under Local Filtering Operations, Phys. Rev. Lett. **89**, 170401 (2002).

[8] Y. C. Liang, T. Vértesi, and N. Brunner, Semi-device-independent bounds on entanglement, Phys. Rev. A **83**, 022108 (2011).

[9] T. Moroder, J. D. Bancal, Y. C. Liang, M. Hofmann, and O. Gühne, Device-Independent Entanglement Quantification and Related Applications, Phys. Rev. Lett. **111**, 030501 (2013).

[10] Z. Wei and L. Lin, Analytic semi-device-independent entanglement quantification for bipartite quantum states, arXiv:1903.05303.

[11] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Bell's theorem without inequalities, Am. J. Phys. **58**, 1131 (1990).

[12] N. D. Mermin, Simple Unified form for the Major No-Hidden-Variables Theorems, Phys. Rev. Lett. **65**, 3373 (1990).

[13] Q. Chen, S. Yu, C. Zhang, C. H. Lai, and C. H. Oh, Test of Genuine Multipartite Nonlocality without Inequalities, Phys. Rev. Lett. **112**, 140404 (2014).

[14] L. Hardy, Nonlocality for Two Particles without Inequalities for Almost All Entangled States, Phys. Rev. Lett. **71**, 1665 (1993).

[15] D. Boschi, S. Branca, F. De Martini, and L. Hardy, Ladder Proof of Nonlocality without Inequalities: Theoretical and Experimental Results, Phys. Rev. Lett. **79**, 2755 (1997).

[16] J. L. Cereceda, Hardy's nonlocality for generalized *n*-particle GHZ states, Phys. Lett. A **327**, 433 (2004).

[17] J. Chen, A. Cabello, Z. Xu, H. Su, C. Wu, and L. C. Kwek, Hardy's paradox for high-dimensional systems, Phys. Rev. A **88**, 062116 (2013).

[18] S.-H. Jiang, Z.-P. Xu, H.-Y. Su, A. K. Pati, and J.-L. Chen, Generalized Hardy's Paradox, Phys. Rev. Lett. **120**, 050403 (2018).

[19] S. S. Bhattacharya, A. Roy, A. Mukherjee, and R. Rahaman, Witnessing genuine mutipartite non-locality, arXiv:1507.07327.

[20] J. L. Cereceda, Cabello's nonlocality for generalized three-qubit GHZ states, Quantum Stud.: Math. Found. **4**, 205 (2017).

[21] J. R. Torgerson, D. Branning, C. H. Monken, and L. Mandel, Experimental demonstration of the violation of local realism without Bell inequalities, Phys. Lett. A **204**, 323 (1995).

[22] J. S. Lundeen and A. M. Steinberg, Experimental Joint Weak Measurement on a Photon Pair as a Probe of Hardy's Paradox, Phys. Rev. Lett. **102**, 020404 (2009).

[23] E. Karimi, F. Cardano, M. Maffei, C. de Lisio, L. Marrucci, R. W. Boyd, and E. Santamato, Hardy's paradox tested in the spin-orbit Hilbert space of single photons, Phys. Rev. A **89**, 032122 (2014).