



## On complexity of single-minded auction

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### Abstract

We consider complexity issues for a special type of combinatorial auctions, the single-minded auction, where every agent is interested in only one subset of the commodities.

First, we present a matching bound on the communication complexity for the single-minded auction under a general communication model. Next, we prove that it is NP-hard to decide whether Walrasian equilibrium exists in a single-minded auction. Finally, we establish a polynomial size duality theorem for the existence of Walrasian equilibrium for the single-minded auction.

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### 1. Introduction

The combinatorial auctions problem has attracted much attention in recent years because of its applications to various interesting problems [6,17,24]. In that model, participating agents have preferences not only for particular item of commodities, but also for all possible bundles. There is, however, an inherent difficulty in applying computational complexity to obtain meaningful results for this model: The value function  $v$  of each agent is defined on  $2^m$  subsets, exponential in the number  $m$  of commodities. That is very similar to a situation in the complexity study of cooperative game theory where it was proposed to consider value functions implicitly described with a number of parameters polynomial in  $m$  [21]. Such a paradigm has turned complexity into a

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major issue in the studies of cooperative games (see, e.g., [8,7]). Obviously, a complexity theory in combinatorial auctions would require such a feature to take off.

A special subclass of combinatorial auctions, the *single-minded auction* [18,1,22,4], where each agent only desires a fixed bundle of commodities, has the property that its input size is  $O(mn)$ , where  $m$  is the number of commodities and  $n$  is the number of participating agents. A useful first step in complexity approach to the studies of combinatorial auctions would be a complete understanding of single-minded auction. In this work, we present the communication complexity of single-minded auction, as well as its computational complexity and the existence condition for Walrasian equilibrium for single-minded auction.

Recently, communication complexity has been proposed as a measure of coordination requirement and applied to study the necessary information exchanged among economic agents at market places. Nisan [23] studied the communication complexity of combinatorial auctions, demonstrated that producing optimal allocation requires an exponential amount of information transfer, and obtained polynomial communication protocols for a truth-telling mechanism when the optimal allocation can be obtained by linear program relaxation. Deng, Papadimitriou and Safra [9] studied the communication complexity for the Arrow–Debreu equilibrium price [2] in an exchange market. Shoham and Tennenholtz [26] studied the communication complexity of rationally computable functions, using the model of auctions of unit-item and multi-items of the commodity.

In the communication complexity model of Nisan for combinatorial auctions [23], we obtain a matching bound for single-minded auction. We present a  $\Theta(mn)$  bound for communication complexity of single-minded auction with  $m$  commodities and  $n$  agents. Note that  $O(mn)$  is a trivial upper bound in this model since every agent can simply submit his bundle to the auctioneer. Our main effort in Section 3 is a proof of the lower bound  $\Omega(mn)$ .

Next, we focus on the *Walrasian equilibrium* for single-minded auction. Walrasian equilibrium specifies a tuple of allocation and price vector of commodities satisfying that (i) the prices of all non-allocated commodities are zero, and (ii) each agent receives a bundle of his highest utility under the price vector, where utility for each agent is the difference between the value function and the price over corresponding allocation. Kelso and Crawford [15] proved that, under *gross substitutes* (GS) condition, Walrasian equilibrium exists. Subsequently, Bikhchandani and Mamer [3] established a powerful sufficient and necessary condition for a Walrasian equilibrium to exist: The total value of the optimal allocation of bundles to individuals, written as an integer program, is equal to the value of the corresponding relaxed linear program. This condition is, however, not easy to verify in polynomial time. Gul and Stacchetti [14] later considered the GS condition again and showed that another condition, the SI condition that valuation functions satisfy *single improvement* condition, is equivalent to the GS condition, and hence is also sufficient for the existence of Walrasian equilibrium.

All the above studies considered the standard linear pricing scheme, i.e., the price of a bundle is defined by the sum of prices of items in the bundle. If non-linear pricing scheme is allowed, Walrasian equilibrium always exists for combinatorial auctions [5,19].

We concentrate on the linear pricing scheme and are interested in the computational complexity for Walrasian equilibrium. In Section 4, we prove that whether a Walrasian equilibrium exists is NP-hard for the single-minded auction problem. Note that, it is known that the optimal allocation problem for single-minded auction is NP-hard [18]. The duality theorem of Bikhchandani and

Mamer [3] (and even ours in Section 5) reveals a relationship between the Walrasian equilibrium and the optimal allocation if Walrasian equilibrium indeed exists. Therefore, there is a probable link between deciding the existence of Walrasian equilibrium and finding the optimal allocation. Our proof of the NP-hardness of the Walrasian equilibrium confirms that.

Finally, we explore the duality relationship for single-minded auction, with its complexity issues in mind. Duality has played a crucial role in many important aspects in economics. Algorithmic design for market equilibrium by Devanur et al. [10] depends on an important duality relations for the prices of commodities and the assets of individual agents. Linear program duality has also been a fruitful tool in cooperative game theory. Literatures dealing with the non-emptiness of the core often result in a statement that an integer program and its corresponding linear program relaxation have the same optimal value, such as the assignment game [25], the cost allocation game on trees [27], the partition game [11], the combinatorial optimization game [7] and the facility location game [13].

In Section 5, we exploit such a duality relationship. Note that the linear program approach established by Bikhchandani and Mamer [3] is exponential in the input size. Thus it is not suitable for our algorithmic study. We establish a formulation that is of polynomial size.

We conclude our work in Section 6 with remarks and discussions on future research.

## 2. The model

An auctioneer sells  $m$  heterogeneous commodities  $\Omega = \{\delta_1 \times \omega_1, \dots, \delta_m \times \omega_m\}$ , with quantity  $\delta_j$  for each commodity  $\omega_j$ , to  $n$  potential buyers  $\mathcal{O} = \{\mathcal{O}_1, \dots, \mathcal{O}_n\}$ . Each buyer  $\mathcal{O}_i$  has a *valuation function*  $v_i: 2^\Omega \rightarrow \mathbb{R}^+ \cup \{0\}$  that describes his true values over the various subsets of commodities. That is, for any  $B \subseteq \Omega$ ,  $v_i(B)$  is the maximal amount of money that  $\mathcal{O}_i$  is willing to pay in order to win  $B$ . We say  $\mathcal{O}_i$  is a *single-minded buyer* if there exists a *basic bundle*<sup>1</sup>  $\Omega_i \subseteq \Omega$  such that for any  $B \subseteq \Omega$ ,  $v_i(B) = v_i(\Omega_i) > 0$  if  $\Omega_i \subseteq B$  and  $v_i(B) = 0$  otherwise. That is,  $\Omega_i$  is the core bundle that  $\mathcal{O}_i$  desires. In this paper, we consider the case that all buyers are restricted to be single-minded, and denote  $\mathcal{A} = (\Omega; \Omega_1, v_1; \dots; \Omega_n, v_n)$  as a *single-minded auction* [18]. Let *type matrix*  $M = [a_{j,i}]_{m \times n}$ , where  $a_{j,i} \in \mathbb{N} \cup \{0\}$  is the number of  $\omega_j$  contained in the bundle  $\Omega_i$ . Unless stated otherwise,  $v_i$  denotes the value  $v_i(\Omega_i)$  throughout the paper.

When receiving the submitted tuple  $(\Omega_i, v_i)$  from each buyer  $i$  (the *input*), the auctioneer specifies the following two parts as the *output* of the auction:

- *Allocation vector*  $X = (X_1, \dots, X_n)$  of  $\Omega$  to all buyers, where  $X_i$  represents the collection of commodities allocated to buyer  $\mathcal{O}_i$ .  $X^* = (X_1^*, \dots, X_n^*)$  is said to be an *optimal allocation* if for any allocation  $X$ , we have

$$\sum_{i=1}^n v_i(X_i^*) \geq \sum_{i=1}^n v_i(X_i).$$

That is,  $X^*$  maximizes total valuations of buyers. Let  $X_0 = \Omega \setminus (\bigcup_{i=1}^n X_i)$  be the set of commodities that are not allocated to any buyers.

<sup>1</sup>Note that we allow multiple items of any commodity contained in a basic bundle, e.g.,  $\Omega_i = \{\omega_1, \omega_1, \omega_2\}$ .

- *Price vector*  $(p(\omega_1), \dots, p(\omega_m))$  (or  $(p_1, \dots, p_m)$ ). In this paper, we assume the prices of the commodities are linear, i.e.,  $p(BC) = p(B) + p(C)$ , for any  $B, C \subseteq \Omega$ .

If  $\mathcal{O}_i$  wins bundle  $B$ , his (quasi-linear) *utility* is  $u_i(B, p) = v_i(B) - p(B)$ . For any non-trading buyer, his *utility* is zero.

### 3. Communication complexity

In this section, we study the communication complexity issues in single-minded auction. For convenience, here we assume that  $\delta_1 = \dots = \delta_m = 1$ .

Following a model in [23], we are interested in the number of bits for the buyers to convey their subsets of interests to the auctioneer. Here, the communication cost is referred to the criteria of Yao [28], i.e., the total number of bits that all buyers transmit to the auctioneer. The upper bound is obviously  $O(mn)$ . Our focus here is thus the proof of the matching lower bound. We study a special case of single-minded auction, and demonstrate the lower bound of communication complexity for this special case is  $\Omega(mn)$ , which implies the same lower bound for the general problem. Specifically, we consider the following single-minded auction:

- There are  $2n$  independent buyers bidding for  $2m$  commodities. For any  $1 \leq i \leq 2n$ , assume the basic bundle  $\Omega_i$  of buyer  $\mathcal{O}_i$  contains exactly  $m$  various commodities (i.e.,  $|\Omega_i| = m$ ), and  $v_i(\Omega_i) = 1$ .
- In terms of the deterministic auction protocol, the auctioneer determines whether there exists the (optimal) allocation  $(X_1, \dots, X_{2n})$  such that  $\sum_{i=1}^{2n} v_i(X_i) = 2$ . That is, whether there exist  $\Omega_i, \Omega_j, i \neq j$ , such that  $\Omega_i \cup \Omega_j = \Omega$ , or equivalently,  $\Omega_i \cap \Omega_j = \emptyset$ . Therefore, we may assume without loss of generality that  $2n \leq \binom{2m}{m} / 2 \triangleq M$ , otherwise, such allocation always exists.
- In any round of the communication, each buyer knows what the other buyers submit to the auctioneer.

Let  $C$  be the set of all  $m$ -collections of commodities, i.e.,  $|C| = \binom{2m}{m} = 2M$ . Therefore any basic bundle can be uniquely encoded by an integer  $x_i \in \{1, 2, \dots, 2M\}$  in such a way that  $\Omega_i \cup \Omega_j = \Omega$  if and only if  $x_i + x_j = 2M + 1$ . It is easy to see such encoding does exist. It can be executed locally and does not require communication. Therefore from the point of view of the communication complexity, the above single-minded auction problem is equivalent to the following multi-party communication problem:

There are  $2n$  players, each holds an integer  $x_i \in \{1, 2, \dots, 2M\}$ , they want to evaluate a  $2n$ -argument function

$$f(x_1, \dots, x_{2n}) = \begin{cases} 1 & \text{if } \exists i \neq j, \text{ s.t. } x_i + x_j = 2M + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Following we construct another two-party communication game  $DISJ_n^M$  [16]: Alice and Bob hold inputs  $Y$  and  $Z$ , respectively, where  $Y, Z \subseteq \{1, 2, \dots, M\}$ , and  $|Y| = |Z| = n$ . Note that  $n \leq M/2$ , define  $DISJ_n^M(Y, Z) = 1$  iff  $Y \cap Z = \emptyset$ .

Now we show that the multi-party communication problem  $f$  is at least as hard as the two-party communication problem  $DISJ_n^M$ , and any communication protocol used to compute  $f$  can be used to computing  $DISJ_n^M$ .

Assume  $Y = \{y_1, \dots, y_n\}$ ,  $Z = \{z_1, \dots, z_n\}$ . Let  $x_i = y_i$ , and  $x_{n+i} = 2M + 1 - z_i$ , for  $i = 1, \dots, n$ . Notice that (i)  $y_i + y_j \leq 2M < 2M + 1$ , since  $y_i \leq M$ , and (ii)  $(2M + 1 - z_i) + (2M + 1 - z_j) \geq 2M + 2 > 2M + 1$ , since  $z_i \leq M$ .

If  $f(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) = 1$ , i.e., there exist  $1 \leq i < j \leq 2n$  such that  $x_i + x_j = 2M + 1$ . From the above arguments, there must be  $x_i \in \{x_1, \dots, x_n\}$ , and  $x_j \in \{x_{n+1}, \dots, x_{2n}\}$ . Therefore, we have  $x_i + x_j = y_i + (2M + 1 - z_{j-n}) = 2M + 1 \Rightarrow y_i = z_{j-n}$ , which implies that  $Y \cap Z \neq \emptyset$ .

If  $Y \cap Z \neq \emptyset$ , then there exist  $i, j$ , such that  $y_i = z_j$ . Thus  $x_i + x_{n+j} = y_i + (2M + 1 - z_i) = 2M + 1$ . Hence,

$$f(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) = 1 \Leftrightarrow DISJ_n^M(Y, Z) = 0.$$

For any protocol  $\mathcal{P}$  that computes  $f(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n})$ , Alice and Bob can simulate it step-by-step. In each step, if player  $i \in \{1, \dots, n\}$  does some computation and broadcasts a bit  $a$ , then we let Alice compute it and send the same bit  $a$  to Bob and the auctioneer (Alice knows all information that player  $i$  knows, so he can simulate the computation player  $i$  does). If player  $j \in \{n + 1, \dots, 2n\}$  does some computation and broadcasts a bit  $b$ , then let Bob compute and send the same bit  $b$  to Alice and the auctioneer (Bob knows all information that player  $j$  knows, so he can simulate player  $j$ 's computation). At the end of the protocol, both players get the value of  $f(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n})$ , so does the auctioneer. That means protocol  $\mathcal{P}$  can be used to computing  $DISJ_n^M$ .

From [16], we know that the  $DISJ_n^M$  problem has the communication lower bound  $\log \binom{M}{n}$ , and

$$\log \binom{M}{n} \geq n \log \frac{M}{n} = n \left( \log \binom{2m}{m} - \log 2n \right) = \Omega(mn),$$

where the last equality holds when  $\log 2n < cm$  for some constant  $c \in (0, 2)$ . That is, in the worst case, the communication complexity is lower bounded by  $\Omega(mn)$ . Hence we get the following conclusion.

**Theorem 3.1.** *For any single-minded auction, the communication complexity that computes the optimal allocation is  $\Theta(mn)$ .*

#### 4. Complexity of Walrasian equilibrium

In this section, we study the computational complexity of Walrasian equilibrium in single-minded auction. Intuitively, Walrasian equilibrium specifies the allocation and price vector such that any remaining commodity is priced at zero and all buyers are satisfied with their corresponding allocations under the given price vector. Formally,

**Definition 4.1** (Walrasian Equilibrium for Unit-Item Auction Gul and Stacchetti [14]). A *Walrasian equilibrium* of single-minded auction  $\mathcal{A} = (\Omega; \Omega_1, v_1; \dots; \Omega_n, v_n)$  with  $\delta_1 = \dots = \delta_m = 1$

is a tuple  $(X, p)$ , where  $X = (X_1, \dots, X_n)$  is an allocation vector,  $p \geq 0$  is a price vector, such that (i)  $p(X_0) = 0$ , where  $X_0 = \Omega \setminus (\bigcup_{i=1}^n X_i)$  and (ii) for any buyer  $\mathcal{O}_i$ ,  $u_i(X_i, p) \geq u_i(B, p)$ , for all  $B \subseteq \Omega$ .

For the auction with multiple items of the same type of commodity, let  $f(\omega_j, X)$  be the number of  $\omega_j$  allocated to buyers in allocation  $X$ . Therefore we must have  $f(\omega_j, X) \leq \delta_j$ .

**Definition 4.2** (Walrasian Equilibrium for Multi-Items Auction).<sup>2</sup> A *Walrasian equilibrium* of multi-items single-minded auction  $\mathcal{A}$  is a tuple  $(X, p)$ , where  $X = (X_1, \dots, X_n)$  is an allocation vector,  $p \geq 0$  is a price vector, such that (i) if  $f(\omega_j, X) < \delta_j$ , then  $p(\omega_j) = 0$ , for any  $1 \leq j \leq m$ , (ii) for any buyer  $\mathcal{O}_i$ ,  $u_i(X_i, p) \geq u_i(B, p)$ , for all  $B \subseteq \Omega$ .

From the above definitions, we may assume without loss of generality that  $X_i = \Omega_i$  or  $X_i = \emptyset$ .

As in the general combinatorial auctions [5], Walrasian equilibrium may not exist in (unit-item) single-minded auction.

**Example 4.1.** Three buyers bid for three commodities with unit quantity each.  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$  desire  $\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_3\}$ , respectively, at valuation 3,2,2. We only consider the optimal allocation here, i.e., assigning  $\{\omega_1, \omega_2\}$  to buyer  $\mathcal{O}_1$ . Hence, the equilibrium price vector  $p$  should satisfy  $p(\omega_1\omega_2) \leq 3$ ,  $p(\omega_2\omega_3) \leq 2$ ,  $p(\omega_1\omega_3) \leq 2$ , and  $p(\omega_3) = 0$ , which cannot be held simultaneously. Therefore the Walrasian equilibrium does not exist.

Note that our value function is defined in a special way for our algorithmic study, different from the standard weakly increasing assumption under which the value of any bundle  $B$  for buyer  $\mathcal{O}_i$  is at least  $v_i$  if  $\Omega_i \subseteq B$ . Therefore, the input size of our model is polynomially bounded by the number of commodities and buyers. That allows us to deal with the computational complexity issues with ease.

**Theorem 4.1.** *Given any single-minded auction, it is NP-hard to determine whether Walrasian equilibrium exists.*

**Proof.** We reduce from EXACT COVER BY 3-SETS (X3C) [12]. That is, given a family  $S = \{s_1, \dots, s_n\}$  of 3-subsets of  $N = \{1, \dots, 3m\}$ , we are asked whether there is a subfamily of  $m$  sets of  $S$  covering all elements in  $N$ .

For the above input of X3C, we construct the following auction with  $3m + 3$  commodities (with unit quantity each) and  $9m + n + 1$  buyers: Let every element of  $N$  correspond to a commodity, i.e.,  $\Omega' = \{\omega_1, \dots, \omega_{3m}\}$ . We add another three special commodities  $\alpha, \beta, \gamma$ , let  $s_{n+1} = \{\alpha, \beta, \gamma\}$  and  $\Omega = \Omega' \cup s_{n+1}$ . The first  $n + 1$  buyers' valuation functions are defined as follows:

$$v_i(B) = \begin{cases} 3 & \text{if } s_i \subseteq B, \\ 0 & \text{otherwise,} \end{cases} \quad i = 1, \dots, n + 1,$$

<sup>2</sup>This definition is from the traditional economic concept, *market clearing* [20]. That is, for any commodity, if there are some items remain, its price must be zero.

where  $s_i$  denotes the set of corresponding commodities. That is,  $s_i$  is the basic bundle of buyer  $\mathcal{O}_i$ ,  $1 \leq i \leq n + 1$ . Let  $\mathcal{O}' = \{\mathcal{O}_1, \dots, \mathcal{O}_n\}$  be the set of first  $n$  buyers. For any 3-collection  $C = \{\phi_1, \phi_2, \omega\}$ , where  $\phi_1, \phi_2 \in s_{n+1}$ ,  $\phi_1 \neq \phi_2$ , and  $\omega \in \Omega'$ , we add one buyer with  $C$  as the basic bundle at valuation 3. Note that the number of such 3-collections is  $9m$ , therefore we just added  $9m$  new buyers.

If there exists a subfamily of  $m$  sets  $\{s_{i_1}, \dots, s_{i_m}\} \subseteq S$  covering all elements in  $N$ , we may allocate  $s_{n+1}$  to buyer  $\mathcal{O}_{n+1}$ , and  $s_{i_k}$  to buyer  $\mathcal{O}_k$ , for  $1 \leq k \leq m$ . Hence when we set the price of each commodity is one, all buyers are satisfied with their corresponding allocations, and then the equilibrium exists.

Conversely, assume  $M$  has an equilibrium  $((X_1, \dots, X_{n+1}, X'), p)$ , where  $X'$  is the vector of allocations to buyers except  $\mathcal{O}_1, \dots, \mathcal{O}_{n+1}$ . We only need to show that the above equilibrium must be allocating all commodities of  $\Omega'$  to buyers in  $\mathcal{O}'$ , that is,  $m$  different buyers in  $\mathcal{O}'$  win with three elements each, which constructs a 3-sets cover. Otherwise, at least three elements of  $\Omega'$  are not allocated to any buyer in  $\mathcal{O}'$ . With a change of notations, we may assume that such three elements are  $\{\omega_1, \omega_2, \omega_3\}$ . There are the following various cases:

- (1) All items in  $s_{n+1}$  are bought by  $\mathcal{O}_{n+1}$ . Then, none of items  $\{\omega_1, \omega_2, \omega_3\}$  can be bought by anyone. They will all be priced at zero. However, at least one of the items in  $\{\alpha, \beta, \gamma\}$  must be priced at least 1. Assume, w.l.o.g., it to be  $\alpha$ . Then the buyer interested in  $\{\beta, \gamma, \omega_1\}$  would be able to buy the bundle at a price no more than 2, a contradiction to the definition of Walrasian equilibrium.
- (2) Some but not all items in  $s_{n+1}$  are bought by some buyer. Without loss of generality, assume that it is bought by a buyer who is interested in  $\{\alpha, \beta, \omega_i\}$  for some  $\omega_i \in \Omega'$ ,  $i \neq 1$ . At least one of  $\alpha$  and  $\beta$  must be priced at no more than  $3/2$ , and assume, w.l.o.g.,  $p(\alpha) \leq 3/2$ . Then no one could buy  $\gamma$  and it must be priced at zero. The buyer interested in  $\{\alpha, \gamma, \omega_1\}$  would be able to buy the bundle at a price no more than  $3/2$ . A contradiction.
- (3) No items in  $s_{n+1}$  is bought by some buyer. This is not possible since all will be priced at zero and buyer  $\mathcal{O}_{n+1}$  would be able to buy his interested bundle  $\{\alpha, \beta, \gamma\}$  at a null price, a contradiction.

In conclusion, no matter how we allocate the commodities in  $s_{n+1}$ , there always exists a buyer whose utility is not maximized for his allocation, which contradicts the definition of Walrasian equilibrium.  $\square$

Furthermore, given any solution of allocation and price vector, it's not hard to see that we can check if it is a Walrasian equilibrium in polynomial time. Hence, the problem of computing Walrasian equilibrium in single-minded auction is NP-complete. As for the general combinatorial auctions, we need exponential steps to check if all buyers are satisfied with their allocations.

### 5. Polynomial size duality theorem for Walrasian equilibrium

The following lemma is on the relation between Walrasian equilibrium and optimal allocation.

**Lemma 5.1.** *If  $(X, p)$  is a Walrasian equilibrium, then  $X$  must be an optimal allocation.*

**Proof.** We need to prove that for any allocation  $Y = (Y_1, \dots, Y_n)$ ,

$$\sum_{i=1}^n v_i(X_i) \geq \sum_{i=1}^n v_i(Y_i).$$

Denote the number of commodity  $w_j$  in bundle  $B$  by  $n_j(B)$ . From the definition of Walrasian equilibrium, we know that

$$v_i(X_i) - \sum_{\omega_j \in X_i} p(\omega_j)n_j(X_i) \geq v_i(Y_i) - \sum_{\omega_j \in Y_i} p(\omega_j)n_j(Y_i).$$

Hence for all buyers, we have:

$$\begin{aligned} \sum_{i=1}^n \left( v_i(X_i) - \sum_{\omega_j \in X_i} p(\omega_j)n_j(X_i) \right) &\geq \sum_{i=1}^n \left( v_i(Y_i) - \sum_{\omega_j \in Y_i} p(\omega_j)n_j(Y_i) \right) \\ &\Rightarrow \sum_{i=1}^n v_i(X_i) - \sum_{i=1}^n \sum_{\omega_j \in X_i} p(\omega_j)n_j(X_i) \geq \sum_{i=1}^n v_i(Y_i) - \sum_{i=1}^n \sum_{\omega_j \in Y_i} p(\omega_j)n_j(Y_i) \\ &\Rightarrow \sum_{i=1}^n v_i(X_i) - \sum_{\omega_j} p(\omega_j)f(\omega_j, X) \geq \sum_{i=1}^n v_i(Y_i) - \sum_{\omega_j} p(\omega_j)f(\omega_j, Y) \\ &\Rightarrow \sum_{i=1}^n v_i(X_i) - \sum_{i=1}^n v_i(Y_i) \geq \sum_{\omega_j} p(\omega_j)(f(\omega_j, X) - f(\omega_j, Y)). \end{aligned}$$

Note that for  $\forall j, f(\omega_j, X) \leq \delta_j$ , and  $f(\omega_j, Y) \leq \delta_j$ . If  $f(\omega_j, X) = \delta_j$ , then

$$p(\omega_j)(f(\omega_j, X) - f(\omega_j, Y)) \geq 0.$$

Otherwise commodity  $\omega_j$  is not clear in allocation  $X$ . Since  $(X, p)$  is a Walrasian equilibrium, due to the condition (i) of Walrasian equilibrium, we have  $p(\omega_j) = 0$ , which implies that

$$p(\omega_j)(f(\omega_j, X) - f(\omega_j, Y)) = 0.$$

Hence we always have  $\sum_{\omega_j} p(\omega_j)(f(\omega_j, X) - f(\omega_j, Y)) \geq 0$ . That is,

$$\sum_{i=1}^n v_i(X_i) \geq \sum_{i=1}^n v_i(Y_i). \quad \square$$

Following we consider the existence of Walrasian equilibrium from the point of view of integer program and its linear program relaxation.

*Integer Program (IP):*

$$\begin{aligned} \max_{x_i} \quad & \sum_{i=1}^n v_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^n a_{j,i} x_i \leq \delta_j, \quad \forall 1 \leq j \leq m, \\ & x_i \in \{0, 1\}, \quad \forall 1 \leq i \leq n, \end{aligned}$$



where  $[a_{j,i}]_{m \times n}$  is the type matrix, and  $x_i$  denotes whether buyer  $\mathcal{O}_i$  wins his desired basic bundle or not.

*Linear Program Relaxation (LPR):*

$$\begin{aligned} \max_{x_i} \quad & \sum_{i=1}^n v_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^n a_{j,i} x_i \leq \delta_j, \quad \forall 1 \leq j \leq m \\ & 0 \leq x_i \leq 1, \quad \forall 1 \leq i \leq n. \end{aligned}$$

*Dual of LPR (DLPR):*

$$\begin{aligned} \min_{y_j, z_i} \quad & \sum_{j=1}^m \delta_j y_j + \sum_{i=1}^n z_i \\ \text{s.t.} \quad & \sum_{j=1}^m a_{j,i} y_j + z_i \geq v_i, \quad \forall 1 \leq i \leq n \\ & y_j \geq 0, \quad \forall 1 \leq j \leq m \\ & z_i \geq 0, \quad \forall 1 \leq i \leq n. \end{aligned}$$

Note that in the economic sense,  $y_j$  represents the price of commodity  $\omega_j$ . Let  $M_{IP}$ ,  $M_{LPR}$  and  $M_{DLPR}$  be the values of the optimal solutions of IP, LPR and DLPR, respectively. Thus,

$$M_{DLPR} = M_{LPR} \geq M_{IP}.$$

**Theorem 5.1.** *Walrasian equilibrium exists if and only if any optimal solution of IP is an optimal solution of LPR, i.e.,  $M_{IP} = M_{LPR}$ .*

**Proof.**  $\Leftarrow$ : Assume  $M_{IP} = M_{LPR}$ . Let  $x^* = (x_i^*)$  be an optimal solution of IP, then  $x^*$  is also an optimal solution of LPR. Let  $y^* = (y_j^*)$ ,  $z^* = (z_i^*)$  be an optimal solution of DLPR. Then

$$\sum_{i=1}^n v_i x_i^* = \sum_{j=1}^m \delta_j y_j^* + \sum_{i=1}^n z_i^* \tag{1}$$

Due to the complementary slackness condition, we have

$$\left( \sum_{j=1}^m a_{j,i} y_j^* + z_i^* - v_i \right) \cdot x_i^* = 0, \quad \forall 1 \leq i \leq n \tag{2}$$

$$\left( \delta_j - \sum_{i=1}^n a_{j,i} x_i^* \right) \cdot y_j^* = 0, \quad \forall 1 \leq j \leq m \tag{3}$$

$$(1 - x_i^*) \cdot z_i^* = 0, \quad \forall 1 \leq i \leq n \tag{4}$$

Now we construct an allocation as follows: If  $x_i^* = 1$ , let  $X_i = \Omega_i$ ; otherwise let  $X_i = \emptyset$ . Let the price of commodity  $\omega_j$  be  $y_j^*$ , i.e.,  $p_j = y_j^*$ . We following prove this is a Walrasian equilibrium. There are four conditions we need to check:

- (1) For each commodity  $\omega_j$ , we should have  $\sum_{i=1}^n a_{j,i}x_i^* \leq \delta_j$ . (It is satisfied since  $x^*$  is IP (LPR) feasible).
- (2) For each commodity  $\omega_j$  that is not clear, we should have  $p_j = 0$ . (If  $\omega_j$  is not clear, i.e.,  $\delta_j - \sum_{i=1}^n a_{j,i}x_i^* > 0$ , by (3), we have  $p_j = y_j^* = 0$ ).
- (3) If buyer  $\mathcal{O}_i$  gets the bundle  $\Omega_i$ , i.e.,  $x_i^* = 1$ , then  $v_i$  should be greater than the price of  $\Omega_i$ . (By (2), we have  $\sum_{j=1}^m a_{j,i}y_j^* + z_i^* - v_i = 0$ , i.e.,  $v_i = \sum_{j=1}^m a_{j,i}y_j^* + z_i^* \geq \sum_{j=1}^m a_{j,i}y_j^* = \sum_{j=1}^m a_{j,i}p_j = p(\Omega_i)$ ).
- (4) If buyer  $\mathcal{O}_i$  does not get the bundle  $\Omega_i$ , i.e.,  $x_i^* = 0$ , then  $v_i$  should be less than the price of  $\Omega_i$ . (By (4), we have  $z_i^* = 0$ . Because  $y^*, z^*$  is DLPR feasible, so  $\sum_{j=1}^m a_{j,i}y_j^* + z_i^* \geq v_i$ . Hence  $\sum_{j=1}^m a_{j,i}y_j^* \geq v_i$ , i.e.,  $p(\Omega_i) = \sum_{j=1}^m a_{j,i}p_j = \sum_{j=1}^m a_{j,i}y_j^* \geq v_i$ ).

From the above conditions,  $(X, p)$  is a Walrasian equilibrium.

$\Rightarrow$ : Assume a Walrasian equilibrium  $(X, p)$  exists. Let  $x_i^* = 1$ , if  $X_i = \Omega_i$ ; otherwise let  $x_i^* = 0$ . Then  $x^* = (x_i^*)$  is IP (LPR) feasible. By Lemma 5.1, we have

$$M_{IP} = \sum_{i=1}^n v_i x_i^*. \quad (5)$$

We following construct vector  $y^* = (y_j^*)$  and  $z^* = (z_i^*)$ . Let  $y_j^* = p_j$ . If  $x_i^* = 0$ , i.e.,  $v_i - p(\Omega_i) \leq 0$ , let  $z_i^* = 0$ ; otherwise let  $z_i^* = v_i - p(\Omega_i)$ . From the above construction, it is clear that for all  $1 \leq i \leq n$ ,

$$\sum_{j=1}^m a_{j,i}y_j^* + z_i^* = \sum_{j=1}^m a_{j,i}p_j + z_i^* = p(\Omega_i) + z_i^* \geq v_i.$$

Hence  $y^*, z^*$  is DLPR feasible. Then,

$$M_{DLPR} \leq \sum_{j=1}^m \delta_j y_j^* + \sum_{i=1}^n z_i^*. \quad (6)$$

Similarly, we have  $x_i^* \cdot (v_i - \sum_{j=1}^m a_{j,i}y_j^* - z_i^*) = 0$ . Hence,

$$\begin{aligned} \sum_{i=1}^n x_i^* \cdot \left( v_i - \sum_{j=1}^m a_{j,i}y_j^* - z_i^* \right) &= 0 \\ \Rightarrow \sum_{i=1}^n v_i x_i^* - \sum_{j=1}^m y_j^* \sum_{i=1}^n a_{j,i}x_i^* - \sum_{i=1}^n x_i^* z_i^* &= 0. \end{aligned}$$

If  $\sum_{i=1}^n a_{j,i}x_i^* < \delta_j$ , then  $y_j^* = p_j = 0$ , and  $y_j^* \cdot \sum_{i=1}^n a_{j,i}x_i^* = 0 = \delta_j y_j^*$ ; otherwise  $\sum_{i=1}^n a_{j,i}x_i^* = \delta_j$ , we also have  $y_j^* \cdot \sum_{i=1}^n a_{j,i}x_i^* = \delta_j y_j^*$ . If  $x_i^* = 0$ , then  $z_i^* = 0$ , and  $x_i^* z_i^* = z_i^*$ ; otherwise  $x_i^* = 1$ , we also

have  $x_i^* z_i^* = z_i^*$ . Therefore,

$$\sum_{i=1}^n v_i x_i^* - \sum_{j=1}^m \delta_j y_j^* - \sum_{i=1}^n z_i^* = 0. \tag{7}$$

Combining (5), (6), and (7), we have  $M_{LPR} = M_{DLPR} \leq M_{IP}$ . That is,  $M_{IP} = M_{LPR}$ .  $\square$

### 6. Conclusion and discussions

In this work, we consider algorithm and complexity issues of Walrasian equilibrium for single-minded auction. Our communication complexity result is the first non-trivial matching bound in communication complexity of computational economics. The reduction technique for NP-hard proof of Walrasian equilibrium may be useful in related equilibrium problems.

A polynomial size duality theorem for single-minded auction is established. In comparison, that of Bikhchandani and Mamer [3] for combinatorial auctions (and their version for single-minded auction) is of exponential size, and thus not suitable for the study of complexity issues. Our discussion is specified to single-minded buyers, what happens if buyers are, for example,  $k$ -minded?

Because Walrasian equilibrium does not necessarily exist, we may consider *relaxed Walrasian equilibrium*, in which we only require condition (i) of Definition 4.2 (i.e., if the commodity is not clear, its price is zero). Trivially, relaxed Walrasian equilibrium always exists. Here, for any single-minded auction, we are asked to select a relaxed Walrasian equilibrium  $(X, p)$  to maximize  $\delta \cdot n$ ,  $0 < \delta \leq 1$ , the number of satisfied buyers, where buyer  $\mathcal{O}_i$  is *satisfied* if his utility is maximized by the corresponding allocation  $X_i$  under price vector  $p$ .

Note that the ordinary Walrasian equilibrium is equivalent to the case  $\delta = 1$ . In Example 4.1,  $\delta = 2/3$ . Thus, there is an instance such that the number of satisfied buyers is at most  $2n/3$ . Specifically, we have the following conjecture.

**Conjecture.** *For any single-minded auction,  $\delta \geq 2/3$ , and then, the bound  $\delta = 2/3$  is tight. That is, there always exists relaxed Walrasian equilibrium  $(X, p)$  such that there are at least  $2n/3$  satisfied buyers under  $(X, p)$ .*

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