Generation of Massive Entanglement Through Adiabatic Quantum Phase Transition in a spinor condensate

Z. Zhang, L.-M. Duan
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA and Center for Quantum Information, IIIS, Tsinghua University, Beijing 100084, China

We propose a method to generate massive entanglement in a spinor Bose-Einstein condensate from an initial product state through adiabatic sweep of magnetic field across a quantum phase transition induced by competition between the spin-dependent collision interaction and the quadratic Zeeman effect. The generated many-body entanglement is characterized by the experimentally measurable entanglement depth in the proximity of the Dicke state. We show that the scheme is robust to practical noise and experimental imperfection and under realistic conditions it is possible to generate genuine entanglement for hundreds of atoms.

Generation of massive entanglement, besides its interest for foundational research of quantum theory, is of great importance for applications in quantum information processing and precision measurements. Entanglement is a valuable resource that can be used to enhance the performance of quantum computation, the security of quantum communication, and the precision of quantum measurements. For these applications, it is desirable to get as many particles as possible into entangled states. However, entanglement is typically fragile and many-particle entangled states can be easily destroyed by decoherence due to inevitable coupling to the environment. As an experimental record, so far fourteen qubits carried by trapped ions have been successfully prepared into genuine entangled states [1]. Pushing up this number represents a challenging goal in the experimental frontier.

The Bose Einstein condensate of ultracold atoms is in a pure quantum mechanical state with strong collision interaction. In a spinor condensate [2–4], the spin-dependent collision interaction can be used to produce spin squeezing [5, 6], which is an indicator of many-particle entanglement [8]. Spin squeezing has been demonstrated in condensates in recent experiments through spin-dependent collision dynamics [6, 7]. A squeezed state is typically sensitive to noise and generation of substantial squeezing requires accurate control of experimental systems, which is typically challenging. In quantum information theory, the Dicke states are known to be relatively robust to noise and they have important applications for quantum metrology [9] and implementation of quantum information protocols [10]. For instance, the three-particle Dicke state, the so-called W state, has been proven to be the most robust entangled state under the particle loss [11]. Because of their applications and nice noise properties, Dicke states represent an important class of many-body states that are pursed in physical implementation. For a few particles, Dicke states have been generated in several experimental systems [12] .

In this paper, we propose a robust method to generate massive entanglement in the proximity of many-particle Dicke states through control of adiabatic passage across a quantum phase transition in a spinor condensate. Using conservation of the magnetic quantum number, we show that sweep of the magnetic field across the polar-ferromagnetic phase transition provides a simple method to generate many-body entanglement in this mesoscopic system. The generated many-body entanglement can be characterized through the entanglement depth, which measures how many particles have been prepared into genuine entangled states [8, 13]. The entanglement depth can be easily measured experimentally for this system through a criterion introduced in Ref. [14]. We quantitatively analyze the entanglement production through the entanglement depth and show that the scheme is robust under noise and experimental imperfection. The scheme works for both the ferromagnetic (such as ^{87}Rb) and the anti-ferromagnetic (such as ^{23}Na) condensates. For the anti-ferromagnetic case, we use adiabatic quantum phase transition in the highest eigenstate of the Hamiltonian instead of its ground state.

The system under consideration is a ferromagnetic (or anti-ferromagnetic) spin-1 Bose Einstein condensate under an external magnetic field, which has been realized with ^{87}Rb (or ^{23}Na) atoms in an optical trap[4]. The Hamiltonian for the spin-1 condensate can be divided into two parts $H = H_0 + H_i$. The non-interacting Hamiltonian H_0 and the interaction Hamiltonian H_i take respectively the following forms [2, 4]

$$\begin{aligned}\hat{H}_0 &= \sum_{m,n=0,\pm1} \int d\mathbf{r} \hat{\psi}_m^\dagger [-\frac{\hbar^2\nabla^2}{2M} \\ &+ U(\mathbf{r}) - p(f_z)_{mn} + q(f_z^2)_{mn}] \hat{\psi}_n, \quad (1)\\ \hat{H}_i &= \frac{1}{2}\int d\mathbf{r}[c_0 : \hat{n}^2(\mathbf{r}) : + c_1 : \hat{F}^2(\mathbf{r}) :]. \quad (2)\end{aligned}$$

where$\hat{\psi}_m(\mathbf{r})$ denote the bosonic filed operators with the spin index $m = 1, 0, -1$, corresponding to annihilation of an atom of mass M in the Zeeman state m on the hyperfine level $F = 1$. The atoms are trapped by the spin-independent optical potential$U(\mathbf{r})$. The linear Zeeman coefficient $p = -g\mu_B B$, where g is the Landé g factor, μ_B is the Bohr magneton, and B is the external magnetic field. The quadratic Zeeman coefficient $q = \frac{(g\mu_B B)^2}{\Delta E_{hf}}$, where ΔE_{hf} is the hyperfine energy splitting. The symbol f_μ ($\mu = x, y, z$) denotes

μ-component of the spin-1 matrix, and $(f_\mu)_{mn}$ is the corresponding (m,n) matrix element. The particle density operator $\hat{n}(\mathbf{r})$ and the spin operator $\hat{F}(\mathbf{r})$ are defined respectively by $\hat{n}(\mathbf{r}) = \sum_{m=-1}^{1} \hat{\psi}_m^\dagger(\mathbf{r})\hat{\psi}_m(\mathbf{r})$ and $\hat{F}_\mu(\mathbf{r}) = \sum_{m,n=-1}^{1} (f_\mu)_{mn}\hat{\psi}_m^\dagger(\mathbf{r})\hat{\psi}_n(\mathbf{r})$. The interaction coefficients $c_0 = 4\pi\hbar^2(a_0+2a_2)/3M$, $c_1 = 4\pi\hbar^2(a_2-a_0)/3M$, where a_s is the s-wave scattering lengths for two colliding atoms with total spin s. We have $c_1 < 0$ ($c_1 > 0$) for ^{87}Rb (^{23}Na), which corresponds to ferromagnetic (anti-ferromagnetic) interaction, respectively.

For typical spinor condensates in experiments such as^{87}Rb and ^{23}Na, we have $c_0 \gg c_1$, so the spin-independent interaction dominates over the spin-dependent interaction. In this case, to describe the ground state in a spin-independent optical trap $U(\mathbf{r})$, it is good approximation to assume that different spin components $\hat{\psi}_m(\mathbf{r})$ of the condensate take the same spatial wave function $\phi(\mathbf{r})$. This is the well-known single mode approximation [3, 4], and under this approximation we have $\hat{\psi}_m \approx \hat{a}_m\phi(\mathbf{r}), (m = 1, 0, -1)$, where a_m is the annihilation operator for corresponding spin mode and $\phi(\mathbf{r})$ is normalized as $\int d\mathbf{r}|\phi(\mathbf{r})|^2 = 1$. We consider a spinor condensate with a fixed total particle number N as in experiments and neglect the terms in the Hamiltonian that are constant under this condition. The spin-dependent part of the Hamiltonian is then simplified to

$$H = c_1' \frac{\mathbf{L}^2}{N} + \sum_{m=-1}^{1} (qm^2 - pm)a_m^\dagger a_m \tag{3}$$

where we have introduced the spin-1 angular momentum operator $\mathbf{L}_\mu = \sum_{m,n} a_m^\dagger (f_\mu)_{mn} a_n$ and defined $c_1' = c_1 N \int d\mathbf{r}|\phi(\mathbf{r})|^4/2$. The linear Zeeman term $\sum_{m=-1}^{1} pma_m^\dagger a_m = pL_z$ typical dominates in the Hamiltonian H, however, this term commutes with all the other terms in the Hamiltonian, and if we start with an initial state that is an eigenstate of L_z, the linear Zeeman term has no effect and thus can be neglected. In this paper, we consider an initial state with all the atoms prepared to the level $|F = 1, m = 0\rangle$ through optical pumping, which is an eigenstate of L_z. The system remains in this eigenstate with magnetization $L_z = 0$, and the effective spin Hamiltonian becomes

$$H = c_1' \frac{\mathbf{L}^2}{N} - qa_0^\dagger a_0. \tag{4}$$

The ratio q/c_1' is the only tunable parameter in this Hamiltonian, and depending on its value, the Hamiltonian has different phases resulting from competition between the quadratic Zeeman effect and the spin-dependent collision interaction.

We first consider the ferromagnetic case with $c_1' < 0$. For the initial state, we tune up the magnetic field to make the quadratic Zeeman coefficient $q \gg |c_1'|$. In this limit, the second term $-qa_0^\dagger a_0$ dominates in the Hamiltonian H. The ground state of the Hamiltonian is given by an eigenstate of $a_0^\dagger a_0$ with the maximum eigenvalue N. This ground state can be prepared by putting all the atoms to the Zeeman level $|F = 1, m = 0\rangle$through optical pumping. Then we slowly ramp down the magnetic field to zero. From the adiabatic theorem, the system remains in the ground state of the Hamiltonian H and the final state is the lowest-energy state of $H_F = c_1'\mathbf{L}^2/N$, which is the Dicke state $|L = N, L_z = 0\rangle$ that maximizes $\mathbf{L}^2$ with the eigenvalue $L(L+1)$. The Dicke state $|L = N, L_z = 0\rangle$ is a massively entangled state of all the particles.

The above simple argument illustrates the possibility to generate massive entanglement through an adiabatic passage. To turn this possibility into reality, however, there are several key issues we need to analyze carefully. First, we need to know what is the requirement of the sweeping speed of the parameter q to maintain an adiabatic passage. In particular, this adiabatic passage goes through a quantum phase transition where the energy gap approaches zero in the thermodynamical limit. So the evolution cannot be fully adiabatic for a large system. It is important to know how the energy gap scales with the particle number N for this mesoscopic system. Second, due to the non-adiabatic correction and other inevitable noise in a real experimental system, the final state is never a pure state and quite different from its ideal form $|L = N, L_z = 0\rangle$. For a many-body system with a large number of particles, the state fidelity is always close to zero with presence of just small noise. So we need to analyze whether we can still generate and confirm genuine many-particle entanglement under realistic experimental conditions.

To analyze the entanglement behavior, first we quantitatively calculate the phase transitions during this adiabatic passage and analyze how the energy gap scales with the particle number N. The mean-field phase diagram for the Hamiltonian (3) is well known [4]. However, in typical mean field calculations one fixes the parameters p, q to obtain the ground state of the Hamiltonian (3), and this ground state in general has varying magnetization $\langle L_z\rangle$. For our proposed adiabatic passage, we should fix $L_z = 0$ and find the ground state of the Hamiltonian (4) instead of (3) as the linear Zeeman term is irrelevant. We perform exact numerical many-body calculation in the Hilbert space with $L_z = 0$ to find the ground state of the Hamiltonian (4) and draw the condensate fraction in the the Zeeman level $|F = 1, m = 0\rangle$, N_0/N with $N_0 \equiv \left\langle a_0^\dagger a_0 \right\rangle$, in Fig.1 as we ramp down the parameter q. Control of the magnetic field can only sweep the parameter q from the positive side to zero. Further sweep of q to the negative side can be obtained through ac-Stack effect induced by a microwave field coupling the hyperfine levels $|F = 1\rangle$ and $|F = 2\rangle$, as demonstrated in experiments [15]. The curve in Fig. 1 shows two second-order phase transitions at the positions $q/|c_1'| = \pm 4$, where the condensate fraction N_0/N drop first from 1 to a positive number r $(0 < r < 1)$ and then to 0. The transition point at $q/|c_1'| = 4$ agrees with the mean field prediction, however, there is a significant discrepancy for the tran-

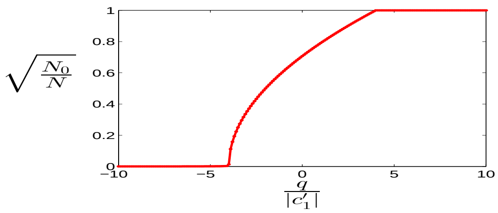

Figure 1: The order parameter $\sqrt{\langle N_0/N\rangle}$ shown as a function of the quadratic Zeeman coefficient q in the unit of $|c_1'|$ for the total atom number $N = 10^5$. Two second-order phase transitions take place at $q/|c_1'| = \pm 4$.

sition at $q/|c_1'| = -4$. Mean field calculation under a fixed parameter $p = 0$ predicts a transition at $q/|c_1'| = 0$, where the magnetization $\langle L_z\rangle$ abruptly changes [4]. For the adiabatic passage considered here, due to the conservation of L_z the transition at $q/|c_1'| = 0$ is postponed to the point $q/|c_1'| = -4$.

Besides prediction of the phase transition points, the exact many-body calculation can show evolution of entanglement for the ground state and scaling of the energy gap with the particle number N at the phase transition points. The scaling of the energy gap is important as it shows the relevant time scale to maintain the adiabatic passage. In Fig. 2(a), we show the energy gap Δ (defined as the energy difference between the ground state and the first excited state) in the unit of $|c_1'|$ as a function of $q/|c_1'|$ for $N = 10^4$ particles. The gap attains the minimum at the phase transition points and is symmetric with respect to the transitions at $q/|c_1'| = \pm 4$. In Fig. 2(b), we show how the energy gap at the phase transition point scales with the particle number N. In the $\log - \log$ plot, the points are on a line, which can be well fit with the polynomial scaling $\Delta = 7.4N^{-1/3}$. The energy gap decreases slowly with increase of the particle number N, which suggests it is possible to maintain an adiabatic passage for typical experimental systems with $N \sim 10^5$.

With this understanding, we now turn to our main task to characterize entanglement generation with this adiabatic passage. For this purpose, we need to have a quantity to measure entanglement in the proximity of the Dicke state and this measure should be accessible to experimental detection. Due to non-adiabatic corrections and inevitable noise in real experiments, we cannot assume that the system is in a pure state and the entanglement measure should work for any mixed states. Many-body entanglement can be characterized in different ways, and a convenient measure is the so-called entanglement depth which measures how many particles in an N-particle system have been prepared into genuine entangled states given an arbitrary mixed state of the system [8, 13, 14]. A quantity to measure the entanglement depth for N spin-1/2 particles has been provided in Ref. [14] based on measurements of the collective spin operators. It is straightforward to generalize

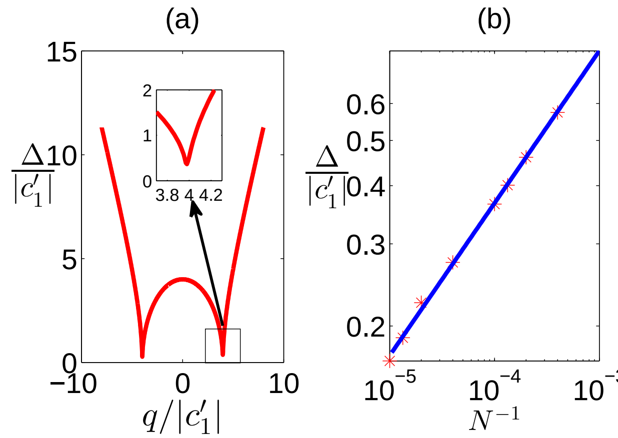

Figure 2: (a) The energy gap Δ in the unit of $|c_1'|$ shown as a function of $q/|c_1'|$ with the total particle number $N = 10^4$. (b) The stars show the scaling of the energy gap $\Delta/|c_1'|$ at the phase transition point with the particle number N in the log-log plot. The solid line is a linear fit to the data points with $\Delta = 7.4N^{-1/3}$.

this quantity to the case of N spin-1 particles. For N spin-1 particles, the collective spin operator is defined by $\boldsymbol{L} = \sum_{i=1}^{N} l_i$, where l_i denotes the individual spin-1operator. In terms of the bosonic mode operators, the collective spin operator has the standard decomposition $\mathbf{L}_\mu = \sum_{m,n} a_m^\dagger (f_\mu)_{mn} a_n$ ($\mu = x, y, z$; $m, n = 0, \pm 1$). To characterize entanglement in the proximity of the Dicke state $|L = N, L_z = 0\rangle$, we measure the quantity

$$\xi = \frac{\langle L_x^2\rangle + \langle L_y^2\rangle}{N(1 + 4\langle(\Delta L_z^2)\rangle)} \tag{5}$$

If $\xi > m$, from the arguments that lead to theorem 1 of Ref. [14] we conclude that the system has at least genuine m-particle entanglement (i.e., the entanglement depth is bounded by m from below). For the ideal Dicke state $|L = N, L_z = 0\rangle$, one can easily verify that $\xi = N + 1 > N$, so all the N particles are in a genuine entangled state. The final state of real experiments is in general a complicated mixed state which is impossible to be read out for many-particle systems. The power of the measure in Eq. (5) is that it gives an experimentally convenient way to bound the entanglement depth in this case through simple detection of the collective spin operators even through the system state remains unknown.

Now we show how the entanglement measure defined in Eq. (5) evolves when we adiabatically sweep the parameter q in the Hamiltonian (4). We ramp down the parameter q linearly from $q = 6|c_1'|$ to 0 with a constant speed, starting from the initial product state with all the particles in the level $|F = 1, m = 0\rangle$. The entanglement depth ξ (in the unit of N) of the final state is shown in Fig. 3(a) and 3(b) as a function of the sweeping speed v (in the unit of $|c_1'|^2$ by taking $\hbar = 1$) for $N = 10^3$ and $N = 10^4$, respectively. We can see that the entangle-

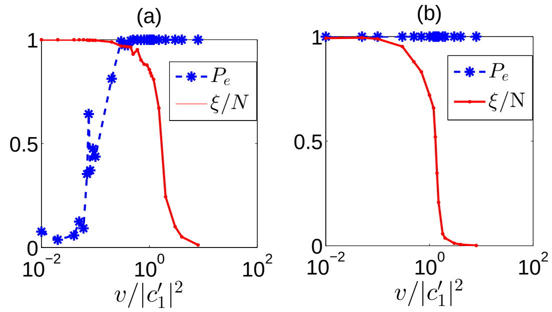

Figure 3: The normalized entanglement depth ξ/N (solid lines) and the excitation probability P_e (star points) for the final state shown as functions of the sweeping speed v (in the unit of $|c_1'|^2$) for the number of particles $N = 10^3$ (a) and $N = 10^4$ (b). The parameter q in the Hamiltonian (4) is ramped down linearly from $q = 6|c_1'|$ to 0 at a constant speed v, starting from the initial product state with all the particles in the level $|m = 0\rangle$.

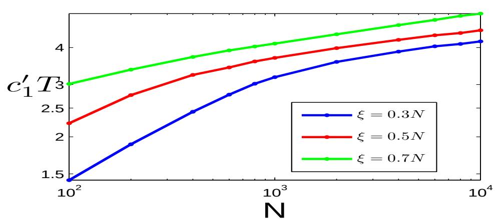

Figure 4: Scaling of the required sweeping time T (in the unit of $1/|c_1'|$) with the particle number N as we fix the entanglement depth of the final state to be$0.3N$ (bottom curve), $0.5N$ (middle curve), and$0.7N$ (top curve), respectively.

ment depth increases abruptly from a few to the order of N when the speed v decreases below $|c_1'|^2$. In the same figure, we also show the excitation probability of the final state (the probability to be not in the ground state). For a small number of particles, the excitation probability typically correlates with the entanglement depth, and they jump roughly around the same value of the sweeping speed. However, for a large number of particles (e.g., $N \geq 10^4$), we can have the entanglement depth of the order of N while the excitation probability is near the unity as shown in Fig. 3(b). This indicates that the entanglement in the proximity of the Dicke state is quite robust. Even when the sweep is not fully adiabatic and most of the atoms are excited to the low-lying excited states (meaning that the state fidelity decrease to almost zero), we can still have the entanglement depth close to N (meaning all the particles are still genuinely entangled).

As the energy gap Δ at the phase transition point decreases with the atom number N, one expects that the required sweeping time T to get substantial entanglement increases with N. However, this increase is very slow. First, Δ increases slowly with N by the scaling $\Delta \propto N^{-1/3}$ as shown in Fig. 2(b). Second, for a large N even when $\Delta T < 1$ and a significant fraction of the atoms get excited during the sweep, we can still observe substantial entanglement as the entanglement depth of the low-lying excited states is still high as shown in Fig. 3(b). To see the the quantitative relation between the required sweeping time T and the particle number N, we fix the entanglement depth of the final state to be a significant number (e.g., with $\xi = 0.3N$, $0.5N$, or $0.7N$) and draw in Fig. 4 the scaling of T (in the unit of $1/|c_1'|$) as a function of N. When $N \geq 10^3$, the curve of $|c_1'|T$ is almost flat, increasing by a modest 20% when the atom number grows by an order of magnitude.

All the calculations above are done for the ferromagnetic case with $c_1' < 0$ by assuming an adiabatic sweep of the Hamiltonian (4) in its ground state. For the anti-ferromagnetic case with $c_1'>0$ (such as ^{23}Na), we can perform an adiabatic sweep along the ground state of the Hamiltonian $-H$ (or the highest eigenstate of the Hamiltonian H in Eq. (4)). Then, all the calculations above equally apply to the anti-ferromagnetic case. The only difference is that initially the parameter q needs to be set to the negative side with $q = -6|c_1'|$ when the atoms are prepared into the level $|m = 0\rangle$. As mentioned before, q can be switched to both the positive and the negative sides, through ac-Stack shift from a π-polarized microwave field that couples the hyperfine levels $|F = 1\rangle$ and $|F = 2\rangle$ [15]. An advantage of using ^{23}Na instead of ^{87}Rb is that it is has a larger spin-dependent collision rate $|c_1'|$ and thus allows a faster sweep of the parameter q. If we take the peak condensate density about $10^{14} cm^{-3}$, $c_1'/\hbar$ is estimated to be about $-2\pi \times 7Hz$ for ^{87}Rb atoms and $2\pi \times 50Hz$ for ^{23}Na atoms.

Finally, we briefly discuss how the noise influence entanglement generation in this scheme. First, in the proximity of the Dicke state the entanglement depth measured through Eq. (5) is very robust to the dephasing noise (dephasing between the Zeeman levels caused by, e.g., a small fluctuating magnetic field). As shown in Ref. [14], even with a dephasing error rate about 50% for each individual atom, the entanglement depth ξ remains about $N/2$, which is still large. The entanglement depth is more sensitive to the bit-flip error that increases $\langle \Delta L_z^2 \rangle$in Eq. (5), which can be caused by imperfect preparation of the initial state, atom loss during the adiabatic sweep, or imperfection in the final measurement of the collective spin operators. The detection error can be corrected through simple data processing using the method proposed in Ref. [16] as long as its error rate has been calibrated. The initial state $|F = 1, m = 0\rangle$ can be prepared efficiently through optical pumping and remaining atoms in the $|F = 1, m = \pm 1\rangle$ levels can be blown away through microwave coupling to the $|F = 2\rangle$ levels that are unstable under atomic collisions. The atomic loss should be small as the sweeping time T is assumed to be much shorter compared with the life time of the condensate. Only loss of atoms in the components $|F = 1, m = \pm 1\rangle$ can increase the fluctuation $\langle \Delta L_z^2 \rangle$. Assume the loss rate is p during the sweep, the resultant $\langle \Delta L_z^2 \rangle$ is estimated by $\langle \Delta L_z^2 \rangle \sim Np(1-p)/6$. For a large number of atoms with $Np \gg 1$, the entanglement depth in Eq. (5) is then estimated by $\xi \sim 3/(2p)$. If we take p about 1%, it is possible to prepare a remarkable number of hundreds of atoms into genuine entangled states.

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