Computer Assisted Proof for Apwenian Sequences

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ABSTRACT

An infinite ± 1-sequence is called *Apwenian* if its Hankel determinant of order n divided by 2^{n-1} is an odd number for every positive integer n. In 1998, Allouche, Peyrière, Wen and Wen discovered and proved that the Thue–Morse sequence is an Apwenian sequence by direct determinant manipulations. Recently, Bugeaud and Han re-proved the latter result by means of an appropriate combinatorial method. By significantly improving the combinatorial method, we find several new Apwenian sequences with Computer Assistance. This research has application in Number Theory to determining the irrationality exponents of some transcendental numbers.

CCS Concepts

•**Computing methodologies** → Symbolic calculus algorithms;

Keywords

Hankel determinant, Thue–Morse sequence, Apwenian sequence, permutation, computer assisted proof

1. INTRODUCTION

For each infinite sequence $\mathbf{c} = (c_k)_{k\geq 0}$ and each nonnegative integer n the *Hankel determinant* of order n of the sequence $\mathbf{c}$ is defined by

$$H_n(\mathbf{c}) := \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix}. \tag{1}$$

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We also speak of the Hankel determinants of the power series $\tilde{\mathbf{c}}(x) = \sum_{k\geq 0} c_k x^k$ and write $H_n(\tilde{\mathbf{c}}(x)) = H_n(\mathbf{c})$. The Hankel determinants are widely studied in Mathematics and, in several cases, can be evaluated by basic determinant manipulation, LU-decomposition, or Jacobi continued fraction (see, e.g., [12, 13, 7]). However, the Hankel determinants studied in the present paper apparently have no closed-form expressions, and require additional efforts to obtain specific arithmetical properties.

An infinite ± 1-sequence $\mathbf{c} = (c_k)_{k\geq 0}$ is called *Apwenian*[1] if its Hankel determinant of order n divided by 2^{n-1} is an odd number, i.e., $H_n(\mathbf{c})/2^{n-1} \equiv 1 \pmod 2$, for all positive integers n. The corresponding generating function or the power series $\tilde{\mathbf{c}}(x)$ is also said to be *Apwenian*. Recall that the *Thue–Morse sequence*, denoted by

$$\mathbf{e} = (e_k)_{k\geq 0} = (1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1 \ldots),$$

is a special ± 1-sequence, defined by the generating function

$$\tilde{\mathbf{e}}(x) = \sum_{k=0}^{\infty} e_k x^k = \prod_{k=0}^{\infty} (1 - x^{2^k}), \tag{2}$$

or equivalently, by the recurrence relations

$$e_0 = 1, \quad e_{2k} = e_k \text{ and } e_{2k+1} = -e_k \text{ for } k \geq 0. \tag{3}$$

The Thue–Morse sequence is also called *Prouhet–Thue–Morse sequence*. For other equivalent definitions and properties related to the sequence, see [3, 2]. In 1998, Allouche, Peyrière, Wen and Wen established a congruence relation concerning the Hankel determinants of the Thue–Morse sequence [1].

THEOREM 1.1 (APWW). *The Thue–Morse sequence on $\{1, -1\}$ is Apwenian.*

Theorem 1.1 has an important application in Number Theory. As a consequence of Theorem 1.1, all the Hankel determinants of the Thue–Morse sequence are nonzero. This property allowed Bugeaud [4] to prove that the irrationality exponents of the Thue–Morse–Mahler numbers are exactly 2.

The goal of the paper is to find more Apwenian sequences. Let d be a positive integer and $\mathbf{v} = (v_0, v_1, v_2, \ldots, v_{d-1})$ a finite ± 1-sequence of length d such that $v_0 = 1$. The generating polynomial of $\mathbf{v}$ is denoted by $\tilde{\mathbf{v}}(x) = \sum_{i=0}^{d-1} v_i x^i$. It is clear that the following power series

$$\Phi(\tilde{\mathbf{v}}(x)) = \prod_{k=0}^{\infty} \tilde{\mathbf{v}}(x^{d^k}) \tag{4}$$

[1] We use the term "*Apwenian*" to honor the four authors Allouche, Peyrière, Wen and Wen for their seminal paper [1].

defines a ± 1-sequence, whose n-th term is equal to

$$f_n = \prod_{i=1}^{d-1} v_i^{\#_i(n)},$$

where $\#_i(n)$ denotes the number of occurrences of the digit i in the base-d representation of n. Thus, the power series displayed in (2) is equal to $\Phi(1-x)$. Our main result is stated next.

THEOREM 1.2. *The following power series are all Apwenian:*

$$\begin{aligned}
F_2(x) &= \Phi(1-x),\\
F_3(x) &= \Phi(1-x-x^2),\\
F_5(x) &= \Phi(1-x-x^2-x^3+x^4),\\
F_{11}(x) &= \Phi(1-x-x^2+x^3-x^4+x^5+x^6+x^7\\
&\qquad +x^8-x^9-x^{10}),\\
F_{13}(x) &= \Phi(1-x-x^2+x^3-x^4-x^5-x^6-x^7\\
&\qquad -x^8+x^9-x^{10}-x^{11}+x^{12}),\\
F_{17a}(x) &= \Phi(1-x-x^2+x^3-x^4+x^5+x^6+x^7+x^8+x^9\\
&\qquad +x^{10}+x^{11}-x^{12}+x^{13}-x^{14}-x^{15}+x^{16}),\\
F_{17b}(x) &= \Phi(1-x-x^2-x^3+x^4+x^5-x^6+x^7+x^8+x^9\\
&\qquad -x^{10}+x^{11}+x^{12}-x^{13}-x^{14}-x^{15}+x^{16}).
\end{aligned}$$

Remarks. Let us make some useful comments about the above theorem.

1. The fact that the generating function $F_2(x)$ for the Thue–Morse sequence is Apwenian has already been proved in [1].

2. By using the Jacobi continued fraction expansion of a power series $F(x)$, we know that $H_n(F(x)) = H_n(F(-x))$ when d is an odd integer. See, for example, [12, 7, 9, 10]. Hence, Theorem 1.2 implies that $F_3(-x) = \Phi(1+x-x^2)$, $F_5(-x) = \Phi(1+x-x^2+x^3+x^4)$, etc. are all Apwenian.

3. There is no F_7 in Theorem 1.2, but two F_{17} (we mean F_{17a} and F_{17b}).

4. For each positive integer j we have

$$\Phi(\tilde{v}(x)) = \Phi(\prod_{k=0}^{j} \tilde{v}(x^{d^k})).$$

   Consequently, $\Phi(1-x-x^2+x^3)$ is Apwenian since it is equal to $\Phi(1-x)$.

5. By exhaustive search, the list $F_2, F_3, \ldots, F_{17b}$ of Theorem 1.2 together with the transformation of Remark 2 yields a complete list of Apwenian sequences for prime integer d.

Actually, Theorem 1.1 has three proofs. The original proof of Theorem 1.1 is based on determinant manipulation by using the so-called *sudoku method* [1, 11]. The second one is a combinatorial proof derived by Bugeaud and Han [5]. The third proof is very short and makes use of Jacobi continued fraction algebra [10]. Unfortunately, the method developed in the short proof cannot be used for proving our main theorem, because the underlying Jacobi continued fractions are not ultimately periodic [10, 9]. However, another analogous result for the sequence $F_3(x)$ when dealing with modulo 3 (instead of modulo 2) is established using the short method, as stated in the next theorem [9].

THEOREM 1.3. *For every positive integer n the Hankel determinant $H_n(F_3(x))$ of the sequence $F_3(x)$ satisfies the following relation*

$$H_n(F_3(x)) \equiv \begin{cases} 1 \pmod 3 & \text{if } n \equiv 1,2 \pmod 4;\\ 2 \pmod 3 & \text{if } n \equiv 3,0 \pmod 4. \end{cases} \tag{5}$$

Combining Theorems 1.2 and 1.3 yields the following result.

COROLLARY 1.4. *For every positive integer n the Hankel determinant $H_n(F_3(x))$ verifies the following relation*

$$\frac{H_n(F_3(x))}{2^{n-1}} \equiv \begin{cases} 1 \pmod 6 & \text{if } n \equiv 0,1 \pmod 4;\\ 5 \pmod 6 & \text{if } n \equiv 2,3 \pmod 4. \end{cases} \tag{6}$$

In the following table we reproduce the first few values of the Hankel determinants of the sequence $F_3(x)$ for illustrating Theorems 1.2, 1.3 and Corollary 1.4.

n	1	2	3	4	5	6	7	8	9
$H_n(\mathbf{f})$	1	-2	-4	8	16	-32	-64	128	4864
$H_n(\mathbf{f}) \pmod 3$	1	1	2	2	1	1	2	2	1
$H_n(\mathbf{f})/2^{n-1}$	1	-1	-1	1	1	-1	-1	1	19
$\frac{H_n(\mathbf{f})}{2^{n-1}} \pmod 2$	1	1	1	1	1	1	1	1	1
$\frac{H_n(\mathbf{f})}{2^{n-1}} \pmod 6$	1	5	5	1	1	5	5	1	1

Recently, Bugeaud and Han re-proved Theorem 1.1 by means of an appropriate combinatorial method [5]. The latter method has been significantly upgraded to prove that $F_3(x), F_5(x), \ldots$ are Apwenian. As can be seen, in Section 3 Step 2, a family of cases (called *types*) is considered for proving the various recurrence relations. Roughly speaking, the types are indexed by words $s_0 s_1 s_2 \cdots s_d$ of length $d+1$ over a d-letter alphabet. Comparing to the original combinatorial method, the upgrading does not provide a shorter proof; however, it involves a systematic proof *by exhaustion* that only consists of checking all the types. The proof of Theorem 1.2 is then achieved with *Computer Assistance*.

In practice, the number of types is very large. For example, as described in [8] for the study of $F_{11}(x)$, there are 2274558 types! Fortunately, the set of permutations of each type can be decomposed into the Cartesian product of so-called *atoms* (see Substep 3(d) in the sequel), and moreover, the cardinality of each atom can be rapidly evaluated by a sequence of tests (see Definition 4.1 and Table 2).

PROBLEM 1.5. *Is the following power series Apwenian:*

$$\begin{aligned}F_{19}(x) = \Phi(1&-x-x^2-x^3+x^4-x^5+x^6-x^7-x^8+x^9\\ &+x^{10}-x^{11}-x^{12}-x^{13}-x^{14}-x^{15}+x^{16}-x^{17}-x^{18})\,?\end{aligned}$$

Find a fast computer assisted proof for Theorem 1.2 to answer the above question.

For proving that $F_{17a}(x)$ is Apwenian, our *C* program has taken about one week by using 24 CPU cores. No hope for $F_{19}(x)$.

PROBLEM 1.6. *Find a human proof of Theorem 1.2 without computer assistance.*

PROBLEM 1.7. *Characterize all the finite* ± 1*-sequences* $\mathbf{v}$ *such that* $\Phi(\tilde{\mathbf{v}}(x))$ *is Apwenian.*

As an application of Theorem 1.2 in Number Theory, the irrationality exponents of $F_5(1/b)$, $F_{11}(1/b)$, $F_{17a}(1/b)$, $F_{17b}(1/b)$ are proved to be equal to 2 (see [6]).

2. PROOF OF THEOREM 1.2

Let d be a positive integer and $\mathbf{v} = (v_0, v_1, v_2, \ldots, v_{d-1})$ be a finite ± 1-sequence of length d with $v_0 = 1$. Let $\mathbf{f} = (f_k)_{k\geq 0}$ be the ± 1-sequence defined by the following generating function

$$\tilde{\mathbf{f}}(x) = \Phi(\tilde{\mathbf{v}}(x)) = \prod_{k=0}^{\infty} \tilde{\mathbf{v}}(x^{d^k}), \tag{7}$$

where $\tilde{\mathbf{v}}(x) = \sum_{i=0}^{d-1} v_i x^i$. The above power series satisfies the following functional equation

$$\tilde{\mathbf{f}}(x) = \tilde{\mathbf{v}}(x) \prod_{k=1}^{\infty} \tilde{\mathbf{v}}\left(x^{d^k}\right) = \tilde{\mathbf{v}}(x)\tilde{\mathbf{f}}(x^d). \tag{8}$$

The sequence $\mathbf{f}$ can also be defined by the recurrence relations

$$f_0 = 1, \quad f_{dn+i} = v_i f_n \quad \text{for } n \geq 0 \text{ and } 0 \leq i \leq d-1. \tag{9}$$

We divide the set $\{1, 2, \ldots, d-1\}$ into two disjoint subsets

$$P = \{1 \leq i \leq d-1 \mid v_{i-1} \neq v_i\},$$
$$Q = \{1 \leq i \leq d-1 \mid v_{i-1} = v_i\}.$$

Two disjoint infinite sets of integers J and K play an important role in the proof of Theorem 1.2.

DEFINITION 2.1. *If* $v_{d-1} = -1$*, define*

$$\begin{aligned} J &= \{(dn+p)d^{2k} - 1 \mid n, k \in \mathbb{N}, p \in P\} \\ &\quad \bigcup \{(dn+q)d^{2k+1} - 1 \mid n, k \in \mathbb{N}, q \in Q\}, \\ K &= \{(dn+q)d^{2k} - 1 \mid n, k \in \mathbb{N}, q \in Q\} \\ &\quad \bigcup \{(dn+p)d^{2k+1} - 1 \mid n, k \in \mathbb{N}, p \in P\}. \end{aligned}$$

If $v_{d-1} = 1$*, define*

$$\begin{aligned} J &= \{(dn+p)d^k - 1 \mid n, k \in \mathbb{N}, p \in P\}, \\ K &= \{(dn+q)d^k - 1 \mid n, k \in \mathbb{N}, q \in Q\}. \end{aligned}$$

From the above definition it is easy to see that $\mathbb{N} = J \cup K$ and following lemma.

LEMMA 2.1. *For each* $t \geq 0$ *the integer* $\delta_t := |(f_t - f_{t+1})/2|$ *is equal to 1 if and only if* t *is in* J.

Let $\mathfrak{S}_m = \mathfrak{S}_{\{0,1,\ldots,m-1\}}$ be the set of all permutations on $\{0, 1, \ldots, m-1\}$. The following theorem which was proved in [5] may be viewed as the combinatorial interpretation of Theorem 1.2.

THEOREM 2.2 ([5]). *Let* $\mathbf{v}$ *be a* ± 1*-sequence of length* d *with* $v_0 = 1$*. The sequence* $\mathbf{f}$ *and the set* J *associated with* $\mathbf{v}$ *are defined by* (7) *and Definition 2.1 respectively. Then, the sequence* $\mathbf{f}$ *is Apwenian if, and only if, the number of permutations* $\sigma \in \mathfrak{S}_m$ *such that* $i + \sigma(i) \in J$ *for* $i = 0, 1, \ldots, m-2$ *(no constraint on* $m - 1 + \sigma(m-1) \in \mathbb{N}$*) is an odd integer for every integer* $m \geq 1$.

For proving that the sequence $\mathbf{f}$ is Apwenian by means of Theorem 2.2, it is convenient to introduce the following notations.

DEFINITION 2.2. *For* $m \geq \ell \geq 0$ *let* $\mathfrak{J}_{m,\ell}$ *(resp.* $\mathfrak{K}_{m,\ell}$*) be the set of all permutations* $\sigma = \sigma_0\sigma_1\cdots\sigma_{m-1} \in \mathfrak{S}_m$ *such that* $i+\sigma_i \in J$ *(resp.* $i+\sigma_i \in K$*) for* $i \in \{0, 1, \ldots, m-1\} \setminus \{\ell\}$*. Let* $n \geq 1$*; for simplicity, write:*

$$\begin{aligned}
&j_{m,\ell} := \#\mathfrak{J}_{m,\ell}, && k_{m,\ell} := \#\mathfrak{K}_{m,\ell}, && \\
&X_n := \sum_{i=0}^{n-1} j_{n,i}, && Y_n := j_{n,n}, && Z_n := j_{n,n-1}, \\
&U_n := \sum_{i=0}^{n-1} k_{n,i}, && V_n := k_{n,n}, && W_n := k_{n,n-1}, \\
&T_n := X_n + X_nY_n + Y_n, && && \\
&R_n := U_n + U_nV_n + V_n. && &&
\end{aligned}$$

Notice that if $\ell = m$, then $\{0, 1, \ldots, m-1\} \setminus \{\ell\} = \{0, 1, \ldots, m-1\}$, so that $j_{m,m}$ (resp. $k_{m,m}$) is the number of permutations $\sigma \in \mathfrak{S}_m$ such that $i + \sigma(i) \in J$ (resp. $\in K$) for all i.

By Theorem 2.2 and Definition 2.2 the sequence $\mathbf{f}$ is Apwenian if and only if $Z_n \equiv 1 \pmod 2$. In Section 4 we describe an algorithm enabling us to *find* and also *prove* a list of recurrence relations between $X_n, Y_n, Z_n, U_n, V_n, W_n$. Then, it is routine to check whether $Z_n \equiv 1 \pmod 2$ or not. Our program `Apwen.py` is an implementation of the latter algorithm in Python.

We now produce the proof of Theorem 1.2 by means of the program `Apwen.py`. Since $F_2(x)$ has been proved to be Apwenian in [1], only the three power series $F_3(x), F_5(x)$ and $F_{11}(x)$ require our attention. We can also prove that $F_{13}(x), F_{17a}(x), F_{17b}(x)$ are Apwenian in the same manner. However, the full proofs are lengthy and are not reproduced in the paper.

Consider $F_3(x)$. Take $\mathbf{v} = (1, -1, -1)$ with $d = 3$ and $v_{d-1} = -1$. Then, the corresponding infinite ± 1-sequence $\mathbf{f}$ is equal to $F_3(x)$. We have $P = \{1\}$, $Q = \{2\}$ and

$$\begin{aligned} J &= \{(3n+1)3^{2k} - 1 \mid n, k \in \mathbb{N}\} \\ &\quad \cup \{(3n+2)3^{2k+1} - 1 \mid n, k \in \mathbb{N}\} \\ &= \{0, 3, 5, 6, 8, 9, 12, 14, 15, 18, \ldots\}, \\ K &= \{(3n+2)3^{2k} - 1 \mid n, k \in \mathbb{N}\} \\ &\quad \cup \{(3n+1)3^{2k+1} - 1 \mid n, k \in \mathbb{N}\} \\ &= \{1, 2, 4, 7, 10, 11, 13, 16, 17, \ldots\} = \mathbb{N} \setminus J. \end{aligned}$$

By enumerating a list of 24 *types* of permutations (see Section 3), the program `Apwen.py` finds and proves the following recurrences.

LEMMA 2.3. *For each* $n \geq 1$ *we have*

$$\begin{aligned}
X_{3n+0} &\equiv U_n, & Y_{3n+0} &\equiv U_n + V_n, \\
X_{3n+1} &\equiv W_{n+1}(U_n + V_n), & Y_{3n+1} &\equiv W_{n+1}V_n, \\
X_{3n+2} &\equiv W_{n+1}(U_{n+1} + V_{n+1}), & Y_{3n+2} &\equiv W_{n+1}V_{n+1},
\end{aligned}$$

$$\begin{aligned}
Z_{3n+0} &\equiv W_n(U_n + U_nV_n + V_n), \\
Z_{3n+1} &\equiv W_{n+1}(U_n + U_nV_n + V_n), \\
Z_{3n+2} &\equiv W_{n+1}.
\end{aligned}$$

As explained in Section 3, the above relations express X, Y, Z in terms of U, V, W since $v_{d-1} = -1$. By exchanging the values of P and Q, J and K, the program `Apwen.py` yields other relations which express U, V, W in terms of X, Y, Z by enumerating a list of 26 types of permutations.

LEMMA 2.4. *For each $n \geq 1$ we have*

$$U_{3n+0} \equiv X_n, \qquad V_{3n+0} \equiv X_n + Y_n,$$
$$U_{3n+1} \equiv Z_{n+1}Y_n, \qquad V_{3n+1} \equiv Z_{n+1}X_n,$$
$$U_{3n+2} \equiv Z_{n+1}Y_{n+1}, \qquad V_{3n+2} \equiv Z_{n+1}X_{n+1},$$

$$W_{3n+0} \equiv Z_n(X_n + X_nY_n + Y_n),$$
$$W_{3n+1} \equiv Z_{n+1}(X_n + X_nY_n + Y_n),$$
$$W_{3n+2} \equiv Z_{n+1}.$$

From Lemmas 2.3 and 2.4 we obtain the following "simplified" recurrence relations based on some elementary calculations.

COROLLARY 2.5. *For each positive integer n we have*

$$Z_{3n+0} \equiv W_nR_n, \qquad W_{3n+0} \equiv Z_nT_n,$$
$$Z_{3n+1} \equiv W_{n+1}R_n, \qquad W_{3n+1} \equiv Z_{n+1}T_n,$$
$$Z_{3n+2} \equiv W_{n+1}, \qquad W_{3n+2} \equiv Z_{n+1},$$
$$T_{3n+0} \equiv R_n, \qquad R_{3n+0} \equiv T_n,$$
$$T_{3n+1} \equiv W_{n+1}R_n, \qquad R_{3n+1} \equiv Z_{n+1}T_n,$$
$$T_{3n+2} \equiv W_{n+1}R_{n+1}, \qquad R_{3n+2} \equiv Z_{n+1}T_{n+1}.$$

Since $Z_1 = 1, T_1 = 3, W_1 = 1, R_1 = 1, Z_2 = 1, T_2 = 1, W_2 = 1$ and $R_2 = 7$, Corollary 2.5 yields $Z_m \equiv T_m \equiv W_m \equiv R_m \equiv 1 \pmod 2$ for every positive integer m by induction. Hence, $F_3(x)$ is Apwenian.

Duo to space constraints, we defer the proofs for $F_5(x)$ and $F_{11}(x)$ to the full version.

3. ALGORITHM FOR FINDING THE RECURRENCES

Keep the same *notations* as in Section 2. We will show how to *find* and also *prove* a list of recurrence relations between the quantities $X_n, Y_n, Z_n, U_n, V_n, W_n$. The set $\mathbb{N}$ of nonnegative integers is partitioned into d disjoint subsets $A_0, A_1, \ldots, A_{d-1}$ according to the value modulo d:

$$A_i = \{dn + i \mid n \in \mathbb{N}\} \quad (i = 0, 1, \ldots, d-1). \tag{10}$$

For an infinite set S let $S|_m$ be the set composed of the m smallest integers in S. Let $\beta : \mathbb{N} \to \mathbb{N}$ denote the transformation $k \mapsto \lfloor \frac{k}{d} \rfloor$. In other words,

$$\beta(k) = (k - i)/d \qquad \text{if } k \in A_i. \tag{11}$$

For simplicity, write

$$\bar{J} = \begin{cases} J & \text{if } v_{d-1} = 1, \\ K & \text{if } v_{d-1} = -1, \end{cases} \quad \text{and} \quad \bar{K} = \begin{cases} K & \text{if } v_{d-1} = 1, \\ J & \text{if } v_{d-1} = -1. \end{cases}$$

Then $\bar{\mathfrak{J}}_{m,\ell}, \bar{X}_n, \bar{Y}_n, \bar{Z}_n$ mean $\mathfrak{J}_{m,\ell}, X_n, Y_n, Z_n$ (resp. $\mathfrak{K}_{m,\ell}$, U_n,V_n,W_n) if $v_{d-1} = 1$ (resp. $v_{d-1} = -1$). It is not hard to verify the following lemma.

LEMMA 3.1. *For each $p \in P$ and $q \in Q$ we have*
(i). $A_{p-1} \subset J$ and $A_{q-1} \subset K$;
(ii). $A_{q-1} \cap J = \emptyset$ and $A_{p-1} \cap K = \emptyset$;
(iii). $\beta(A_{d-1} \cap J) = \bar{J}$ and $\beta(A_{d-1} \cap K) = \bar{K}$.

Let $i, j \in [0, d-1]$ and $x \in A_i, y \in A_j$. For determining the condition of i and j such that the sum $x + y$ belongs to J or K, there are three cases to be considered.
(S1) If $i + j + 1 \pmod d \in P$, then, $x + y \in J$;
(S2) If $i + j + 1 \pmod d \in Q$, then, $x + y \in K$;
(S3) If $i + j + 1 \pmod d = 0$, then, $x + y \in A_{d-1}$. In this case, the sum $x + y$ may belong to J or K.

Let $m \geq \ell \geq 0$. We want to enumerate the permutations in $\mathfrak{J}_{m,\ell}$ modulo 2. Each permutation $\sigma = \sigma_0\sigma_1 \cdots \sigma_{m-1} \in \mathfrak{S}_m$ may be written as the *two-line representation*

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & m-1 \\ \sigma_0 & \sigma_1 & \sigma_2 & \cdots & \sigma_{m-1} \end{pmatrix}.$$

The columns $\binom{i}{\sigma_i}$ are called *bi-letters.* Let $\{a_0, a_1, \ldots, a_{d-1}\}$ be an alphabet of d letters. For each $\sigma \in \mathfrak{J}_{m,\ell}$ a bi-letter $\binom{i}{\sigma_i}$ in σ is said to be of (*normal*) *form* $\binom{a_j}{a_k}$ (resp. *specific form* $\binom{\ell}{a_k}$) if $i \neq \ell$ and $(i, \sigma_i) \in A_j \times A_k$ (resp. $i = \ell$ and $\sigma_i \in A_k$). To count the permutations from $\mathfrak{J}_{m,\ell}$ modulo 2, we proceed in several steps. In most cases the calculations are illustrated with $d = 5$.

Step 1. Occurrences of bi-letters. Since we want to enumerate permutations modulo 2, we can delete suitable pairs of the permutations and the result will not be changed. Let $i \in \mathbb{N}|_d$, if a permutation $\sigma \in \mathfrak{J}_{m,\ell}$ contains more than two bi-letters of form $\binom{a_i}{a_j}$ such that $i+j+1 \pmod d \in P$, select the first two such bi-letters $\binom{i_1}{j_1}$ and $\binom{i_2}{j_2}$. We define another permutation τ obtained from σ by exchanging j_1 and j_2 in the bottom line. This procedure is reversible. By (S1), it is easy to verify that τ is also in $\mathfrak{J}_{m,\ell}$, so that we can delete the pair σ and τ. Then, there only remain the permutations containing 0 or 1 bi-letter of form $\binom{a_i}{a_j}$ such that $i + j + 1 \pmod d \in P$.

Let $\mathfrak{J}'_{m,\ell}$ be the set of permutations $\sigma \in \mathfrak{J}_{m,\ell}$ which, for each $i \in \mathbb{N}|_d$, contains 0 or 1 bi-letter of form $\binom{a_i}{a_j}$ such that $i + j + 1 \pmod d \in P$. We have $j_{m,\ell} = \#\mathfrak{J}_{m,\ell} \equiv \#\mathfrak{J}'_{m,\ell} \pmod 2$. By (S2), each permutation $\sigma \in \mathfrak{J}'_{m,\ell}$ does not contain any bi-letter of form $\binom{a_i}{a_j}$ such that $i + j + 1 \pmod d \in Q$. Thus, most of the bi-letters are of form $\binom{a_i}{a_{d-i-1}}$. In conclusion, the number of occurrences of each form is summarized in Table 1. A bi-letter of form $\binom{a_i}{a_j}$ such that $i + j + 1 \pmod d \in P$ is said to be *unsociable.* A bi-letter of form $\binom{a_j}{a_{d-j-1}}$ is said to be *friendly.* By Table 1, each permutation in $\mathfrak{J}'_{m,\ell}$ contains only a few unsociable bi-letters.

form	total times
$\{\binom{a_k}{a_j} \mid k + j + 1 \pmod d \in Q\}$	0
$\{\binom{a_i}{a_j} \mid i + j + 1 \pmod d \in P\} \quad \forall i \in \mathbb{N}\vert_d$	$0, 1$
$\{\binom{a_j}{a_{d-j-1}} \mid j \in \mathbb{N}\vert_d\}$	$0, 1, 2, 3, \ldots$
$\{\binom{\ell}{a_j} \mid j \in \mathbb{N}\vert_d\} \quad (\ell = m)$	0
$\{\binom{\ell}{a_j} \mid j \in \mathbb{N}\vert_d\} \quad (0 \leq \ell \leq m-1)$	1

Table 1: Number of occurrences of bi-letters

Step 2. Form and type. The two-line representation of a permutation can be seen as a word of bi-letters. In fact, the order of the bi-letters does not matter. Let $m \geq 2d$. The *form* $f(\sigma)$ of a permutation $\sigma \in \mathfrak{J}'_{m,\ell}$ is obtained from σ by replacing each bi-letter of σ by its (normal or specific) form. From Table 1, the form $f(\sigma)$ of a permutation $\sigma \in \mathfrak{J}'_{m,\ell}$ is

$$\left(\begin{array}{cccc|cccc|c|cccc} a_0 & \cdot & a_0 & a_0 & a_1 & \cdot & a_1 & a_1 & \cdots & a_{d-1} & \cdot & a_{d-1} & a_{d-1} \\ a_{d-1} & \cdot & a_{d-1} & s_0 & a_{d-2} & \cdot & a_{d-2} & s_1 & \cdots & a_0 & \cdot & a_0 & s_{d-1} \end{array}\right),$$

for $\ell = m$, or

$$\left(\begin{array}{cccc|cc|c|cccc|c} a_0 & \cdot & a_0 & a_0 & a_1 & \cdots & \cdots & a_{d-1} & \cdot & a_{d-1} & a_{d-1} & \ell \\ a_{d-1} & \cdot & a_{d-1} & s_0 & \cdot & \cdots & \cdots & a_0 & \cdot & a_0 & s_{d-1} & s_d \end{array}\right),$$

for $0 \le \ell \le m-1$, where

$$\begin{cases} s_i \in \{a_j \mid i+j+1 \pmod{d} \in P \cup \{0\}\}, \ (i<d) \\ s_d \in \{a_0, a_1, \ldots, a_{d-1}\}. \end{cases} \tag{12}$$

Consequently, it can be characterized by a word $t(\sigma) = s_0 s_1 \ldots s_{d-1}$ or $s_0 s_1 \ldots s_{d-1} s_d$, of length d or $d+1$ respectively. The word $t(\sigma)$ is called the *type* of the permutation σ. We classify the permutations from the set $\mathfrak{J}'_{m,\ell}$ according to the type $t = s_0 s_1 \ldots s_{d-1}$ (resp. $t = s_0 s_1 \ldots s_{d-1} s_d$) by defining

$$\mathfrak{J}^t_{m,\ell} = \{\sigma \in \mathfrak{J}'_{m,\ell} \mid t(\sigma) = t\}. \tag{13}$$

Hence,

$$j_{m,\ell} \equiv \sum_t \#\mathfrak{J}^t_{m,\ell} \pmod{2}. \tag{14}$$

Some types do not have any contribution for counting the permutations modulo 2, as stated in the following two lemmas.

Lemma 3.2. *Let $\ell = m$ and $t = s_0 s_1 s_2 \ldots s_{d-1}$ (resp. $\ell \in \mathbb{N}|_m$ and $t = s_0 s_1 s_2 \ldots s_{d-1} s_d$). If there are $i < j \in [0, d-1]$ (resp. $i < j \in [0,d]$), such that $s_i = s_j$, $s_i \neq a_{d-i-1}$ and $s_j \neq a_{d-j-1}$, then*

$$\#\mathfrak{J}^t_{m,\ell} \equiv 0 \pmod{2}. \tag{15}$$

Proof. If $\mathfrak{J}^t_{m,\ell} = \emptyset$, then (15) holds. Otherwise, each permutation $\sigma \in \mathfrak{J}^t_{m,\ell}$ has two bi-letters $\binom{i_1}{i_2}$ and $\binom{j_1}{j_2}$ of forms $\binom{a_i}{a_k}$ and $\binom{a_j}{a_k}$, respectively, where $a_k = s_i = s_j$. We define another permutation τ obtains from σ by exchanging i_2 and j_2 in the bottom line. This procedure is reversible. By Lemma 3.1(i) or Table 1, it is easy to verify that τ is also in $\mathfrak{J}^t_{m,\ell}$. Thus, the transformation $\sigma \leftrightarrow \tau$ is an *involution* on $\mathfrak{J}^t_{m,\ell}$. Hence, $\#\mathfrak{J}^t_{m,\ell} \equiv 0 \pmod{2}$. $\square$

Lemma 3.3. *Let $\ell \in \mathbb{N}|_m$ and $t = s_0 s_1 s_2 \ldots s_d$. If there is $i \in \mathbb{N}|_m$ such that $s_i \neq a_{d-i-1}$, then*

$$\sum_{\ell \in \mathbb{N}|_m \cap A_i} \#\mathfrak{J}^t_{m,\ell} \equiv 0 \pmod{2}. \tag{16}$$

Proof. For any $\ell \in \mathbb{N}|_m \cap A_i$, each permutation $\sigma \in \mathfrak{J}^t_{m,\ell}$ contains two bi-letters $\binom{i_1}{i_2}$ and $\binom{\ell}{\sigma_\ell}$ of forms $\binom{a_i}{a_j}$ and $\binom{\ell}{a_k}$, respectively. We define another permutation τ by exchanging i_2 and σ_ℓ in the bottom line. This procedure is reversible. By Lemma 3.1(i) it is easy to verify that $\tau \in \mathfrak{J}^t_{m,\ell'}$, where $\ell' = i_1 \in \mathbb{N}|_m \cap A_i$. Thus, the transformation $\sigma \leftrightarrow \tau$ is an *involution* on $\sum_{\ell \in \mathbb{N}|_m \cap A_i} \mathfrak{J}^t_{m,\ell}$. Hence, (16) holds. $\square$

Let $m = dn + h$ ($n \ge 2$, $h \in \mathbb{N}|_d$), $k \in \mathbb{N}|_d$ and

$$\begin{cases} \mathfrak{P}_Y(m;t) & := \mathfrak{J}^t_{m,m}, \\ \mathfrak{P}_Z(m;t) & := \mathfrak{J}^t_{m,m-1}, \\ \mathfrak{P}_X(m;t,k) & := \sum_{\ell \in \mathbb{N}|_m \cap A_k} \mathfrak{J}^t_{m,\ell}. \end{cases} \tag{17}$$

From Definition 2.2, we have

$$\begin{cases} Y_{dn+h} & := \sum_t \#\mathfrak{P}_Y(dn+h;t) \\ Z_{dn+h} & := \sum_t \#\mathfrak{P}_Z(dn+h;t) \\ X_{dn+h} & :\equiv \sum_t \sum_{k \in \mathbb{N}|_d} \#\mathfrak{P}_X(dn+h;t,k). \end{cases} \tag{18}$$

By using relation (18), the recurrence relations listed in Lemmas 2.3 and 2.4, can be generated by Algorithm 1. The procedure `EvalAtoms(P,t, h,k)` appearing in Algorithm 1 evaluates the cardinality of the set $\mathfrak{P} := \mathfrak{P}_Y$, $\mathfrak{P}_Z$ or $\mathfrak{P}_X$ for each type t, and will be discussed in Section 4 (see Algorithm 2). Actually, Algorithm 1 delivers a recurrence relation because the output of `EvalAtoms(P,t,h,k)` is an algebra expression in terms of $X_n, X_{n+1}, Y_n, Y_{n+1}, Z_n, Z_{n+1}$.

Algorithm 1 Finding the recurrences

```
for P in ['PX', 'PY', 'PZ']:
  for h in range(d):
    Val=0
    for k in range(d) if P=='PX' else range(1):
      for t in PossibleTypes(P,h,k):
        Val=Val+EvalAtoms(P,t,h,k)
    print P,h,k,Val
```

Step 3. Counting permutations. Throughout this step we fix $m = dn + h$ ($h \in \mathbb{N}|_d$). Counting permutations from $\mathfrak{J}^t_{m,\ell}$ is lengthy; it is made in several substeps. We illustrate the entire calculations by means of four well-selected examples, using some *compressed and intuitive notation*. Then, we explain the maining of this compressed notation mean in full detail. The examples are given for $d = 5$. We write A, B, C, D, E instead of A_0, A_1, A_2, A_3, A_4 and a, b, c, d, e instead of a_0, a_1, a_2, a_3, a_4, respectively.

Example 3.1. *Consider $m = \ell = 5n + 1$ and the type 'adbca' which satisfies condition* (12). *We have*

$$\begin{aligned} &\mathfrak{J}^{adbca}_{5n+1,5n+1} \\ &\overset{w}{=} \left(\begin{array}{cccc|ccc|ccc|ccc|ccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 2 & \tilde{7} & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 \\ e & a & e & e & d & d & d & c & b & c & c & b & b & a & a & a \end{array}\right) \\ &\overset{a}{=} \left(\begin{array}{cccc|ccc|ccc|ccc|cccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 2 & \tilde{7} & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 & 19 \\ e & a & e & e & d & d & d & c & b & c & c & b & b & a & a & a & \underline{19} \end{array}\right) \\ &\overset{e}{=} \left(\begin{array}{cccc|ccc|ccc|ccc|cccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 2 & \tilde{7} & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 & 19 \\ e & \underline{19} & e & e & d & d & d & c & \underline{c} & c & \underline{b} & b & b & a & a & a & \underline{a} \end{array}\right) \\ &\overset{d}{=} \begin{pmatrix} 0 & \tilde{5} & 10 & 15 \\ e & \underline{19} & e & e \end{pmatrix} \begin{pmatrix} 1 & 6 & 11 \\ d & d & d \end{pmatrix} \begin{pmatrix} 2 & \tilde{7} & 12 \\ c & \underline{c} & c \end{pmatrix} \begin{pmatrix} \tilde{3} & 8 & 13 \\ \underline{b} & b & b \end{pmatrix} \begin{pmatrix} 4 & 9 & 14 & 19 \\ a & a & a & \underline{a} \end{pmatrix} \\ &\overset{b}{=} Z_{n+1} \times Y_n \times X_n \times X_n \times Z_{n+1}. \end{aligned}$$

Example 3.2. *Consider $m = 5n + 2$, $\ell = 5n + 1$ and the type 'dcbbaa' which satisfies condition* (12). *We have*

$$\begin{aligned} &\mathfrak{J}^{dcbbaa}_{5n+2,5n+1} \\ &\overset{w}{=} \left(\begin{array}{cccc|cccc|ccc|ccc|ccc} 0 & \tilde{5} & 10 & 15 & \tilde{1} & 6 & 11 & 16 & 2 & \tilde{7} & 12 & 3 & 8 & 13 & 4 & 9 & 14 \\ e & d & e & e & c & d & d & \underline{a} & c & b & c & b & b & b & a & a & a \end{array}\right) \end{aligned}$$

$$\stackrel{a}{=}\left(\begin{array}{cccc|cccc|ccc|cccc|cccc} 0 & \tilde{5} & 10 & 15 & \tilde{1} & 6 & 11 & 16 & 2 & \tilde{7} & 12 & 3 & 8 & 13 & 18 & 4 & 9 & 14 & 19 \\ e & d & e & e & c & d & d & \underline{a} & c & b & c & b & b & b & \underline{18} & a & a & a & \underline{19} \end{array}\right)$$

$$\stackrel{e}{=}\left(\begin{array}{cccc|cccc|ccc|cccc|cccc} 0 & \tilde{5} & 10 & 15 & \tilde{1} & 6 & 11 & 16 & 2 & \tilde{7} & 12 & 3 & 8 & 13 & 18 & 4 & 9 & 14 & 19 \\ e & \underline{19} & e & e & \underline{d} & d & d & \underline{18} & c & \underline{c} & c & b & b & b & \underline{b} & a & a & a & \underline{a} \end{array}\right)$$

$$\stackrel{d}{=}\begin{pmatrix} 0 & \tilde{5} & 10 & 15 \\ e & \underline{19} & e & e \end{pmatrix}\begin{pmatrix} \tilde{1} & 6 & 11 & 16 \\ \underline{d} & d & d & \underline{18} \end{pmatrix}\begin{pmatrix} 2 & \tilde{7} & 12 \\ c & \underline{c} & c \end{pmatrix}\begin{pmatrix} 3 & 8 & 13 & 18 \\ b & b & b & \underline{b} \end{pmatrix}\begin{pmatrix} 4 & 9 & 14 & 19 \\ a & a & a & \underline{a} \end{pmatrix}$$

$$\stackrel{b}{=} Z_{n+1} \times X_n \times X_n \times Z_{n+1} \times Z_{n+1}.$$

EXAMPLE 3.3. *Consider $m = 5n+4$, $\ell \in C|_{n+1}$ and the type 'adcbac' which satisfies condition* (12). *We have*

$$\sum_{\ell \in C|_{n+1}} \mathfrak{J}^{adcbac}_{5n+4,\ell}$$

$$\stackrel{w}{=}\left(\begin{array}{cccc|cccc|cccc|cccc|ccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 16 & 2 & \tilde{7} & 12 & 17 & 3 & 8 & 13 & 18 & 4 & 9 & 14 \\ e & a & e & e & d & d & d & d & c & \underline{c} & c & c & b & b & b & b & a & a & a \end{array}\right)$$

$$\stackrel{a}{=}\left(\begin{array}{cccc|cccc|cccc|cccc|cccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 16 & 2 & \tilde{7} & 12 & 17 & 3 & 8 & 13 & 18 & 4 & 9 & 14 & 19 \\ e & a & e & e & d & d & d & d & c & \underline{c} & c & c & b & b & b & b & a & a & a & \underline{19} \end{array}\right)$$

$$\stackrel{e}{=}\left(\begin{array}{cccc|cccc|cccc|cccc|cccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 16 & 2 & \tilde{7} & 12 & 17 & 3 & 8 & 13 & 18 & 4 & 9 & 14 & 19 \\ e & \underline{19} & e & e & d & d & d & d & c & \underline{c} & c & c & b & b & b & b & a & a & a & \underline{a} \end{array}\right)$$

$$\stackrel{d}{=}\begin{pmatrix} 0 & \tilde{5} & 10 & 15 \\ e & \underline{19} & e & e \end{pmatrix}\begin{pmatrix} 1 & 6 & 11 & 16 \\ d & d & d & d \end{pmatrix}\begin{pmatrix} 2 & \tilde{7} & 12 & 17 \\ c & \underline{c} & c & c \end{pmatrix}\begin{pmatrix} 3 & 8 & 13 & 18 \\ b & b & b & b \end{pmatrix}\begin{pmatrix} 4 & 9 & 14 & 19 \\ a & a & a & \underline{a} \end{pmatrix}$$

$$\stackrel{b}{=} Z_{n+1} \times Y_{n+1} \times X_{n+1} \times Y_{n+1} \times Z_{n+1}.$$

EXAMPLE 3.4. *Consider $m = 5n+1$, $\ell = 5n$ and the type 'edcaab' which satisfies condition* (12). *We have*

$$\mathfrak{J}^{edcaab}_{5n+1,5n}$$

$$\stackrel{w}{=}\left(\begin{array}{cccc|ccc|ccc|ccc|ccc} 0 & 5 & 10 & 15 & 1 & 6 & 11 & 2 & 7 & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 \\ e & e & e & \underline{b} & d & d & d & c & c & c & a & b & b & a & a & a \end{array}\right)$$

$$\stackrel{a}{=}\left(\begin{array}{cccc|ccc|ccc|ccc|cccc} 0 & 5 & 10 & 15 & 1 & 6 & 11 & 2 & 7 & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 & 19 \\ e & e & e & \underline{b} & d & d & d & c & c & c & a & b & b & a & a & a & \underline{19} \end{array}\right)$$

$$\stackrel{e}{=}\left(\begin{array}{cccc|ccc|ccc|ccc|cccc} 0 & 5 & 10 & 15 & 1 & 6 & 11 & 2 & 7 & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 & 19 \\ e & e & e & \underline{19} & d & d & d & c & c & c & \underline{b} & b & b & a & a & a & \underline{a} \end{array}\right)$$

$$\stackrel{d}{=}\begin{pmatrix} 0 & 5 & 10 & 15 \\ e & e & e & \underline{19} \end{pmatrix}\begin{pmatrix} 1 & 6 & 11 \\ d & d & d \end{pmatrix}\begin{pmatrix} 2 & 7 & 12 \\ c & c & c \end{pmatrix}\begin{pmatrix} \tilde{3} & 8 & 13 \\ \underline{b} & b & b \end{pmatrix}\begin{pmatrix} 4 & 9 & 14 & 19 \\ a & a & a & \underline{a} \end{pmatrix}$$

$$\stackrel{b}{=} Y_n \times Y_n \times Y_n \times X_n \times Z_{n+1}.$$

Notation 1. In the above compressed writing, the letter w, a, e, d, b over the symbol " $=$ " means that the equality is obtained by substep 3(w), 3(a), 3(e), 3(d), 3(b) respectively.

Notation 2. In the compressed writing the integer n is represented by the explicit value 3. Hence, the second block in the first equality in Example 3.1 has the following meaning:

$$\left|\begin{array}{ccc} 1 & 6 & 11 \\ d & d & d \end{array}\right| := \left|\begin{array}{cccccc} 1 & 6 & 11 & 16 & \cdots & 5n-4 \\ d & d & d & d & \cdots & d \end{array}\right|.$$

Also, the added bi-letter $\binom{19}{19}$ (see Substep 3(a)) in the second equality in Example 3.1 means $\binom{5n+4}{\underline{5n+4}}$.

Substep 3(w). Rewrite the set. For each permutation σ from $\mathfrak{J}^t_{m,\ell}$, we reorder the bi-letters of σ such that $\binom{i}{\sigma_i}$ is on the left of $\binom{j}{\sigma_j}$ if $i \mod d < j \mod d$, or if $i \equiv j \mod d$ and $i < j$. Then, we replace each letter $y \in a_k$ in the bottom line by a_k. To facilitate readability, vertical bars are inserted between the bi-letters $\binom{i}{\sigma_i}$ and $\binom{j}{\sigma_j}$ such that $i \not\equiv j \pmod d$. We get a biword w, denoted by $\rho(\sigma) = w$, called *shape* of σ.

Applying this operation to the following permutation $\sigma \in \mathfrak{J}^{adbca}_{5n+1,5n+1}$ considered in Example 3.1

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 4 & 3 & 2 & 7 & 15 & 0 & 13 & 1 & 11 & 10 & 14 & 8 & 12 & 6 & 5 & 9 \end{pmatrix}, \tag{19}$$

we get the shape $\rho(\sigma) = w$, where

$$w = \left(\begin{array}{cccc|ccc|ccc|ccc|ccc} 0 & 5 & 10 & 15 & 1 & 6 & 11 & 2 & 7 & 12 & 3 & 8 & 13 & 4 & 9 & 14 \\ e & a & e & e & d & d & d & c & b & c & c & b & b & a & a & a \end{array}\right). \tag{20}$$

Notation 3. In the compressed writing, the above shape w also represents the set $\rho^{-1}(w)$ of all the permutations σ such that $\rho(\sigma) = w$.

Each permutation $\sigma \in \mathfrak{J}^{adbca}_{5n+1,5n+1}$ contains exactly three unsociable bi-letters of form $\binom{a}{a}$, $\binom{c}{b}$, $\binom{d}{c}$, denoted by $\binom{i_0}{j_0}$, $\binom{i_1}{j_1}$, $\binom{i_2}{j_2}$, respectively. So that, for example, in the block

$$\left|\begin{array}{ccc} 2 & 7 & 12 \\ c & b & c \end{array}\right|$$

there is exactly one letter 'b' in the bottom line. All other letters are 'c'. However, the position of the letter 'b' is not fixed. The shape of another permutation may contain the block

$$\left|\begin{array}{ccc} 2 & 7 & 12 \\ b & c & c \end{array}\right| \quad \text{or} \quad \left|\begin{array}{ccc} 2 & 7 & 12 \\ c & c & b \end{array}\right|.$$

Notation 4. The underlined bi-letters $\binom{i}{\underline{a_j}}$ in the shape of a permutation σ means that there is no constraint $i + \sigma_i \in J$ for the corresponding bi-letters $\binom{i}{\sigma_i}$ of σ. All other bi-letters of σ must satisfy the latter constraint.

In the first equality of each calculation, there is no underlined bi-letter if $m = \ell$ (Example 3.1) or exactly one underlined bi-letter if $0 \le \ell \le m-1$ (Examples 3.2 and 3.3). In the latter case, the underscore sign indicates the position of ℓ.

Notation 5. The shape w, with a tilde sign ˜ over a bi-letter $\binom{\tilde{i}}{a_j}$, represents the sum of all shapes w' which are obtained from w by moving the letter a_j, including the underscore sign if it is underlined, to other non-underlined position in the block. For example, we write (see Example 3.1)

$$\left|\begin{array}{ccc} 2 & \tilde{7} & 12 \\ c & b & c \end{array}\right| := \left|\begin{array}{ccc} 2 & 7 & 12 \\ b & c & c \end{array}\right| + \left|\begin{array}{ccc} 2 & 7 & 12 \\ c & b & c \end{array}\right| + \left|\begin{array}{ccc} 2 & 7 & 12 \\ c & c & b \end{array}\right|,$$

and (see Example 3.2)

$$\left|\begin{array}{cccc} \tilde{1} & 6 & 11 & 16 \\ c & d & d & \underline{a} \end{array}\right| := \left|\begin{array}{cccc} 1 & 6 & 11 & 16 \\ c & d & d & \underline{a} \end{array}\right| + \left|\begin{array}{cccc} 1 & 6 & 11 & 16 \\ d & c & d & \underline{a} \end{array}\right| + \left|\begin{array}{cccc} 1 & 6 & 11 & 16 \\ d & d & c & \underline{a} \end{array}\right|,$$

$$\left|\begin{array}{cccc} \tilde{1} & 6 & 11 & 16 \\ \underline{d} & d & d & \underline{18} \end{array}\right| := \left|\begin{array}{cccc} 1 & 6 & 11 & 16 \\ \underline{d} & d & d & \underline{18} \end{array}\right| + \left|\begin{array}{cccc} 1 & 6 & 11 & 16 \\ d & \underline{d} & d & \underline{18} \end{array}\right| + \left|\begin{array}{cccc} 1 & 6 & 11 & 16 \\ d & d & \underline{d} & \underline{18} \end{array}\right|.$$

Notice that there is at most one tilde in each block by Lemma 3.3

Substep 3(a). Add bi-letters. For each $\sigma \in \mathfrak{J}^t_{dn+h,\ell}$, we add all bi-letters $\binom{i}{i}$ such that $\max\{dn+h, dn+d-h\} \le i \le dn+d-1$. Thus, the number of occurrences of a_j in the bottom row becomes the same as the number of occurrences of a_{d-j-1} for any $j \in \mathbb{N}|_d$.

For instance, the bottom row of the right-hand side of $\stackrel{w}{=}$ in Example 3.1 contains $4 \times a, 3 \times b, 3 \times c, 3 \times d, 3 \times e$. By adding the bi-letter $\binom{19}{19}$ to the shape the number of occurrences of a in the bottom row becomes the same as the number of occurrences of e (since 19 is also an 'e'). The added bi-letter in the shape is still represented by $\binom{19}{19}$, instead of $\binom{19}{e}$. Notice that it is underlined (see Notation 4).

Substep 3(e). Exchange. Consider all the bi-letters of the permutation σ, which are unsocial, or which were added in Substep 3(a), or still which have the specific form $\binom{\ell}{\underline{a_k}}$ with $0 \le \ell \le m-1$. Exchange the bottom letters of those bi-letters in such a way that all the bi-letters will become friendly. In most of the cases, each block contains zero or one bad bi-letter. The only exception is the block containing the specific form $\binom{\ell}{\underline{a_k}}$ with $\ell = m-1$, and another unsocial bi-letter $\binom{i}{a_j}$. In such a case we put the appropriate *explicit letter*, which was added in Substep 3(a), under the letter ℓ when the exchange was made. The whole procedure is reversible.

In Examples 3.1 and 3.2, the exchanges of the bad bi-letters are realized respectively as follows:

$$\begin{pmatrix} \cdots & \tilde{5} & \tilde{7} & \tilde{3} & 19 & \cdots \\ & a & b & c & \underline{19} & \end{pmatrix} \qquad \mapsto \qquad \begin{pmatrix} \cdots & \tilde{5} & \tilde{7} & \tilde{3} & 19 & \cdots \\ & \underline{19} & \underline{c} & \underline{b} & \underline{a} & \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \tilde{5} & \tilde{1} & 16 & \tilde{7} & 18 & 19 & \cdots \\ & d & c & \underline{a} & b & \underline{18} & \underline{19} & \end{pmatrix} \qquad \mapsto \qquad \begin{pmatrix} \cdots & \tilde{5} & \tilde{1} & 16 & \tilde{7} & 18 & 19 & \cdots \\ & \underline{19} & \underline{d} & \underline{18} & \underline{c} & \underline{b} & \underline{a} & \end{pmatrix}$$

In the second example, the block $\left| \begin{smallmatrix} \tilde{1} & 16 \\ c & \underline{a} \end{smallmatrix} \right|$ contains two bad bi-letters. We put the *explicit letter* 18 instead of the symbol 'd' under the letter $\ell = 16$.

Substep 3(d). Decomposition. After Substep 3(e) Exchange, the set $\mathfrak{I}^t_{dn+h,\ell}$ is decomposed, in a natural way, into the Cartesian product of d sets of biwords, which are called *atoms* in the sequel.

Substep 3(b). Beta transformation. The cardinalities of the atoms can be derived by means of the transformation β defined in (11). Applying the transformation β to each letter in the top and bottom rows of each element, we can get a new permutation. We defer more details to the full version.

4. ALGORITHM FOR EVALUATING THE ATOMS

Keep the same notations as in Section 3, in particular, $m = dn + h$ ($h \in \mathbb{N}|_d$). For each numerical type t, numerical integers $h, k < d$, and *symbolic* integer n, the cardinality of the set $\mathfrak{P} := \mathfrak{P}_Y(m;t), \mathfrak{P}_Z(m;t)$ or $\mathfrak{P}_X(m;t,k)$ is evaluated by the substeps 3(w), 3(a), 3(e), 3(d), 3(b), which are fully described in Section 3. As a consequence, the latter cardinality is equal to the products of d factors (see Examples 3.1–3.4) corresponding to the d atoms respectively. The evaluation yields an algebra expression in terms of $X_n, X_{n+1}, Y_n, Y_{n+1}, Z_n, Z_{n+1}$. In this section, we show that the substeps in Step 3 can be combined into one super-step. In fact, each factor can be evaluated directly by using a prefabricated dictionary.

DEFINITION 4.1. *Let $i \in \mathbb{N}|_d$ be a fixed integer. We define several parameters depending on i, k, d, m, t, where $t = s_0 s_1 \ldots s_{d-1}$ (if $\mathfrak{P} = \mathfrak{P}_Y$) or $s_0 s_1 \ldots s_{d-1} s_d$ (if $\mathfrak{P} = \mathfrak{P}_X$ or $\mathfrak{P}_Z$):*

$$\eta_0 = \begin{cases} 1, & \textit{if } i+1 \le h, \\ 0, & \textit{otherwise}; \end{cases} \qquad \eta_1 = \begin{cases} 1, & \textit{if } d-i \le h, \\ 0, & \textit{otherwise}; \end{cases}$$

$$\eta_2 = \begin{cases} 1, & \textit{if } s_i = a_{d-i-1}, \\ 0, & \textit{otherwise}; \end{cases}$$

$$\eta_3 = \begin{cases} 1, & \textit{if } \mathfrak{P} \ne \mathfrak{P}_Y \textit{ and } s_d = a_{d-i-1}, \\ 0, & \textit{otherwise}. \end{cases}$$

$$\nu = \begin{cases} \text{'}Z\text{'}, & \textit{if } \mathfrak{P} = \mathfrak{P}_Z \textit{ and } m-1 \in A_i, \\ \text{'}X\text{'}, & \textit{if } \mathfrak{P} = \mathfrak{P}_X \textit{ and } k = i, \\ \text{'}G\text{'}, & \textit{otherwise}; \end{cases}$$

$$\mu_i = \begin{cases} \Psi_Z(\eta_0, \eta_1, \eta_2, \eta_3), & \textit{if } \nu = \text{'}Z\text{'}, \\ \Psi_X(\eta_0, \eta_1, \eta_2, \eta_3), & \textit{if } \nu = \text{'}X\text{'}, \\ \Psi_G(\eta_0, \eta_1, \eta_2), & \textit{if } \nu = \text{'}G\text{'}, \end{cases}$$

where the explicit values of the functions Ψ_Z, Ψ_X, Ψ_G are given in Table 2.

η	000	001	010	011	100	101	110	111
$\Psi_G(\eta)$	$\bar{X}_n$	$\bar{Y}_n$	0	$\bar{Z}_{n+1}$	$\bar{Z}_{n+1}$	0	$\bar{X}_{n+1}$	$\bar{Y}_{n+1}$
η	0000	0010	0100	0110	1000	1010	1100	1110
$\Psi_Z(\eta)$	0	$\bar{Z}_n$	0	0	$\bar{X}_n$	$\bar{Y}_n$	0	$\bar{Z}_{n+1}$
η	0001	0011	0101	0111	1001	1011	1101	1111
$\Psi_Z(\eta)$	0	$\bar{Z}_n$	0	0	$\bar{X}_n$	0	0	$\bar{Z}_{n+1}$
η	0000	0010	0100	0110	1000	1010	1100	1110
$\Psi_X(\eta)$	0	$\bar{X}_n$	0	0	0	$\bar{Z}_{n+1}$	0	$\bar{X}_{n+1}$
η	0001	0011	0101	0111	1001	1011	1101	1111
$\Psi_X(\eta)$	0	$\bar{X}_n$	0	0	0	0	0	$\bar{X}_{n+1}$

Table 2: Explicit values of the functions Ψ_Z, Ψ_X, Ψ_G

Notice that each permutation contains $n+\eta_0$ (resp. $n+\eta_1$) letters in A_i (resp. in A_{d-i-1}).

EXAMPLE 4.1. *Consider $\mathfrak{P} = \mathfrak{I}^{adbca}_{5n+1,5n+1}$, studied in Example 3.1. In this case, $d = 5, m = 5n+1, h = 1, \ell = m = 5n+1, t = s_0 s_1 s_2 s_3 s_4 =$ 'adbca'. For $i = 1$ we have $\eta_0 = 0, \eta_1 = 0, \eta_2 = 1$. Hence, $\mu_1 = \Psi_G(0,0,1) = \bar{Y}_n$.*

EXAMPLE 4.2. *Consider $\mathfrak{P} = \mathfrak{I}^{dcbbaa}_{5n+2,5n+1}$, studied in Example 3.2. In this case, $d = 5, m = 5n+2, h = 2, \ell = m-1 = 5n+1 \in A_1, t = s_0 s_1 s_2 s_3 s_4 =$ 'dcbbaa'. For $i = 1$ we have $\eta_0 = 1, \eta_1 = 0, \eta_2 = 0, \eta_3 = 0$. So that $\mu_1 = \Psi_Z(1,0,0,0) = \bar{X}_n$.*

EXAMPLE 4.3. *Consider $\mathfrak{P} = \sum_{\ell \in C|_{n+1}} \mathfrak{I}^{adcbac}_{5n+4,\ell}$, studied in Example 3.3. In this case, $d = 5, m = 5n+4, h = 4, \ell \in A_2, t = s_0 s_1 s_2 s_3 s_4 =$ 'adcbac'. For $i = 2$ we have $\eta_0 = 1$, $\eta_1 = 1, \eta_2 = 1, \eta_3 = 1$ and $\mu_2 = \Psi_X(1,1,1,1) = \bar{X}_{n+1}$.*

THEOREM 4.1. *With the above notation, the cardinality of the set $\mathfrak{P} := \mathfrak{P}_Y, \mathfrak{P}_Z, \mathfrak{P}_X$ is equal to*

$$\#\mathfrak{P} = \mu_0 \times \mu_1 \times \mu_2 \times \cdots \times \mu_{d-1}. \tag{21}$$

For example, the set $\mathfrak{P} = \sum_{\ell \in C|_{n+1}} \mathfrak{I}^{adcbac}_{5n+4,\ell}$, studied in Example 3.3, is evaluated by means of Theorem 4.1 as follows:

$$\begin{aligned} &\sum_{\ell \in C|_{n+1}} \mathfrak{I}^{adcbac}_{5n+4,\ell} \\ &= \mu_0\, \mu_1\, \mu_2\, \mu_3\, \mu_4 \\ &= \Phi_G(1,0,0)\, \Phi_G(1,1,1)\, \Psi_X(1,1,1,1)\, \Psi_G(1,1,1)\, \Phi_G(0,1,1) \\ &= Z_{n+1}\, Y_{n+1}\, X_{n+1}\, Y_{n+1}\, Z_{n+1}. \end{aligned}$$

By Theorem 4.1, the procedure `EvalAtoms(P,t,h,k)` figured in Algorithm 1, which evaluates the cardinality of the set $\mathfrak{P} := \mathfrak{P}_Y$, $\mathfrak{P}_Z$ or $\mathfrak{P}_X$ for each type t, is described in Algorithm 2.

Algorithm 2 Evaluating the atoms

```
def EvalAtoms(P,t,h,k):
  Prod=1
  for i in Ch:
    nu='G'
    if P=='PZ' and i==(h+d-1)%d: nu='Z'
    if P=='PX' and i==k:  nu='X'
    eta=(i+1<=h, d-i<=h, t[i]==d-i-1)
    if nu=='X' or nu=='Z': eta=eta+(t[d]==d-i-1,)
    Prod=Prod*Psi(nu, eta)
  return Prod
```

Proof of Theorem 4.1. When we speak of *case*, we refer to a tuple $(\nu = \text{`}G\text{'}, \eta_0, \eta_1, \eta_2)$, $(\nu = \text{`}Z\text{'}, \eta_0, \eta_1, \eta_2, \eta_3)$ or $(\nu = \text{`}X\text{'}, \eta_0, \eta_1, \eta_2, \eta_3)$, which depends on $i, k, d, m, t, \mathfrak{P}$ by Definition 4.1. The case is reproduced without non-significant symbols. For example, we write $X1000$ for the case $(\text{`}X\text{'}, 1, 0, 0, 0)$.

In fact, the cases $G101, Z1011, X1011$ do not appear in product (21) and can take any value, in particular, zero. In the cases $X1000$ and $X1001$, we have $\#\mathfrak{P} = 0$ by Lemma 3.3, so that Identity (21) is true. In the cases $(\text{`}G\text{'}, 0, 1, 0)$ and (ν, η) for

$$\begin{aligned} \nu &= \text{`}Z\text{'}, \text{`}X\text{'}; \\ \eta &= (0,0,0,0), (0,1,0,0), (0,1,1,0), (1,1,0,0), \\ &\quad\ (0,0,0,1), (0,1,0,1), (0,1,1,1), (1,1,0,1), \end{aligned}$$

Lemma 3.2 implies that $\#\mathfrak{P} = 0$. Hence, Identity (21) is true. All other cases are proved as follows.

The evaluations of product (21) are explained in Section 3, see Examples 3.1–3.4. The factors $\mu_0, \mu_1, \ldots, \mu_{d-1}$ are obtained at the same time by proceeding with the substeps $3(w)$, $3(a)$, $3(e)$, $3(d)$, $3(b)$. In fact, we can evaluate each sole factor μ_i without keeping in mind the others. For this purpose, we extract all bi-letters such that either their top letter are in A_i or their bottom letter are a_{d-i-1} in the first two substeps $3(w)$ and $3(a)$.

Again, consider $i = 1$ and $\mathfrak{P} = \mathfrak{J}^{dcbbaa}_{5n+2,5n+1}$. We extract all bi-letters such that either their top letter are in $\{1, 6, 11, 16, \ldots\}$ or their bottom letter are d in the first two substeps $3(w)$ and $3(a)$ of Example 3.2. We have

$$\mathfrak{J}^{dcbbaa}_{5n+2,5n+1} \stackrel{w}{=} \left(\begin{array}{c|cccc|c|c|c} \tilde{5} & \tilde{1} & 6 & 11 & 16 & ? & ? & ? \\ d & c & d & d & \underline{a} & & & \end{array}\right) \stackrel{a}{=} \left(\begin{array}{c|cccc|c|c|c} \tilde{5} & \tilde{1} & 6 & 11 & 16 & ? & 18 & ? \\ d & c & d & d & \underline{a} & & \underline{18} & \end{array}\right),$$

and

$$\mathfrak{J}^{dcbbaa}_{5n+2,5n+1} \stackrel{e}{=} \left(\begin{array}{c|cccc|c|c|c} \tilde{5} & \tilde{1} & 6 & 11 & 16 & ? & 18 & ? \\ \underline{19} & \underline{d} & d & d & \underline{18} & & \underline{b} & \end{array}\right) \stackrel{d}{=} ? \begin{pmatrix} \tilde{1} & 6 & 11 & 16 \\ \underline{d} & d & d & \underline{18} \end{pmatrix} ? \stackrel{b}{=} ? X_n ?.$$

It means that $\mu_1 = X_n = \bar{X}_n$. On the other hand, the case corresponds to the tuple $(\text{`}Z\text{'}, 1, 0, 0, 0)$ that takes the value $\bar{X}_n$, as shown in Example 4.2. Then we can verify Table 2 one by one by hand (see full version for more detail). $\square$

5. IMPLEMENTATION AND OUTPUTS

Our program `Apwen.py` is an implementation of Algorithms 1 and 2 in `Python`. [2] The proof that F_{13} is Apwenian takes 11 hours by using the program `Apwen.py` on a modern personal computer. For proving that F_{17a} and F_{17b} are Apwenian, it was necessary to rewrite the program in the `C` language with some optimizations. The running times of the two programs are reproduced in the following table:

f	F_3	F_5	F_{11}	F_{13}	F_{17a}, F_{17b}	F_{19}
`Python`	$< 1s$	$< 1s$	$11m$	$11h$	∞	∞
`C`	$< 1s$	$< 1s$	$16s$	$29m$	7 days $\times$ 24 CPUs	∞

6. REFERENCES

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[2] A extended version of this paper, the program `Apwen.py` and its outputs are available at `www-irma.u-strasbg.fr/~guoniu/papers/p93apwen/`