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Ascending sequences with neighboring elements add up to perfect square numbers

Kai Jin

Institute for Interdisciplinary Information Sciences
Tsinghua University
Beijing 100084, China
e-mail: cscjjk@gmail.com

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Abstract: We consider the problem of partitioning the numbers $1..n$ to ascending sequences as few as possible, so that every neighboring pair of elements in each sequence add up to some perfect square number. We prove that the minimum number of sequences is $\left\lceil \sqrt{2(n+1)} \right\rceil - 1$. We hope that this paper exhibits an interesting property of the perfect square numbers.
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1 Introduction

For a sequence of numbers, we call it a *good sequence* if its elements are sorted in ascending order and each neighboring pair of elements add up to a perfect square number.

We can always partition the first n natural numbers to several good sequences. Take $n = 11$ for example. We can partition the numbers in $\{1, \ldots, 11\}$ to four good sequences:

$$(1, 8), (2, 7, 9), (3, 6, 10), (4, 5, 11).$$

However, it is impossible to partition the numbers in $\{1, \ldots, 11\}$ to three good sequences (see the proof below). Thus, it raises the following questions.

Question 1: Given a positive integer n, what is the smallest number k, denoted by $g(n)$, such that 1 to n can be partitioned to k good sequences?

Question 2: Given a positive integer k, what is the largest integer n, denoted by $f(k)$, such that 1 to n can be partitioned to k good sequences?

In this paper, we prove the following formulas.

$$g(n) = \left\lceil \sqrt{2(n+1)} \right\rceil - 1, \tag{1}$$

$$f(k) = \left\lfloor \frac{1}{2}(k+1)^2 \right\rfloor - 1. \tag{2}$$

From the definition of $f(k)$ and $g(n)$, we can see that $g(n)$ is equal to the smallest number k such that $f(k) \geq n$. Therefore, (2) implies (1). Thus, we only need to prove (2).

2 The proof of (2)

For convenience, denote

$$a_k = \left\lfloor \frac{1}{2}(k+1)^2 \right\rfloor, \quad b_k = \left\lceil \frac{1}{2}(k+1)^2 \right\rceil.$$

We first show that $f(k) \geq a_k - 1$.

Lemma 1. *For any positive number k, we can partition $\{1, \ldots, a_k - 1\}$ to k good sequences.*

Proof. We shall instead prove the following enhanced statement: *We can partition $\{1, \ldots, a_k - 1\}$ to k good sequences so that the largest k numbers are all in distinct sequences.*

To prove this statement, we give a simple method for constructing such partitions.

When $k = 1$, we have $a_k - 1 = 1$, and the unique partition of $\{1\}$ satisfies our requirement. Now, assume that Π_k is a partition of $\{1, \ldots, a_k - 1\}$ to k good sequences in which the largest k numbers are in distinct sequences. We construct a partition Π_{k+1} of $\{1, \ldots, a_{k+1} - 1\}$ as follows.

1) Let $\Pi_{k+1} = \Pi_k$.

2) For each sequence in Π_{k+1}, append a new element $(k+1)^2 - X$ to its end, where X denotes the original tail of this sequence.

3) Add a new sequence to Π_{k+1}, which equals

$$\begin{array}{ll} (\frac{1}{2}(k+1)^2), & \text{when } k \text{ is odd;} \\ (a_k, b_k), & \text{when } k \text{ is even.} \end{array}$$

For example,

$$\begin{aligned}
\Pi_1 &= (1) \\
\Pi_2 &= (1, 3), (2) \\
\Pi_3 &= (1, 3, 6), (2, 7), (4, 5) \\
\Pi_4 &= (1, 3, 6, 10), (2, 7, 9), (4, 5, 11), (8)
\end{aligned}$$

We need to prove the following properties of Π_{k+1}.

(i) It is a partition of $\{1, \ldots, a_{k+1} - 1\}$.

(ii) All of its $k+1$ sequences are good.

(iii) The largest $k+1$ numbers in $\{1,\ldots,a_{k+1}-1\}$ are all in distinct sequences.

First of all, since Π_k has the property that the largest k numbers are on the tails, the numbers on the tails of Π_k are

$$a_k-k,\ldots,a_k-1. \tag{3}$$

Therefore, the number that are appended to the tails in Step 2) are

$$(k+1)^2-a_k+1,\ldots,(k+1)^2-a_k+k;$$

equivalently, they are

$$b_k+1,\ldots,b_k+k. \tag{4}$$

Further, noticing that the new sequence added in Step 3) contains exactly $a_k,\ldots,b_k$, and noticing that the numbers in Π_k are $1\ldots a_k-1$, all the elements in the $k+1$ sequences of Π_{k+1} are distinct, and they are precisely

$$1,\ldots,b_k+k.$$

By the definition of b_k and a_{k+1}, we get

$$b_k+k=\left\lceil\frac{(k+1)^2}{2}+k\right\rceil=\left\lceil\frac{(k+2)^2-3}{2}\right\rceil=\left\lceil\frac{(k+2)^2-1}{2}\right\rceil-1=\left\lfloor\frac{(k+2)^2}{2}\right\rfloor-1=a_{k+1}-1.$$

Therefore, we get (i).

Due to (4), the appended numbers are larger than the largest number in Π_k. Thus, all the $k+1$ sequences of Π_{k+1} are ascending sequences. Further, according to the rule we construct Π_{k+1}, every neighboring pair of elements add up to a perfect square number. Therefore, we get (ii).

Notice that the largest $k+1$ numbers in $\{1,\ldots,a_{k+1}-1\}$ are precisely $b_k,\ldots,b_k+k$, and clearly all of them are on the tail of the $k+1$ sequences of Π_{k+1}, we get (iii). $\square$

Next, we show that $f(k)\le a_k-1$.

Lemma 2. *The numbers from 1 to a_k cannot be partitioned to k good sequences.*

Proof. Suppose to the contrary that there is a partition Π^* of $\{1,\ldots,a_k\}$ to k good sequences. Let us consider the largest $k+1$ numbers in $\{1,\ldots,a_k\}$, which are $a_k-k,\ldots,a_k$. Since these $k+1$ numbers are partitioned to k sequences, and the elements in each sequence of Π^* are sorted in ascending order, there exist two numbers among these numbers, e.g. x and y, such that they are partitioned to the same sequence under Π^* and that they are neighboring elements within the sequence. This follows that the sum $x+y$ is a perfect square number.

Since x,y are distinct numbers in $\{a_k-k,\ldots,a_k\}$, we have

$$(a_k-k)+(a_k-k+1)\le x+y\le(a_k-1)+(a_k).$$

From the left inequality, we get

$$x+y\ge 2\left\lfloor\frac{1}{2}(k+1)^2\right\rfloor-2k+1\ge 2\left(\frac{1}{2}(k+1)^2-\frac{1}{2}\right)-2k+1=k^2+1.$$

From the right inequality, we get

$$x + y \le 2\left\lfloor \frac{1}{2}(k+1)^2 \right\rfloor - 1 \le (k+1)^2 - 1.$$

Together, we have

$$k^2 < x + y < (k+1)^2,$$

and this immediately implies that $x + y$ is not a perfect square number, which is contradictory. Therefore, there is no way to partition $\{1, \ldots, a_k\}$ to k good sequences. $\square$

Combining the above two lemmas, we get $f(k) = a_k - 1$, and hence prove (2).

Note that $f(3) = 7 < 11$. There is no way to partition the numbers in $\{1, \ldots, 11\}$ to three good sequences, as mentioned at the beginning of this paper.

3 Concluding remarks

In this paper, we propose and solve a problem of partition the first n natural number, which to the best of our knowledge has not been studied before. As a result, we find some cute formulas for this problem, which exhibit an interesting property of the perfect square numbers.

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