

# Stability of Generalized Two-sided Markets with Transaction Thresholds\*

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## ABSTRACT

We study a type of generalized two-sided markets where a set of sellers, each with multiple units of the same divisible good, trade with a set of buyers. Possible trade of each unit between a buyer and a seller generates a given welfare (to be split among them), indicated by the weight of the edge between them. What makes the markets interesting is a special type of constraints, called transaction threshold constraints, which essentially mean that the amount of goods traded between a pair of agents can either be zero or above a certain edge-specific threshold. This constraints has originally been motivated from the water-right market domain by Liu et al. where minimum thresholds must be imposed to mitigate administrative and other costs. The same constraints have been witnessed in several other market domains.

Without the threshold constraints, it is known that the seminal result by Shapley and Shubick holds: the social welfare maximizing assignments between buyers and sellers are in the core. In other words, by algorithmically optimizing the market, one can obtain desirable incentive properties for free. This is no longer the case for markets with threshold constraints: the model considered in this paper.

We first demonstrate a counterexample where no optimal assignment (with respect to any way to split the trade welfare) is in the core. Motivated by this observation, we study the stability of the optimal assignments from the following two perspectives: 1) by relaxing the definition of core; 2) by restricting the graph structure. For the first line, we show that the optimal assignments are pairwise stable, and no coalition can benefit twice as large when they deviate. For the second line, we exactly characterize the graph structure

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for the nonemptiness of core: the core is nonempty if and only if the market graph is a tree. Last but not least, we compliment our previous results by quantitatively measuring the welfare loss caused by the threshold constraints: the optimal welfare without transaction thresholds is no greater than constant times of that with transaction thresholds. We evaluate and confirm our theoretical results using real data from a water-right market.

## Keywords

Assignment games; Core; Computational sustainability; Water right market

## 1. INTRODUCTION

One of the main research themes at the interface of computer science and game theory, called *algorithmic mechanism design*, is to design algorithms that solve optimization problems and satisfy incentive constraints at the same time [16]. For certain problems, incentive properties are obtained for free. For example, in order to allocate valuable resources to a set of agents that maximizes the social welfare, it turns out that one only needs to have an optimal algorithm and a truthful welfare-maximizing mechanism (aka. the VCG mechanism) is resulted, by complementing the optimal algorithm with the VCG payment [23, 5, 8].

This is also the case in some domain without money, where a designer wants to locate a facility that minimizes the sum of all agents' distances to the facility on a line [17, 22, 13, 4]. It turns out again that, the designer only needs to follow the optimal algorithm, which in this case is to trivially locate the median agent, and each agent will find it in his best interest to report the truthful location.

Similar analogue extends to the realm of cooperative game theory, where a principal wants to match a set of sellers, each of who provides one unit of indivisible item, to a set of buyers, who have unit demand for the items provided by the set of sellers. Visualized as a bipartite graph between sellers and buyers, the edges in between denote which transactions are feasible and the weight of each edge denotes the welfare improvement (to be split) from conducting the transaction between the corresponding seller and buyer. For

such a simple market model, also known as the assignment games, Shapley and Shubik proved in their seminal paper [21] that by algorithmically maximizing (via LP) social welfare among all agents, i.e., to find the maximum matching in the weighted bipartite graph, one also obtains for free (as the solution of the LP dual) a split of the welfare improvement of each edge to the ending agents so that the resulting utility vector is in the *core*, a very desirable incentive property [19, Chapter 7 and 8] that says no subset of agents can profitably deviate by trading among themselves.

Following the agenda set by Shapley and Shubik, there is a literature that aims to extend this elegant result to more general market models. Shapley and Scarf presented an algorithmic approach to find a utility vector in the core of an indivisible good exchange market [20]. Blume et. al. [2] showed that, in models with intermediate traders who set prices strategically, the game always has a subgame perfect Nash equilibrium, and that all equilibria lead to an efficient allocation of goods. It is also well known that, in models without threshold as mentioned in the abstract of the paper, a Shapley-Shubik style claim still holds: a maximum matching induces a utility vector that is in the core. When the market reveals a varying welfare profile depending on the choices of trading, modeled from labor markets [6, 9], adjusting processes are proposed which guarantee to converge into stable allocations. For multi-sided generalizations of Shapley-Shubik markets, stability results were also settled [18, 12, 10].

In this paper, we aim to study a type of generalized two-sided markets where a set of sellers, each with multiple units of the same divisible good, trade with a set of buyers. Possible trade of each unit between a buyer and a seller generates a given welfare (to be split among them), indicated by the weight of the edge between them. What makes the markets interesting is a special type of constraints, called transaction threshold constraints, which essentially mean that the amount of goods traded between a pair of agents can either be zero or above a certain edge-specific threshold. This constraint has originally been motivated from the water-right market domain by Liu et. al. where minimum thresholds must be imposed to mitigate administrative and other costs. In the water right market domain, it is considered not worth the effort to set up a trade if the transaction amount doesn't meet the threshold, since for an actual trade to take place, one must set up pipes and pumps for water to transmit. The same constraints have been witnessed in several other market domains.

Our model is a natural generalization of the Shapley-Shubik assignment game and the aforementioned model where each agent has a maximum supply or demand. Our model also generalizes the water-right market model proposed by [11] by allowing different unit profit for transaction between different pair of agents, instead of setting a selling(buying) price for each. In that paper, they consider the computational aspect of finding the optimal assignment in such a threshold model, however, it is unclear whether such an assignment induces a utility vector that is in the core.<sup>1</sup>

As the first contribution of this paper, we answer the above question negatively by showing that this is not always the case: there are market instances where the core is empty. Thus, the elegance of Shapley and Shubik Theorem

<sup>1</sup>To the best of our knowledge, there is no other related work on the threshold model besides [11].

fails to prevail to this setting. Motivated by this observation, we propose to study the following important question:

*How stable are the optimal assignments in generalized markets?*

We investigate the question from two perspectives: 1) Relaxation of the definition of core. 2) Restriction on the structure of the market graph.

In particular, our contribution can be summarized as follows:

- We borrow the notion of  $\alpha$ -core from [3] which states that no subset of agents can benefit more than  $\alpha$  times. We show that the core can be empty for some market instances, however, the 2-core of any market instance is non-empty. Moreover, we prove that any general market always has a pairwise stable optimal trading assignment.
- We show that the damaging on social welfare by imposing threshold constraints is limited. In particular, the optimal social welfare with threshold constraints is at least 1/4 of the optimal welfare without threshold constraints. This lower bound can be improved to 1/2 when each transaction has the same weight, and 1/2 is tight.
- We give a complete characterization on the graph structure of the market with respect to the non-emptiness of core. For any fixed bipartite graph, the core is non-empty for any quantity, threshold and weights vectors, if and only if each component of the graph is either an even cycle or a tree.
- By experiments on water right market instances in China, we show that the gap between optimal social welfare and the core is very close to 1, giving evidences that our generalized market model is stable in real-world circumstances.

## 2. PRELIMINARIES

### 2.1 The generalized two-sided market model

We model a generalized market by a bipartite graph  $\mathcal{G} = (N, E)$ , where  $N = U \cup V$  and  $U, V$  denote the set of sellers and buyers respectively. An agent  $i$  has a maximum amount of goods  $q_i$  that it demands/supplies. Each directed arc  $ij \in E$  indicates that the seller  $i$  can sell goods to  $j$ , and  $\alpha_{ij}$  be the social welfare created by unit transaction from agent  $i$  to  $j$ . Agents  $i$  and  $j$  will then split the unit profit  $\alpha_{ij}$ .

For each arc  $ij$ , we require that the amount of the good traded to be either 0 or no less than some threshold  $th_{ij}$ . This transaction threshold ensures that each transaction covers its cost that is not explicitly reflected in the current model. Here we assume  $th_{ij}$  is no larger than  $\min(q_i, q_j)$ , otherwise we can safely delete this edge without loss of generality.

A trading assignment of the market can be described by a *flow* from sellers to buyers. Let  $f_{ij}$  denote the trading amount between seller  $i$  and buyer  $j$ . The optimal social welfare with threshold constraints,  $\text{TS}$ , is defined as the objective value of the following linear program:

$$\begin{aligned}
& \text{maximize} && \sum_{ij \in E} \alpha_{ij} f_{ij}, \\
& \text{subject to} && \sum_{i \in U} f_{ij} \leq q_j, && j \in V \\
& && \sum_{j \in V} f_{ij} \leq q_i, && i \in U \\
& && f_{ij} = 0 \text{ or } f_{ij} \geq th_{ij}, && ij \in E \\
& && f_{ij} = 0, && ij \notin E
\end{aligned}$$

We also define the optimal social welfare when only a subset of agents  $C \subseteq U \cup V$  are involved in the trade as  $\text{TS}(C)$ . For instance,  $\text{TS}(\emptyset) = 0$ ,  $\text{TS}(\{i\}) = 0$ ,  $\text{TS}(\{i, j\}) = \min(q_i, q_j)\alpha_{ij}$  for  $ij \in E$  and  $\text{TS}(N) = \text{TS}$ . In this way, we can view a water right market  $\langle N, \text{TS} \rangle$  as a *cooperative game* where  $\text{TS}$  is the characteristic function. Similarly, we define  $\text{FS}$  as the optimal social welfare without threshold constraints, and  $\text{FS}(C)$  for  $C \subseteq N$ . Clearly,  $\text{TS} \leq \text{FS}$ .

## 2.2 The core

We use  $\mathbf{x} \in \mathbb{R}^N$  to denote the utility vector of game  $\langle N, \text{TS} \rangle$ , where  $x_i$  is the utility of agent  $i$  gained from the trading. Among all the utility vectors, those in the *Core* are of special interest, which characterize the stable outcomes of the game.

**Definition 2.1.** The *core*  $C(N, \text{TS})$  of the cooperative game  $\langle N, \text{TS} \rangle$  is the set of utility vectors,

$$\begin{aligned}
C(N, \text{TS}) = \{ \mathbf{x} \in \mathbb{R}^N \mid \\
\sum_{i \in N} x_i = \text{TS}(N), \sum_{i \in C} x_i \geq \text{TS}(C), \forall C \subseteq N \}
\end{aligned}$$

Intuitively, the core is the set of feasible utility vectors in which no subset of agents want to deviate and trade among themselves. Clearly, the core of cooperative game  $\langle N, \text{TS} \rangle$  is non-empty iff  $\text{P1}$ , the value of **LP1**, is equal to  $\text{TS}(N)$ .

$$\begin{aligned}
\text{LP1:} \quad & \text{minimize} && \sum_{i \in N} x_i \\
& \text{subject to} && \sum_{i \in C} x_i \geq \text{TS}(C), \quad \forall C \subseteq N.
\end{aligned} \tag{1}$$

The above definition of core lies in the realm of cooperative game with transferable utility (TU). For non-transferable utility (NTU) games (see, e.g. [1] for more detailed discussions), the above core definition doesn't extend well because given trading assignment  $f_{ij}$ , not all efficient utility vectors  $\mathbf{x}$  are realizable without side payments. Given trading assignment  $f_{ij}$ , we say a utility vector  $\mathbf{x}$  is *realizable without TU*, iff there exists  $g_{ij} \in \mathbb{R}^+$ , s.t.

$$\sum_{j:ij \in E} g_{ij} = x_i, \quad \forall i \in N$$

and

$$g_{ij} + g_{ji} = \alpha_{ij} f_{ij}, \quad \forall ij \in E.$$

In the lemma below, we show that in the market games, all the utility vectors in the core are realizable without TU, thus in the remainder part of this paper we only care about the existence of core solution.

**Lemma 2.2.** Given optimal trading assignment  $f_{ij}$ , if a utility vector  $\mathbf{x} \in \mathbb{R}^N$  is in the core  $C(N, \text{TS})$ , then  $\mathbf{x}$  is realizable without TU.

Define  $\text{CS}$  to be the value of the following LP, which is the dual LP to (1). By strong duality,  $\text{CS} = \text{P1}$ . Thus,  $\text{CS} = \text{P1} \geq \text{TS}(N) = \text{TS}$ .

$$\begin{aligned}
\text{LP2} \quad & \text{maximize} && \sum_{C \subseteq N} \text{TS}(C) \lambda_C \\
& \text{subject to} && \sum_{C \ni i} \lambda_C = 1, \quad \forall i \in N \\
& && \lambda_C \geq 0, \quad \forall C \subseteq N
\end{aligned}$$

We call a group of non-negative coefficients  $\{\lambda_C\}$  *balanced coefficients* iff  $\sum_{C \ni i} \lambda_C = 1$ ,  $\forall C \subseteq N$  and denote the collection  $\{C \mid \lambda_C > 0\}$  by  $\text{supp}(\lambda)$ .

According to [21], the cooperative game  $\langle N, \text{FS} \rangle$  has a non-empty core. By definition,  $\text{TS} \leq \text{FS}$ . Thus for optimal balanced coefficients  $\lambda_S$  in (2.2), we have

$$\text{CS} = \sum_{C \subseteq N} \lambda_C \text{TS}(C) \leq \sum_{C \subseteq N} \lambda_S \text{FS}(C) \leq \text{FS}(N) = \text{FS}.$$

**Lemma 2.3.** The following inequality holds:

$$\text{TS} \leq \text{CS} \leq \text{FS} \tag{2}$$

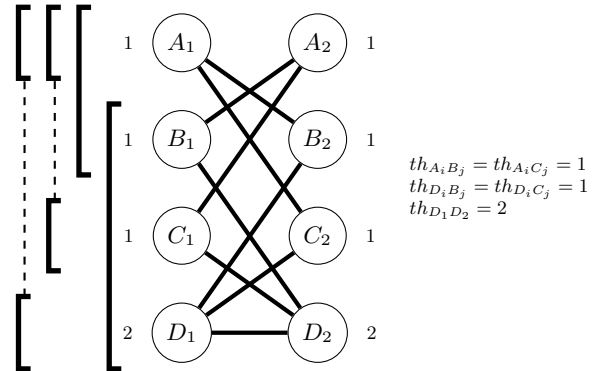
## 3. STABILITY ON GENERAL MARKETS

Now that the core can sometimes be empty, it is natural to consider solutions where the core constraint in (1) are relaxed. Given  $\alpha > 1$ , [3] proposes the constraints  $\alpha \sum_{i \in S} x_i \geq \text{TS}(S)$ ,  $\forall S \subseteq N$ . These constraints imply that a coalition  $S$  will not deviate from the current trading assignment unless it can unilaterally improve its total wealth by more than an  $\alpha$  factor. We denote the set of feasible solutions by  $\alpha$ -core. The minimum  $\alpha$  for which  $\alpha$ -core is non-empty arises when  $\alpha^* = \frac{\text{CS}}{\text{TS}}$ . We also call  $\alpha^*$ -core the least core.

**Theorem 3.1** (Shapley-Shubik 71'). *The 1-core is non-empty for any market instance without thresholds  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} = \mathbf{0} \rangle$ . That is ,*

$$\text{TS} \geq \text{CS}.$$

However, it is not the case when threshold exists. We have the following unweighted counter example, where  $\frac{\text{CS}}{\text{TS}} = \frac{7}{6}$ . This counter example shows that Theorem 1 fails to extend to the general threshold model.



**Figure 1:** An unweighted market where  $\frac{\text{CS}}{\text{TS}} = \frac{7}{6}$ . The numbers near the nodes are the quantities of supplies/demands.

Let  $f_{ij}$  be any trading assignment and  $p_i = \sum_{j \in N} f_{ij}$  be the total trading volume of agent  $i$ ,  $\forall i \in N$ . We show the maximal social welfare  $\text{TS}$  is at most 4, in both following cases.

- $f_{D_1 D_2} = 2$ . In this case,  $f_{B_2 D_1} = f_{C_2 D_1} = 0$ , thus  $p_{B_2} + p_{C_2} \leq 1$ . As a result, the total social welfare, which is exactly the total trading volume in this instance, is at most  $p_{A_2} + p_{D_2} + p_{B_2} + p_{C_2} \leq 1 + 2 + 1 = 4$ .
- $f_{D_1 D_2} = 0$ . In this case, we abandon edge  $D_1 D_2$ , and therefore  $p_{A_2} + p_{D_2} = f_{C_1 D_2} + f_{C_1 A_2} + f_{B_1 D_2} + f_{B_1 A_2} \leq q_{B_1} + q_{C_1} = 2$ . Thus  $\text{TS}(N) \leq p_{A_2} + p_{D_2} + p_{B_2} + p_{C_2} \leq 2 + 1 + 1 = 4$ .

On other hand, consider balanced coefficients  $\lambda$ ,  $\lambda_{A \cup D} = \lambda_{A \cup B} = \lambda_{A \cup C} = \frac{1}{3}$ ,  $\lambda_{B \cup C \cup D} = \frac{2}{3}$ . With simple calculation, we have  $\text{TS}(A \cup D) = 2$ ,  $\text{TS}(A \cup B) = 2$ ,  $\text{TS}(A \cup C) = 2$ ,  $\text{TS}(B \cup C \cup D) = 4$ , which indicates  $\text{CS} \geq \frac{1}{3}(2 + 2 + 2) + \frac{2}{3} \times 4 = \frac{14}{3}$ .

Therefore,  $\frac{\text{CS}}{\text{TS}} \geq \frac{7}{6}$ .

### 3.1 Threshold Gaps

It is of natural interest to inspect the gap between the optimal threshold solution  $\text{TS}$  and the optimal fractional solution  $\text{CS}$ . On one hand, we hope the threshold constraint imposed by transaction cost doesn't harm social welfare too much. On the other hand, the threshold gap  $\frac{\text{FS}}{\text{TS}}$  is an upper bound for  $\alpha^* = \frac{\text{CS}}{\text{TS}}$ .

We will first study the *unweighted* case where  $\alpha_{ij} = 1$ ,  $\forall ij \in E$ . In this case, the optimal social welfare becomes the maximum trading volume.

**Proposition 3.2.** *The threshold gap  $\frac{\text{FS}}{\text{TS}}$  for any unweighted water market instance  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} \rangle$  is at most 2, which is almost tight. Namely for any  $\varepsilon > 0$ ,*

$$2 - \varepsilon \leq \frac{\text{FS}}{\text{TS}} \leq 2$$

The instance of gap  $2 - \varepsilon$  simply contains 3 agents, seller 1 and buyer 2,3, with  $q_1 = 2 - \varepsilon$ ,  $q_2 = q_3 = \text{th}_{12} = \text{th}_{13} = 1$ , and  $\alpha_{12} = \alpha_{13} = 1$ . Then  $\text{FS} = 2 - \varepsilon$ , while  $\text{TS} = 1$ , because seller 1 can only trade with at most 1 buyer.

**Corollary 3.3.** *For the unweighted case where  $\alpha_{ij} = 1$ ,  $\forall ij \in E$ , the 2-core of the threshold trading assignment problem is non-empty.*

For general case where arbitrary  $\alpha_{ij}$  are allowed, we show that the gap is upper bounded by 4.

**Proposition 3.4.** *The threshold gap  $\frac{\text{FS}}{\text{TS}}$  for any water market instance  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} \rangle$  is at most 4.*

$$\frac{\text{FS}}{\text{TS}} \leq 4$$

### 3.2 Non-emptiness of 2-core

As a by-product of the upper bounds of threshold gap, we can derive that the 4-core is non-empty for general weighted water market without threshold constraints. In other words, there always exists an efficient, almost-stable solution, such that no coalition can gain more than 4 times by deviating and trading among themselves. Below we show for weighted case, the bound can also be improved to 2, which is the same as the unweighted case. We prove  $\alpha^* \leq 2$  by direct construction of allocation.

**Theorem 3.5.** *The 2-core is non-empty for any water market instance  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} \rangle$ . That is ,*

$$\text{TS} \geq \frac{1}{2} \text{CS}.$$

**PROOF.** Given an optimal threshold solution  $\mathbf{f}$ , let  $x_i = \frac{1}{2} \sum_{j \in V} f_{ij} \alpha_{ij}$  for  $i \in U$ , and  $x_j = \frac{1}{2} \sum_{i \in U} f_{ij} \alpha_{ij}$  for  $j \in V$ .

Notice that

$$\begin{aligned} \sum_{i \in C} x_i &= \sum_{i \in C \cap U} x_i + \sum_{i \in C \cap V} x_j \\ &= 2 \sum_{i \in C, j \in C} f_{ij} \alpha_{ij} + \sum_{i \in C, j \notin C} f_{ij} \alpha_{ij} + \sum_{i \notin C, j \in C} f_{ij} \alpha_{ij} \\ &\geq \sum_{i \in C, j \in C} f_{ij} \alpha_{ij} + \sum_{i \in C, j \notin C} f_{ij} \alpha_{ij} + \sum_{i \notin C, j \in C} f_{ij} \alpha_{ij} \\ &= \sum_{i \in C \text{ or } j \in C} f_{ij} \alpha_{ij}, \end{aligned} \quad (3)$$

for consider any coalition  $C$ , if  $2 \sum_{i \in C} x_i < \text{TS}(C)$  then

$$\sum_{i \in C \text{ or } j \in C} f_{ij} \alpha_{ij} \leq 2 \sum_{i \in C} x_i < \text{TS}(C),$$

and we can replace all the transactions  $f_{ij}$  with  $i \in C$  or  $j \in C$  by the optimal threshold solution on  $C$ , which increases the social welfare and thus contradicts to the optimality of  $\mathbf{f}$ .

Therefore,  $2 \sum_{i \in C} x_i \geq \text{TS}(C)$  for every  $C \subseteq U \cup V$ , and  $\{2x_i\}$  is a feasible solution to LP (1), whose minimum is  $\text{CS}$ . Thus  $2\text{TS} \geq \sum_i x_i \geq \text{CS}$ .

### 3.3 Pairwise Stability

Pairwise stability is another natural relaxation of the definition of core, in the sense that we only allow subsets of size 2 to deviate. We show that there always exists a pairwise stable assignment (which may not optimize social welfare). Furthermore, the optimal solution is always pairwise stable in the markets with transferable utility.

Let's first consider a more restricted market, where each agent can trade with at most one buyer/seller. Now that the trading assignment is a matching in the bipartite graph  $G = (U \cup V, E)$ , and we can ignore the threshold constraints without loss of generality. This is because in this case a pair of seller/buyer will always trade in their smaller quantity, the threshold constraints are satisfied trivially.

The optimal trading assignment is thus the maximal matching of the bipartite graph, with weight on edge  $ij$  being  $\text{TS}(\{i, j\}) = \min(q_i, q_j) \alpha_{ij}$ . The following LP relaxation gives the maximal matching.

$$\begin{aligned} \text{LP3} \quad & \text{maximize} && \sum_{i \in U, j \in V} \text{TS}(\{i, j\}) \lambda_{ij} \\ & \text{subject to} && \sum_{j: ij \in E} \lambda_{\{i, j\}} \leq 1, \quad \forall j \in V \\ & && \sum_{i: ij \in E} \lambda_{\{i, j\}} \leq 1, \quad \forall i \in U \\ & && \lambda_{\{i, j\}} \geq 0, \quad \forall \{i, j\} \in U \times V \end{aligned}$$

Let  $\lambda^*$  be an integral maximal matching, and  $\mathbf{x}^* \in \mathbb{R}^N$  be any optimal solution of the dual LP of LP 3.

$$\begin{aligned} \text{LP4} \quad & \text{minimize} && \sum_{i \in U} x_i + \sum_{j \in V} x_j \\ & \text{subject to} && x_i + x_j \geq \text{TS}(\{i, j\}), \quad \forall i \in U, j \in V \\ & && x_i \geq 0, \quad \forall i \in U \cup V \end{aligned}$$

By slackness,  $\lambda_{\{i,j\}}^* = 1$  implies  $x_i^* + x_j^* = \text{TS}(\{i,j\})$ , and  $x_i^* > 0$  implies that  $\sum_{j:i,j \in E} \lambda_{\{i,j\}}^* = 1$ , which indicates  $i$  is in the matching. Therefore we can conclude that  $\mathbf{x}^*$  is a realizable utility vector without TU, by splitting the profit gained from each pair of transaction in the maximal matching to both sides. Moreover,  $\mathbf{x}^*$  is pairwise stable by the constraints in LP 4.

Let  $f_{ij} = \lambda_{\{i,j\}}^* \min(q_i, q_j)$ , we have the following theorem.

**Theorem 3.6.** *For any market instance with non-transferable utility  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} \rangle$ , there exists a trading assignment  $f_{ij}$  and a corresponding realizable pairwise stable utility vector  $\mathbf{x}$ , namely*

$$x_i + x_j \geq \text{TS}(\{ij\}) \quad \forall ij \in E.$$

**Theorem 3.7.** *For any market instance with transferable utility  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} \rangle$ , there exists an optimal pairwise stable utility vector  $\mathbf{x}$ , namely*

$$x_i + x_j \geq \text{TS}(\{ij\}) \quad \forall ij \in E,$$

and

$$\sum_i x_i = \text{TS}(N)$$

PROOF. Since  $\lambda^*$  represents a matching,  $\text{supp}(\lambda^*)$  is a collection of disjoint 2-people coalitions. Thus,

$$\sum_{C \in \text{supp}(\lambda^*)} \text{TS}(C) \leq \text{TS} \left( \bigcup_{C \in \text{supp}(\lambda^*)} C \right) \leq \text{TS}(N)$$

Therefore, optimized social welfare  $\text{TS}(N)$  is enough to keep each pair stable.

**Definition 3.8.** A market instance is  $k$ -way stable iff there exists utility vector  $\mathbf{x}$  such that  $\sum_{i \in S} x_i \geq \text{TS}(S), \forall S \subseteq N, |S| \leq k$ , and  $\sum_{i \in N} x_i = \text{TS}(N)$ .

Specifically, we also call 2-way stable as pairwise stable.

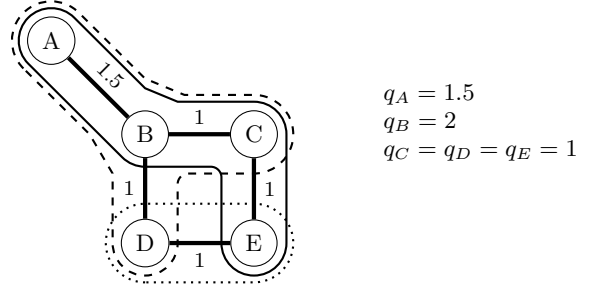
**Proposition 3.9.** *There exists market instances that are not 4-way stable.*

It is easy to verify the social welfare is at most 2.5 in Figure 2. However, consider the balanced coefficients  $\lambda$  with 3 supports:  $\lambda_{\{A,B,C,D\}} = \lambda_{\{A,B,C,E\}} = \lambda_{\{D,E\}} = 0.5$ , with  $\text{TS}(\{A,B,C,D\}) = 2.5, \text{TS}(\{A,B,C,E\}) = 2, \text{TS}(\{D,E\}) = 1$ . Clearly  $\text{TS}(N) \leq \sum_S \lambda_S \text{TS}(S) = 2.75$ .

Yet we still don't know whether transferable utility is necessary to guarantee the pairwise stability of the optimal solution. It is formulated as a linear program in the following conjecture, and we will resume further discussion on this topic in the experiment section.

**Conjecture 3.10.** *For any market instance  $\mathcal{G} = \langle G, \alpha, \mathbf{q}, \mathbf{th} \rangle$  with any optimal trading assignment  $f_{ij}$ , the value of the following LP is at least 1:*

$$\begin{aligned} \text{LP5} \quad & \text{maximize} \quad \beta \\ & \text{subject to} \quad \sum_{j'} g_{ij'} + \sum_{i'} g_{ji'} \geq \beta \text{TS}(\{ij\}), \\ & \quad \quad \quad g_{ij} + g_{ji} \leq \alpha_{ij} f_{ij}, \\ & \quad \quad \quad g_{ij} \geq 0, g_{ji} \geq 0, \forall ij \in E \end{aligned}$$



**Figure 2:** The counter example for 4-way stability. The numbers on the edges are the thresholds, and  $\alpha_{ij} = 1, \forall ij$ .

## 4. GRAPH RESTRICTED MARKETS

In the previous section we showed that for every instance of water right markets, the 2-core is nonempty. In this section, we study the impact of the underlying graph structure on the stability of the market. This graph restricted model was studied in the seminal work of Myerson [15]. We prove that when the underlying graph  $G$  is a tree or a cycle, the game has nonempty core, so that no coalition will deviate from the optimal social welfare solution. We begin with the concept of connected balanced coefficients.

**Definition 4.1.** A group of balanced coefficients  $\lambda$  is *connected*, if for every  $C \in \text{supp}(\lambda)$ , the induced subgraph of  $G$  on  $C$  (denoted as  $G_C$ ) is connected.

It is clear that there always exists a group of connected optimal balanced coefficients, since  $\text{TS}(A \cup B) = \text{TS}(A) + \text{TS}(B)$  when there is no edge between  $A$  and  $B$  in  $G$ . Therefore, for the rest of this section we only consider connected balanced coefficients.

Define the 2-norm of a group of balanced coefficients  $\|\lambda\|_2 = (\sum_C \lambda_C |C|^2)^{1/2}$ . The following lemma turns out to be extremely useful in the proofs later on.

**Lemma 4.2.** *Consider all groups of connected optimal balanced coefficients, amongst which  $\lambda$  is the one maximizing  $\|\lambda\|_2$ . Then for any  $A \in \text{supp}(\lambda)$  and  $ij \in E$  such that  $i \in A, j \notin A$ , there exists  $B \in \text{supp}(\lambda)$  such that  $i \notin B, j \in B$  and  $A \cap B \neq \emptyset$ .*

PROOF. Since  $\sum_{C \ni i} \lambda_C = \sum_{C \ni j} \lambda_C = 1$ , there must exist  $B \in \text{supp}(\lambda)$  that  $i \notin B$  and  $j \in B$ . Notice that  $A \cup B$  also induces a connected subgraph of  $G$  because of the edge  $ij$ . Suppose  $A \cap B = \emptyset$ , then we can design another group of connected balanced coefficients  $\lambda'$  with  $\delta = \min(\lambda_A, \lambda_B) > 0$ :

$$\lambda'_C = \begin{cases} \lambda_C + \delta & \text{if } C = A \cup B, \\ \lambda_C - \delta & \text{if } C = A \text{ or } C = B, \\ \lambda_C & \text{otherwise.} \end{cases}$$

Since  $\text{TS}(A \cup B) \geq \text{TS}(A) + \text{TS}(B)$ , balanced coefficients  $\lambda'$  must also be optimal. On the other hand,  $\|\lambda'\|_2^2 - \|\lambda\|_2^2 = \delta(|A \cup B|^2 - |A|^2 - |B|^2) > 0$ , which contradicts to our assumption on  $\lambda$ . So it must be the case that  $A \cap B \neq \emptyset$ .

**Proposition 4.3.** *If  $G$  is a tree (a connected acyclic graph), then  $\text{TS} = \text{CS}$ .*

PROOF. Let  $\lambda$  be the group of connected optimal balanced coefficients which maximizes  $\|\lambda\|_2$ . Suppose there exists  $A \in \text{supp}(\lambda) - \{N\}$ , then there must be an edge  $ij \in E$  such that  $i \in A, j \notin A$ . However, as  $ij$  is a cut edge in  $G$ , every connected subgraph of  $G$  containing  $j$  and not containing  $i$  must be disjoint from  $A$ , which contradicts to Lemma 4.2. So  $\text{supp}(\lambda) = \{N\}$  and  $\text{TS} = \text{CS}$ .

We note that the result of Proposition 4.3 was already known since [7], and was also implied from the more general treewidth results in [14, 3].

**Proposition 4.4.** *If  $G$  is an even cycle, then  $\text{TS} = \text{CS}$ .*

PROOF. Let  $\lambda$  be the group of connected optimal balanced coefficients which maximizes  $\|\lambda\|_2$ . Every coalition  $C$  in  $\text{supp}(\lambda)$  consists of the vertices in a consecutive chain in  $G$ . We first prove a conditional convexity: if  $G_A$  and  $G_B$  are two chains, such that every component in  $G_{A \cap B}$  consists of even number of vertices, then  $\text{TS}(A) + \text{TS}(B) \leq \text{TS}(A \cup B) + \text{TS}(A \cap B)$ . Here we only prove this convexity in the case where  $G_{A \cap B}$  has only one component, and other cases can be dealt in the exact same way.

Let the vertices in  $A \cap B$  being  $v_1, v_2, \dots, v_{2k}$  in order, and suppose  $x_0 \in A \setminus B, v_{2k+1} \in B \setminus A$  such that  $v_0 v_1, v_{2k} v_{2k+1} \in G$ . Let  $f$  and  $f'$  be the optimal threshold solution of  $\text{TS}(A)$  and  $\text{TS}(B)$  respectively. We propose  $2k$  designs for the threshold solutions on  $A \cup B$  and  $A \cap B$ : the  $i$ -th design is that

$$f_{v_j v_{j+1}}^{A \cup B} = \begin{cases} f_{v_j v_{j+1}} & 0 \leq j < i \\ f'_{v_j v_{j+1}} & i \leq j \leq 2k \end{cases}$$

$$f_{v_j v_{j+1}}^{A \cap B} = \begin{cases} f'_{v_j v_{j+1}} & 1 \leq j < i \\ f_{v_j v_{j+1}} & i \leq j < 2k \end{cases}$$

Notice that for each design, the summation of social welfare on  $f^{A \cup B}$  and  $f^{A \cap B}$  is the same as  $\text{TS}(A) + \text{TS}(B)$ . The  $i$ -th design is not feasible, only when the transaction quantity on  $v_i$  is out of limit. When  $1 < i < 2k$ , that means

$$f_{v_{i-1} v_i} + f'_{v_i v_{i+1}} > q_i \text{ or } f'_{v_{i-1} v_i} + f_{v_i v_{i+1}} > q_i,$$

and since  $q_i \geq f_{v_{i-1} v_i} + f_{v_i v_{i+1}}, q_i \geq f'_{v_{i-1} v_i} + f'_{v_i v_{i+1}}$ , we know that

$$(f'_{v_i v_{i+1}} > f_{v_i v_{i+1}} \text{ and } f_{v_{i-1} v_i} > f'_{v_{i-1} v_i}),$$

$$\text{or } (f_{v_i v_{i+1}} > f'_{v_i v_{i+1}} \text{ and } f'_{v_{i-1} v_i} > f_{v_{i-1} v_i}).$$

This indicates that the sign of  $(f - f')$  must be alternative along the edges from  $v_1$  to  $v_{2k}$ . However, for the 1st and  $2k$ -th design to fail, it is necessary that  $f_{v_1 v_2} < f'_{v_1 v_2}$  and  $f_{v_{2k-1} v_{2k}} > f'_{v_{2k-1} v_{2k}}$ , leading to a direct contradiction, which means there must be at least one feasible design.

The conditional convexity implies that no two mutually exclusive  $A$  and  $B$  in  $\text{supp}(\lambda)$  can have an intersection consisting of only even consecutive parts: otherwise since

$$|A|^2 + |B|^2 < |A \cup B|^2 + |A \cap B|^2,$$

replacing a fraction of  $A$  and  $B$  by  $A \cup B$  and  $A \cap B$  always give rise to larger  $\|\lambda\|_2$ .

Now suppose  $C_1 \in \text{supp}(\lambda)$  and  $C_1 \neq N$ . We label the vertices in the even cycle as  $1, 2, \dots, 2n$ , and assume  $C_1 = [1, m]$  with  $m < 2n$ . According to Lemma 4.2, there must be another  $C_2 = [r, 2n] \in \text{supp}(\lambda)$  with  $1 < r \leq m$ . Furthermore, according to the arguments above,  $|C_1 \cap C_2| = m - r + 1$  is odd.

By symmetry we assume that both  $m$  and  $r$  are odd. Applying Lemma 4.2 on vertices  $r$  and  $r - 1$ , there must be another  $C_3 = [1, r - 1] \cup [l, 2n] \in \text{supp}(\lambda)$ , and  $|C_2 \cap C_3| = 2n - l + 1$  is odd. However, whether  $C_1 \cap C_3 = [1, r - 1]$  (if  $m < l$ ) or  $[1, r - 1] \cup [l, m]$  (if  $m \geq l$ ), it always contradicts to the conditional convexity. That means at the very beginning,  $C_1 \neq N$  does not exist, so the balanced coefficients  $\lambda$  is simply  $\lambda_N = 1$ . Thus  $\text{TS} = \text{CS}$ .

A directed corollary of Proposition 4.3 and 4.4 is that when every connected component of  $G$  is a tree or an even cycle, the trading assignment game on  $G$  has nonempty core. On the other hand, Figure 2 already presented a market instance with empty core, whose underlying bipartite graph is a minor of every connected graph which is neither a tree nor an cycle. That leads to the following characterization of bipartite graphs ensuring nonempty core:

**Theorem 4.5.** *Given a bipartite graph  $G$ , every possible market instance  $\mathcal{G} = \langle G, \alpha, q, \mathbf{th} \rangle$  has nonempty core if and only if every connected component of  $G$  is a tree or an even cycle.*

We define the set of *thresholded edges*  $F \subset E$  consists of edges  $ij$  with  $th_{ij} > 0$ . The result of Shapley and Shubik essentially says that the market is stable when  $F$  is empty. We make a further observation that a larger class of underlying graphs could ensure stability given constraints on  $F$ . That is concluded in the following proposition:

**Proposition 4.6.** *If  $G$  is a unicyclic graph with the even cycle  $H$ , and all thresholded edges are in  $H$ , then the game is stable.*

## 5. EXPERIMENTS VIA WATER-RIGHT MARKET DATA

Our first experiment studies the gap of TS and CS in real water market in Gansu Province, China. Intuitively, CS is the least amount of profit to keep any coalition stable, which is at least TS, the optimal social welfare. In other words, no coalition can earn  $\alpha^* = \frac{\text{CS}}{\text{TS}}$  times what they earn by deviating. We show a theoretic upper bound of 2 for  $\alpha^*$  for any market, and construct a market instance with  $\alpha^* = \frac{7}{6}$ . However, the construction for market instance with empty core is not easy, which needs careful design for the graph structure, weights, quantity and thresholds. In the following experiment, we show markets with empty core do exist in the real world, but they typically have  $\alpha^*$  with close to 1, that is,  $\alpha^* = 1 + \epsilon$  for some  $\epsilon \sim 10^{-3}$ , which is small enough to be ignored. We calculate  $\alpha^*$  by first solving the MIP for TS (the same MIP in [11]) and LP2 for CS.

Based on the real trading data in the water right market in Xiying Irrigation, Gansu Province, China, we generate data to evaluate the relationship between core solutions and the optimal assignment. The original data consists two parts. The first part is the connectivity data between villages in the market. The second part is the trade records from 2008 to 2015. Each record consists of its price, volume and date. When sampling an instance with  $n$  agents. We run the following process  $n$  times as the  $n$  agents: sample a unit bid/ask (0.2-0.4 yuan) according to the historical prices; sample a volume from historical records; sample a village as its location. The connectivity between villages is based on our data. For simplification, the threshold on an

arc is 1/3 of the smaller one between seller's volume and buyer's volume.

In the second experiment, we study the validity of the conjecture proposed in the end of Section 3. We run LP 5 on the same water right market instances in Gansu Province, China and find that all the objective values  $\beta$  of those LP are at least 1, which means the optimal solution has an efficient utility vector without TU. We vary the size of the market from 8 to 50, and for each size, we solve the LP on 500 randomly generated instances. The result is presented below in Table 2.

## APPENDIX

### A. PROOF FOR LEMMA 1

**Theorem A.1** (Generalized Hall's Theorem). *Given a bipartite graph  $(A \cup B, \mathcal{E})$ , and vectors  $\mathbf{a} \in \mathbb{R}^A, \mathbf{b} \in \mathbb{R}^B$ , there exists  $u_{ij}, i, j \in E$ , such that*

$$\sum_j u_{ij} \geq a_i \quad \text{and} \quad \sum_i u_{ij} \leq b_j,$$

if and only if

$$\forall C \subseteq A, \sum_{j \in \mathcal{N}(C)} b_j \geq \sum_{i \in C} a_i,$$

where  $\mathcal{N}(C)$  is the neighbour of  $C$  in  $B$ .<sup>2</sup>

PROOF LEMMA 1.

Let  $A = N, B = E, \mathcal{E} = \{(i, e) | i \text{ is a node of } e\}$ , where  $G = (N, E)$  is the bipartite graph for buyers and sellers and  $a_i = x_i, b_e = \alpha_{ij} f_{ij}$ , where  $e = ij$ .

Note that by the definition of core, for any coalition  $C \subseteq N$ ,

$$\sum_{i \in N/C} x_i \geq \text{TS}(N/C) \geq \sum_{i, j \notin C} \alpha_{ij} f_{ij},$$

in other words,

$$\begin{aligned} \sum_{i \in C} a_i &= \sum_{i \in C} x_i = \text{TS}(N) - \sum_{i \in N/C} x_i \\ &= \sum_{ij \in E} \alpha_{ij} f_{ij} - \sum_{i \in N/C} x_i \\ &\leq \sum_{ij \in E} \alpha_{ij} f_{ij} - \sum_{i, j \notin C} \alpha_{ij} f_{ij} \\ &= \sum_{i \in C \vee j \in C} \alpha_{ij} f_{ij} = \sum_{e \in \mathcal{N}(C)} b_e. \end{aligned}$$

By Generalized Hall Theorem, there exists  $g_{ij}$  and  $g_{ji}, i \in U, j \in V$ , such that

$$\sum_{e \in E} g_{ie} \geq x_i, \forall i \in N \quad \text{and} \quad \sum_{e = ij \in E} g_{ie} + g_{je} \leq \alpha_{ij} f_{ij}.$$

### B. PROOFS FOR THRESHOLD GAPS

The proof of upper bound part of Proposition 3.2 is based on the following observation that there's always a tree-structured optimal fractional solution and classical result in 0-1 knapsack.

<sup>2</sup>The proof is left to the full version

**Lemma B.1.** *There exists a fractional solution  $f$  with maximum social welfare, along with a rooted forest  $\mathcal{T}$  on  $U \cup V$ , such that:*

- An edge  $ij \in \mathcal{T}$  if and only if  $f_{ij} > 0$ ;
- For any non-root vertex  $i$  in  $\mathcal{T}$ ,  $q_i = \sum_j f_{ij}$ .

**Lemma B.2.** *An 0-1 knapsack problem always have a feasible solution with total value which is at least half of the optimal value of its linear relaxation.*

PROOF OF PROPOSITION 3.2. Let  $f$  and  $\mathcal{T}$  be the fractional solution and rooted forest given in Lemma B.1. Within  $\mathcal{T}$ , let  $T(i)$  be the vertices in the subtree rooted at vertex  $i \in U \cup V$ ,  $S(i)$  be the children of  $i$  and  $\text{par}(i)$  be the parent vertex of  $i$  (if exists). We call a vertex  $i$  *introverted*, if

$$f_{\text{par}(i)i} \geq \sum_{j \in S(i)} f_{ij}.$$

For instance, a leaf vertex  $i$  in  $\mathcal{T}$  must be introverted, since  $f_{\text{par}(i)i} = \sum_j f_{ij}$  in this case. Oppositely, a root vertex  $i$  with at least one child cannot be introverted (in this case, we assume  $f_{\text{par}(i)i} = 0$ ). Consider any introverted vertex  $i$ , we have

$$q_i \geq \sum_j f_{ij} \geq 2 \sum_{j \in S(i)} f_{ij} \geq \sum_{j \in S(i)} \sum_k f_{jk} = \sum_{j \in S(i)} q_j.$$

The last equality holds since every  $j$  in the summation is a non-root vertex. Therefore, even in a threshold solution  $f'$  it is feasible to have  $f'_{ij} = q_j$  for all  $j \in S(i)$ .

We prove  $\text{TS}(N) \geq \frac{1}{2} \text{FS}(N)$  by induction on the number of edges. The base case, when there is no edge and every vertex is isolated, is trivial. When  $|E| > 0$ , there must be a vertex  $i$  which is not introverted, but every other vertex in  $T(i)$  is introverted. By the above arguments on leaves and roots,  $|T(i)|$  must be larger than 1, and therefore induction hypothesis holds on  $N - T(i)$ .

Consider an 0-1 knapsack problem with capacity  $W = \sum_j f_{ij}$ . For each  $j \in S(i)$ , there is an item of weight  $w_j = q_j$  and value  $v_j = f_{ij}$ . If  $\sum w_j \geq W$ , in the linear relaxation the optimal value is at least  $W - \sum (w_j - v_j)$  and we apply Lemma B.2. Otherwise when  $\sum w_j < W$ , we can select all items and since  $i$  is not introverted we know  $2 \sum v_j \geq W$ . So in either case, there is a solution  $S \subseteq S(i)$  such that

$$\sum_{j \in S} f_{ij} \geq \frac{1}{2} \sum_j f_{ij} - \frac{1}{2} \sum_{j \in S(i)} (q_j - f_{ij}).$$

We can then design a feasible threshold solution  $f'$ : the transactions outside  $T(i)$  is the same as the optimal solution  $\text{TS}(N - T(i))$ , which by induction hypothesis is at least  $\frac{1}{2} \text{FS}(N - T(i))$ ; and assuming  $i \in U$  by symmetry, we set  $f'_{\text{par}(k)k} = q_k$  for all  $k$  in

$$\left( \bigcup_{j \in S} V \cap T(j) \right) \cup \left( \bigcup_{j \in S(i)-S} U \cap T(j) \right).$$

**Table 1: The gap between the maximal social welfare and core in water markets in Gansu Province, China.**

Number of agents	8	16	24	32	40	48
Percentage of empty core	0%	2%	6%	3%	21%	23%
Maximum $\epsilon = \alpha^* - 1$	0	$3.373 \times 10^{-3}$	$1.118 \times 10^{-3}$	$1.270 \times 10^{-3}$	$1.731 \times 10^{-3}$	$5.694 \times 10^{-4}$
Average $\epsilon$	0	$3.466 \times 10^{-5}$	$2.543 \times 10^{-3}$	$2.035 \times 10^{-5}$	$5.281 \times 10^{-5}$	$1.992 \times 10^{-5}$
Median $\epsilon$	0	0	0	0	0	0

**Table 2: The objective value  $\beta$  on instances of water markets in Gansu Province, China.**

Number of agents	8	12	16	20	24	28	32	36	40	44	48
Minimum	1	1	1.0072	1	1	1	1	1	1.0023	1	1
Median	1.0936	1.1015	1.0917	1.1126	1.0923	1.0757	1.0698	1.0605	1.0662	1.0551	1.0606
Maximum	2.1731	1.9549	1.4496	1.4350	1.3729	1.3367	1.3585	1.2310	1.3172	1.2305	1.1994

Then it holds

$$\begin{aligned}
& \text{TS}(N) \\
& \geq \frac{1}{2} \text{FS}(N - T(i)) + \sum_{j \in S} \sum_{k \in V \cap T(j)} q_k + \sum_{j \in S(i)} \sum_{k \in U \cap T(j)} q_k \\
& = \frac{1}{2} \text{FS}(N - T(i)) + \sum_{j \in S} f_{ij} + \sum_{j \in S(i)} \sum_{k, l \in T(j)} f_{kl} \\
& \geq \frac{1}{2} \sum_{k, l \notin T(i)} f_{kl} + \sum_{j \in S} f_{ij} + \sum_{j \in S(i)} \sum_{k, l \in T(j)} f_{kl} \\
& \geq \frac{1}{2} \sum_{k, l} f_{kl} - \frac{1}{2} \sum_j f_{ij} + \frac{1}{2} \sum_{j \in S(i)} (q_j - f_{ij}) + \sum_{j \in S} f_{ij} \\
& \geq \frac{1}{2} \text{FS}(N).
\end{aligned}$$

The proof for Proposition 3.4 also relies on the Lemma B.2, but in a simpler manner.

**PROOF OF PROPOSITION 3.4.** Let  $f$  and  $\mathcal{T}$  be the fractional solution and rooted forest given in Lemma B.1. For each vertex  $i$ , let  $S(i)$  be the children of  $i$ , and we propose a 0-1 knapsack problem  $\text{KS}(i)$ : the capacity  $W = q_i$ , item weights are  $w_j = \min(q_i, q_j)$  and item unit values are  $\alpha_{ij}$  for every  $j \in S(i)$ . It is clear that  $f_{ij}/w_j$  is a feasible solution of the relaxed problem of  $\text{KS}(i)$ , so there exists a solution  $S_{\text{KS}}(i) \subseteq S(i)$  such that

$$\sum_{j \in S_{\text{KS}}(i)} w_j \alpha_{ij} \geq \frac{1}{2} \sum_{j \in S(i)} f_{ij} \alpha_{ij}.$$

Now we design two threshold solutions  $f'$  and  $f''$ :

$$f'_{ij} = \min(q_i, q_j), \forall i \in U, j \in S_{\text{KS}}(i),$$

$$f''_{ij} = \min(q_i, q_j), \forall i \in V, j \in S_{\text{KS}}(i).$$

And therefore,

$$\begin{aligned}
\text{TS}(N) & \geq \frac{1}{2} \sum_{i,j} f'_{ij} \alpha_{ij} + \frac{1}{2} \sum_{i,j} f''_{ij} \alpha_{ij} \\
& \geq \frac{1}{4} \sum_{i \in U} \sum_{j \in S(i)} f_{ij} \alpha_{ij} + \frac{1}{4} \sum_{i \in V} \sum_{j \in S(i)} f_{ij} \alpha_{ij} = \frac{1}{4} \text{FS}(N).
\end{aligned}$$

### C. PROOF FOR PROPOSITION 4.6

**PROOF.** It suffices to show stability when  $G$  is connected. Let  $\lambda$  be the group of connected optimal balanced coefficients which maximizes  $\|\lambda\|_2$ , and we are going to combine

the arguments used in the two proofs of Proposition 4.3 and 4.4.

For every vertex  $v$  in the cycle  $H$ , let  $T(v)$  denote the tree outside  $H$  rooted at  $v$ . With the same argument for trees, we know that every  $C \in \text{supp}(\lambda)$  either  $T(v) \subset C$  or  $T(v) \cap C = \emptyset$ . Therefore, the coalition  $C$  could still be represented by a consecutive chain in  $H$ . What is left is to prove a similar convexity: if  $A, B \in \text{supp}(\lambda)$ , and even connected component in  $G_{A \cap B} \cap H$  consists of even number of vertices, then  $\text{TS}(A) + \text{TS}(B) \leq \text{TS}(A \cup B) + \text{TS}(A \cap B)$ .

We inherit the same scheme and notations from the proof of Proposition 4.4. The difference is that, the  $i$ -design is now parameterized. We abuse the notations for a solution  $f$ , using  $f_{T(v)}$  to denote transaction amounts on any edge in  $T(v)$ , and  $f_{vT(v)}$  to denote the summation of all transactions between  $v$  and its children in  $T(v)$ . The  $i$ -design with parameter  $t \in [0, 1]$  is:

$$f_{v_j v_{j+1}}^{A \cup B} = \begin{cases} f_{v_j v_{j+1}} & 0 \leq j < i \\ f'_{v_j v_{j+1}} & i \leq j \leq 2k \end{cases},$$

$$f_{T(v_j)}^{A \cup B} = t f_{T(v_j)} + (1-t) f'_{T(v_j)}, 1 \leq j \leq 2k.$$

$$f_{v_j v_{j+1}}^{A \cap B} = \begin{cases} f'_{v_j v_{j+1}} & 1 \leq j < i \\ f_{v_j v_{j+1}} & i \leq j < 2k \end{cases},$$

$$f_{T(v_j)}^{A \cap B} = (1-t) f_{T(v_j)} + t f'_{T(v_j)}, 1 \leq j \leq 2k.$$

We can construct such solutions because there is no thresholded edge in  $T(v)$ , and thus we can do convex combinations. The  $i$ -th design is not feasible for any  $t \in [0, 1]$ , only when for every  $t$ ,

$$\max(f_{v_{i-1} v_i} + f'_{v_i v_{i+1}} + t f_{v_i T(v_i)} + (1-t) f'_{v_i T(v_i)},$$

$$f'_{v_{i-1} v_i} + f_{v_i v_{i+1}} + (1-t) f_{v_i T(v_i)} + t f'_{v_i T(v_i)}) > q_i.$$

Notice that the two terms adds up to

$$f_{v_{i-1} v_i} + f'_{v_i v_{i+1}} + f'_{v_{i-1} v_i} + f_{v_i v_{i+1}} + f_{v_i T(v_i)} + f'_{v_i T(v_i)} \leq 2q_i,$$

so it must be the case when one of them is always dominating and larger than  $q_i$  throughout every  $t \in [0, 1]$ . Computing the two cases for  $t = 0, 1$ , we could get the same alternating pattern of  $f - f'$ :

$$(f'_{v_i v_{i+1}} > f_{v_i v_{i+1}} \text{ and } f_{v_{i-1} v_i} > f'_{v_{i-1} v_i}),$$

$$\text{or } (f_{v_i v_{i+1}} > f'_{v_i v_{i+1}} \text{ and } f'_{v_{i-1} v_i} > f_{v_{i-1} v_i}).$$

And everything remaining follows the proof of Proposition 4.4.



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