Efficiency, Fairness and Competitiveness in Nash Bargaining Games

Deeparnab Chakrabarty[1], *Gagan Goel*[2], *Vijay V. Vazirani*[2⋆], *Lei Wang*[2] *and Changyuan Yu*[3⋆⋆]

[1] Department of Combinatorics and Optimization, University of Waterloo, Waterloo.
[2] College of Computing, Georgia Institute of Technology, Atlanta, GA 30332–0280
[3] Institute for Theoretical Computer Science, Tsinghua University, Beijing,China.

Abstract. Recently, [8] defined the class of Linear Nash Bargaining Games (LNB) and obtained combinatorial, polynomial time algorithms for several games in this class. [8] also defines two natural subclasses within LNB, UNB and SNB, which contain a number of natural Nash bargaining games. In this paper we define three basic game theoretic properties of Nash bargaining games: price of bargaining, fairness and full competitiveness. We show that for each of these properties, a game in UNB has this property iff it is in SNB.

1 Introduction

The bargaining game was first modeled in John Nash's seminal 1950 paper [5] using the framework of game theory given a few years earlier by von Neumann and Morgenstern [9]. Since bargaining is perhaps the oldest situation of conflict of interest, and since game theory develops solution concepts for negotiating in such situations, it is not surprising that this paper led to a theory (of bargaining) that lies today at the heart of game theory (e.g., see [3, 7, 6]).

In a recent paper, Vazirani [8] initiated a study of Nash bargaining games via combinatorial, polynomial time algorithms. [8] defines LNB (Linear Nash Bargaining Games) – the class of games whose feasible set of utilities is defined by finitely many packing constraints. [8] also defines two natural subclasses within LNB: UNB and SNB. These classes contain a number of natural Nash bargaining games. In this paper we define three basic game theoretic properties of Nash bargaining games and show that for each of these properties, a game in UNB has this property iff it is in SNB. Below we intuitively define the classes UNB and SNB and then state the three properties; formal definitions appear in Section 2.

UNB is the subclass in which for each available resource, each agent who uses this resource uses it in the same way, i.e., the coefficients in the packing constraints are 0/1. Clearly, only $2^{|A|}$ such constraints are needed, where A is the set of agents – one for each subset of A. We can now view the right hand

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sides of these constraints as being given by a set valued function over the power set of A. If this function is submodular, the game is said to be in the subclass SNB of UNB.

We define price of bargaining in a way that is analogous to the notion of price of anarchy [4], i.e., it measures the loss in efficiency in resorting to the Nash bargaining solution as compared to the most efficient solution that can be obtained in a centralized manner. We will say that a Nash bargaining game is *fully competitive* if whenever one player increases his disagreement utility, no other player's utility can increase in the resulting Nash bargaining solution. We consider max-min and min-max fairness of the Nash bargaining solution in relation to all feasible solutions that are Pareto optimal. For all these properties, we gave a complete characterization of bargaining games in UNB. That is, any bargaining game in UNB has one of these properties if and only if it is in SNB.

2 Uniform Nash Bargaining Games

For a set of agents A, a Nash bargaining game is defined by a pair $(\mathbf{c}, \mathrm{P})$, where $\mathrm{P} \subseteq R_+^{|A|}$ is a compact and convex set which defines the feasible set of utilities of all the agents, and $\mathbf{c} \in \mathrm{P}$ is known as the disagreement point which defines the amount of utility each agent will get if the bargaining process fails.

Nash [5] defined the bargaining solution $\mathbf{u} \in \mathrm{P}$ of this game to be the one which satisfies four axioms: Pareto optimality, Invariance under affine transformation, Symmetry, and Independence of irrelevant alternatives. Nash proved that there is a unique point in P which satisfies these axioms, and moreover this point ($\mathbf{u} \in \mathrm{P}$) is the one that maximizes $\prod_{i\in A}(u_i - c_i)$ or equivalently $\sum_{i \in A} \log(u_i - c_i)$.

The class Linear Nash Bargaining Games (LNB), defined in [8], consists of games whose feasible set P is defined by a finite number of linear packing constraints. The main focus of our paper will be on a natural subclass of LNB called Uniform Nash bargaining games (or UNB) which was also defined in [8]. In these games, the coefficients of the variables in the linear packing constraints are either 0 or 1. Clearly there can be at most $2^{|A|}$ such constraints, thus a function of the form $v : 2^A \to R^+$ uniquely encodes a feasible set in UNB games.

Now given a disagreement point $\mathbf{c}$, and a fixed set of agents $T \subseteq A$, the solution to a UNB game can be captured by the following convex program:

$$\begin{aligned} \max \ & \sum_{i \in T} \log(u_i - c_i) \\ s.t. \ & \forall S \subset T : \sum_{i \in S} u_i \leq v(S) \\ & \forall i \in T : u_i \geq 0 \end{aligned} \tag{1}$$

For a fixed function $v : 2^A \to R^+$, we will define a family of games $F(v)$ to be the set of all Nash bargaining games for various choices of disagreement points $\mathbf{c}$ and set $T \subseteq A$. An instance $(\mathbf{c}, T) \in F(v)$ will refer to a particular Nash bargaining game in $F(v)$ with a fixed set T and disagreement point $\mathbf{c}$. A **UNB** game is called an **SNB** if the function v is a submodular function.

We will assume that the following two natural conditions are satisfied by the function v :

1. **Non degenerate:** $v(\emptyset) = 0$.
2. **Non redundancy of sets:** $\forall S \subseteq A$, there exists a feasible utility vector u such that set S is *tight* w.r.t. u, i.e. $\sum_{i \in S} u_i = v(S)$.

We will call such functions to be *valid* functions. Note that the second condition above implies 1)*Monotonicity:* for any $Z_1 \subset Z_2 \subseteq A$, we have $v(Z_1) \leq v(Z_2)$, and 2)*Complement freeness:* $v(Z_1 \cup Z_2) \leq v(Z_1) + v(Z_2)$.

In this paper, we are interested in the following three game theoretic properties of UNB games:

Price of Bargaining: For any valid function $v : 2^A \rightarrow R^+$, we define the *Price of bargaining* of $F(v)$ to be $\min\limits_{(\mathbf{c},T) \in F(v)} \frac{u(\mathbf{c},T)}{v(T)}$, where $u(\mathbf{c}, T)$ is the total utility obtained by set T of agents in the bargaining solution of the instance $(\mathbf{c}, T)$.

Full competitiveness: For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *fully competitive* if, for all games in $F(v)$, the following property holds: On increasing the disagreement utility c_i of an agent i, the bargaining solution doesn't increase the utility for any other agent j, where $j \neq i$.

Fairness: For any instance $I = (\mathbf{c}, T) \in F(v)$, define *core*$(I)$ to be the set of all feasible Pareto optimal solutions. For any vector $\mathbf{u}$, let $\mathbf{u}_{dec}$ be the vector obtained by sorting the components of $\mathbf{u}$ in decreasing order. A vector $\mathbf{x}$ *min-max dominates* $\mathbf{y}$ if $\mathbf{x}_{dec}$ is lexicographically smaller than $\mathbf{y}_{dec}$. Also let $\mathbf{u}^*$ be bargaining solution of instance I. Instance I is said to be *min-max fair* if the vector $\mathbf{u}^* - \mathbf{c}$ *min-max dominates* $\mathbf{y} - \mathbf{c}$ for all $\mathbf{y} \in$ *core*(I). $F(v)$ is said to be *min-max fair* if all the instances in $F(v)$ are min-max fair. Similarly we define the notion of *max-min fairness*.

Main results of this paper are described in theorems 1, 2, and 3.

3 Preliminaries

For any valid function v, we say that S is *tight* w.r.t. $\mathbf{u}$ if $\sum_{i \in S} u_i = v(S)$. Let $\mathbf{u}^*$ be the solution to the convex program given in (1). Then by KKT conditions, there must exist variables $\{p_S, \ \forall S \subseteq T\}$ such that:

1. $\forall S \subseteq T, \ p_S \geq 0$.
2. $\forall S \subseteq T, \ p_S > 0 \Rightarrow \mathbf{u}^*$ makes set S tight.
3. $\forall k \in T$, we have $\sum\limits_{S: k \in S} p_S = \frac{1}{u_k^* - c_k}$.

We will call p_S to be the price of set S.

Now we give some properties of the submodular and non-submodular functions which will be used in our proofs.

Property 1. Given a valid submodular function $v : 2^A \to R_+$, and a utility vector $\mathbf{u}$, if $Z_1, Z_2 \subseteq A$ are tight sets w.r.t. $\mathbf{u}$, then $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are also tight sets w.r.t. $\mathbf{u}$.

By using the uncrossing argument and the above property, we get the following corollary.

Corollary 1. *Given any SNB instance specified by v, $\mathbf{c}$ and T, we can choose the prices for all subsets of T in the KKT conditions, such that the tight sets with positive prices form a nested set family, i.e. $T = T_1 \supset T_2 \supset \cdots T_k \supset T_{k+1} = \emptyset$.*

Also, we will use the following property of non-submodular functions which is similar to the one given in [1]. Proof is given in the full version.

Property 2. Given a valid non-submodular function $v : 2^A \to R_+$, there exists set $S \subset A$, $i, j \in A \setminus S$, $l \in S$ and a feasible utility vector $\mathbf{u}$ such that:

1. $S \cup \{i\}$, $S \cup \{j\}$ are both tight w.r.t. $\mathbf{u}$.
2. Let $T = S \cup \{i, j\}$, $\mathcal{F}_k = \{Z \subseteq T : k \in Z, \text{ and } Z \text{ is tight w.r.t. } \mathbf{u}\}$. Then following holds
$$\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j \ , \quad \mathcal{F}_i \cap \mathcal{F}_j = \emptyset \ .$$
3. $u_k > 0$, $\forall k \in T$.

4 Price of Bargaining

Theorem 1. *For any valid function v, F(v) has Price of bargaining equal to* 1 *if and only if v is submodular.*

Proof. $\Leftarrow$: Suppose v is submodular. We want to show that for any disagreement point $\mathbf{c}$, and set $S \subseteq A$, if we restrict to the subproblem among agents in S, the Nash bargaining solution $\mathbf{u}^*$ satisfies $\sum_{i \in S} u_i^* = v(S)$.

Since $\mathbf{u}^*$ is the solution of Nash bargaining game, it must be Pareto optimal. Therefore every agent i is in some tight set T_i. Therefore by Property 1, we have $S = \cup_{i \in S} T_i$ is also tight, which means $\sum_{i \in S} u_i^* = v(S)$.

$\Rightarrow$: Suppose v is not submodular. By Property 2, there is a set $T = S \cup \{i, j\}$ and a feasible utility vector $\mathbf{u} = (u_k)_{k \in T}$ such that: (1) $u_k > 0, \forall k \in T$, (2) $S \cup i$ and $S \cup j$ are tight w.r.t. $\mathbf{u}$, (3) T is not tight w.r.t. $\mathbf{u}$. This is obtained from $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

Now for any $k \in T$, k is in some tight set w.r.t $\mathbf{u}$, hence by the lemma 1 below (proof in full version), there exists $\mathbf{c}$ such that $\mathbf{u}$ is the Nash bargaining solution corresponding to $\mathbf{c}$.

By condition 3 above, we have $\sum_{k \in T} u_k < v(T)$, which implies the Price of bargaining is strictly less than 1.

Lemma 1. *Given any valid function v, an instance $(\mathbf{c}, T)$ in $F(v)$, and a utility vector $\mathbf{u}$ with $u_i > 0, \forall i \in T$, $\mathbf{u}$ is Pareto optimal if and only if there exists a vector $\mathbf{c}$, with $c_i > 0$ $\forall i \in T$, such that $\mathbf{u}$ is the bargaining solution for the instance $(\mathbf{c}, T)$.*

5 Full Competitiveness

Theorem 2. *For any valid function v, $F(v)$ is fully competitive if and only if v is submodular.*

Proof. $\Leftarrow$: We first describe the algorithm for finding the optimal solution to the convex program (1) when the function v is a submodular function. Let $\mathbf{y} := \mathbf{u} - \mathbf{c}$. Then, an equivalent convex program is $\max\{\sum_i \log y_i : \ y(S) \leq f(S); \mathbf{y} \geq \mathbf{0}\}$, where $f(S) := v(S) - c(S)$. Call a set tight if $y(S) = f(S)$. Call an agent free if w.r.t the current $\mathbf{y}$ it is not in any tight set. The algorithm maintains a set of tight sets $\mathcal{T}$ initially empty. For all agents i which are free increase y_i simultaneously until some new set X gets tight. If X intersects with any set in $\mathcal{T}$, then since v is submodular, their union must be tight. Pick X to be the maximal (inclusion-wise) tight set and put it in $\mathcal{T}$. Continue till T (the set of all agents) becomes tight. We have the following lemma (Proof in the full version):

Lemma 2. *The utility allocation returned by the above algorithm is an optimal solution to the convex program.*

Now we prove that SNB games are fully competitive. Suppose the disagreement of agent i goes from c_i to $c_i + \delta$. Call the new disagreement vector $\mathbf{c}'$. Let $f'(S) := v(S) - c'(S)$ for all S. To show full competitiveness, it suffices to show that the optimum, $\mathbf{y}'$ of the convex program $\max\{\sum_i \log y_i : \ y(S) \leq f'(S); \mathbf{y} \geq \mathbf{0}\}$ is dominated by $\mathbf{y}$, the solution to the original convex program with $f()$. We will use the continuous time algorithm above to prove this.

Firstly, note that $f'(S) = f(S)$ for all sets not containing i and $f'(S) = f(S) - \delta$ for all others. This implies, that there is at least one agent j with $y'_j < y_j$. Secondly, observe from the description of the algorithm that for any agent j with $y'_j < y_j$, there must be a corresponding tight set in $\mathcal{T}'$ which contains both j and i.

We now show that if an agent j became non-free at time t in the original run (which means $y_j = t$), then by time t it must be in a tight set in the new run. We do so by showing that at time t if $y'_j = t$, then some set containing j at that time is tight (or over-tight which would imply $y'_j < t$).

Let A be the set containing j which went tight in the original run of the algorithm. Consider the set A in the new run of the algorithm at time t. Let $Q := \{j \in A : y'_j < y_j\}$. Note that if $j \in Q$, we are done. Assume $j \notin Q$. By the second observation made above and using the submodularity of v (to show union of intersecting tight sets is tight), we know there must exist a set Z which contains both Q and i, and which is tight. That is, $y'(Z) = f'(Z) = f(Z) - \delta$. We claim that $y'(Z \cup A) \geq g'(Z \cup A)$ and thus we are done.

This is because

$$y'(Z \cup A) = y'(A \setminus Q) + y'(Z) \geq y(A \setminus Q) + f'(Z) = y(A) - y(Q) + f'(Z)$$
$$\geq f(A) - f(Q) + f(Z) - \delta \geq f(A \cup Z) - \delta = f'(A \cup Z)$$

The first inequality follows from definition of Q, the second from the tightness of A under y and feasibility of y and the last follows from submodularity of f.

$\Rightarrow$: Suppose v is not submodular, then by property 2 there must exist a set S and agents $i, j \in A \setminus S$, $l \in S$, and a feasible utility vector $\mathbf{u}$ such that: (1) $S \cup \{i\}$, $S \cup \{j\}$ are both tight w.r.t. $\mathbf{u}$, (2)$\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$, (3) $u_k > 0$, $\forall k \in T$, where $T = S \cup \{i, j\}$.

We will now construct an instance $(c, T) \in F(v)$ which is not fully competitive. Let $\delta = \min_{k \in T} u_k > 0$. For tight sets $S \cup \{i\}$, $S \cup \{j\}$, we set their prices to be $p_{S,i}$, $p_{S,j}$ respectively, where $p_{S,i} = p_{S,j} = P = \frac{2}{\delta}$. For any other set $Z \subseteq T$, we set its price p_Z to be zero.

Let

$$\forall k \in T \quad c_k = u_k - \frac{1}{\sum_{Z \subseteq T, k \in Z} p_Z}$$

Since $S \cup \{i\}$ and $S \cup \{j\}$ are both tight, so for any $k \in T$, there exist at least one $Z \subseteq S$ such that $p_Z = P$, and we have

$$c_k \geq u_k - \frac{\delta}{2} > 0.$$

By the definition of $\mathbf{c}$, all the KKT conditions hold, thus $\mathbf{u}$ is the bargaining solution w.r.t. $(\mathbf{c}, T)$.

We will now construct a $\mathbf{c}'$ and its corresponding bargaining solution $\mathbf{u}'$, such that: (1) $\forall k \in T, k \neq j, c'_k \geq c_k$, and (2) $c'_j = c_j$ and $u'_j > u_j$.

Note that if the above conditions hold, then we can show that there exists a game in F(v) which is not fully competitive. This is because $\mathbf{c}'$ can be obtained from $\mathbf{c}$ by increasing only the coordinates other than j. If $F(v)$ is fully competitive, then each time a coordinate of $\mathbf{c}$ is increased utility allocated to j shouldn't increase. But if $u'_j > u_j$ is true then we get a contradiction.

Now we give the construction of $\mathbf{u}'$ and $\mathbf{c}'$. Let $\mathbf{u}'$ equals $\mathbf{u}$ except that $u'_j = u_j + \epsilon$, $u'_i = u_i + \epsilon$, $u'_l = u_l - \epsilon$. Using arguments similar to the proof of property 2, one can show that there exists small enough ϵ (given below) so that $\mathbf{u}'$ is feasible.

$$\epsilon < \min\{\epsilon_0, u_l/2\}, \text{ where } \epsilon_0 := \min_{\text{non-tight } Z \subseteq T} \frac{(v(Z) - \sum_{k \in Z} u_k)}{2}.$$

Now we construct $\mathbf{c}'$, so that it satisfies the condition mentioned above and the KKT conditions.

Note that for the KKT conditions, if we only assign positive price to tight sets $S \cup i$, $S \cup j$, say $p'_{S,i}$ and $p'_{S,j}$ respectively, then $\mathbf{u}', \mathbf{c}'$ satisfy the KKT conditions and the above requirements iff

- $c'_i = u'_i - \frac{1}{p'_{S,i}} = u_i + \epsilon - \frac{1}{p'_{S,i}} \geq u_i - \frac{1}{p_{S,i}} = c_i$;
- $c'_j = u'_i - \frac{1}{p'_{S,i}} = u_j + \epsilon - \frac{1}{p'_{S,j}} = u_j - \frac{1}{p_{S,j}} = c_j$;
- $c'_l = u'_l - \frac{1}{p'_{S,i}+p'_{S,j}} = u_l - \epsilon - \frac{1}{p'_{S,i}+p'_{S,j}} \geq u_l - \frac{1}{p_{S,i}+p_{S,j}} = c_l$;
- $c'_k = u'_k - \frac{1}{p'_{S,i}+p'_{S,j}} = u_k - \frac{1}{p'_{S,i}+p'_{S,j}} \geq u_k - \frac{1}{p_{S,i}+p_{S,j}} = c_k, \forall k \neq l, k \in S.$

The above conditions can be reduced to the following:

$$p'_{S,j} = \frac{1}{\epsilon + \frac{1}{p_{S,j}}},\ p'_{S,i} \geq \frac{1}{\frac{1}{p_{S,i}+p_{S,j}} - \epsilon} - p'_{S,j}$$

This can be satisfied as long as $\epsilon < \frac{1}{p_{S,i}+p_{S,j}} = \frac{\delta}{4}$.

To sum up, by setting $\epsilon = \min\{\epsilon_0/2, \delta/8\}$, we can find $p'_{S,i}, p'_{S,j}$ such that:

$$p'_{S,j} = \frac{1}{\epsilon + \frac{1}{p_{S,j}}},\ p'_{S,i} \geq \frac{1}{\frac{1}{p_{S,i}+p_{S,j}} - \epsilon} - p'_{S,j}$$

Note that this value of ϵ is consistent with the previous mentioned upper bound on it. Therefore, we can construct $\mathbf{c}'$ such that $\mathbf{u}'$ is the bargaining solution w.r.t. $\mathbf{c}'$ and $c'_k \geq c_k, \forall k \in T, c'_j = c_j$. Thus $(c, T) \in F(v)$ is not fully competitive.

6 Fairness

Theorem 3. *For any valid function v, $F(v)$ is min-max and max-min fair if and only if v is submodular.*

Proof. $\Leftarrow$: Suppose v is submodular. let $\mathbf{u}^*$ be the Nash bargaining solution for $(\mathbf{c}, T)$ where $T \subseteq A$. By corollary 1, we can choose the prices such that the tight sets w.r.t $\mathbf{u}^*$ with positive price form a nested set family, $T = T_1 \supset T_2 \supset \ldots\ldots \supset T_t \supset \emptyset$.

Pick any element $\mathbf{g}$ in *core*$((\mathbf{c}, T))$ i.e. $\mathbf{g}$ is Pareto optimal. If $\mathbf{u}^*$ does not min-max dominate $\mathbf{g}$, then $\mathbf{g}$ min-max dominates $\mathbf{u}^*$. In this case we will show that $\mathbf{g} = \mathbf{u}^*$, which leads to a contradiction.

Since $\mathbf{g}$ is Pareto optimal therefore every agent is in some tight set w.r.t $\mathbf{g}$. Hence by property 1, T_1 is tight, i.e. $\sum_{k \in T_1} g_k = v(T_1)$. Since $\mathbf{g}$ is feasible, we have

$$\sum_{k \in T_2} g_k \leq v(T_2)$$

T_1 and T_2 are tight sets w.r.t $\mathbf{u}^*$, so we have

$$\sum_{k \in T_1 - T_2} g_k \geq \sum_{k \in T_1 - T_2} u^*_k \tag{2}$$

Since each agent i in $T_1 - T_2$ has the highest $u_i - c_i$ among all the agents, if $\mathbf{g}$ min-max dominates $\mathbf{u}^*$, then for any $k \in T_1 - T_2$, we have $g_k \leq u^*_k$. Then by

(2), we have $g_k = u_k^*, \forall k \in T_1 - T_2$. Then we can use induction to show for any $1 \le i \le t$ and any $k \in T_i - T_{i+1}$, $g_k = u_k^*$. Hence $\mathbf{g} = \mathbf{u}^*$.

This proof also shows that $\mathbf{u}^*$ is the unique min-max fair utility vector. By using an argument similar to [2], we can show that any unique min-max fair utility vector is also max-min fair.

$\Rightarrow$: Suppose v is not submodular, then by property 2, there is a set $T = S \cup \{i, j\}$ and a $\mathbf{g} = (g_k)_{k \in T}$ such that: (1) $g_k > 0, \forall k \in T$, (2) $S \cup \{i\}$ and $S \cup \{j\}$ are tight w.r.t $\mathbf{g}$, (3) $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

For each $k \in T$, let $c_k = g_k - \epsilon$, where $0 < \epsilon < \min_{k \in T}\{g_k\}$. Clearly $\mathbf{g}$ is a feasible core element corresponding to $\mathbf{c}$, since each k is in a tight set (either $S \cup \{i\}$ or $S \cup \{j\}$).

Let $\mathbf{u}^*$ be the Nash bargaining solution corresponding to $(\mathbf{c}, T)$. Since $\mathbf{g}$ is the unique min-max and max-min feasible solution, thus if $\mathbf{u}^*$ min-max and max-min dominates $\mathbf{g}$, then $\mathbf{u}^*$ must equal $\mathbf{g}$. Next we show that this is not possible.

Suppose $\mathbf{u}^* = \mathbf{g}$, by KKT conditions, we can price all the subsets of T such that:

$$\frac{1}{g_l - c_l} = \sum_{Z \in \mathcal{F}_l} p_Z = \sum_{Z \in \mathcal{F}_i} p_Z + \sum_{Z \in \mathcal{F}_j} p_Z = \frac{1}{g_i - c_i} + \frac{1}{g_j - c_j}$$

which contradicts the fact that $g_l - c_l = g_i - c_i = g_j - c_j = \epsilon$.

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