Entanglement Detection in the Vicinity of Arbitrary Dicke States

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Dicke states represent a class of multipartite entangled states that can be generated experimentally with many applications in quantum information. We propose a method to experimentally detect genuine multipartite entanglement in the vicinity of arbitrary Dicke states. The detection scheme can be used to experimentally quantify the entanglement depth of many-body systems and is easy to implement as it requires measurement of only three collective spin operators. The detection criterion is strong as it heralds multipartite entanglement even in cases where the state fidelity goes down exponentially with the number of qubits.

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Quantum entanglement provides the most useful resource for implementation of many quantum information protocols. To test fundamentals of quantum mechanics and to realize quantum information processing, a big experimental drive is to get more and more particles prepared into massively entangled states [1–4]. There are different types of entangled states for many qubits [5–7]. Experiments so far typically center around two kinds of entangled states [1–4]. The first kind is the graph states, including the Greenberger-Horne-Zeilinger states as a special case [1]; the second kind is the Dicke states, including the *W* states as a special case [2–4]. Both types of entangled states have interesting properties and important applications in quantum information [5–7], and they have been generated from a number of experimental systems [1–4]. One can never get a perfect entangled state in any experiment. A critical question is thus to experimentally prove that the prepared state still contains genuine multipartite entanglement similar to the target state. For graph states, some powerful witness operators have been known which significantly simplifies the experimental entanglement detection [1,8,9]. For the Dicke type of states, however, the entanglement detection is more challenging. The experiments so far use either quantum state tomography [2], which requires measurements in an exponentially large number of experimental settings and thus is limited to only small systems, or some clever tricks that apply to only particular Dicke states [3,4,10,11], and are hard to be generalized to arbitrary Dicke states of many qubits.

In this Letter, we propose a general method to detect genuine multipartite entanglement in the vicinity of arbitrary Dicke states and to characterize the entanglement depth of the system. The proposed scheme has several favorable features. First, it only requires measurement of the collective spin operators and thus is straightforward for experimental implementation. Independent of the number of qubits, we only need to measure three operators with no requirement of separately addressing individual qubits. This is particularly convenient for entanglement detection in many-particle systems (such as a spinor condensate) where individual addressing is almost impossible. Second, the proposed detection criterion is strong and universally applicable to arbitrary Dicke states. It not only detects entanglement, but also quantifies the entanglement depth of the system [12,13]. The detection scheme is robust to experimental noise and can demonstrate a significant entanglement depth even in cases where the state fidelity has become exponentially small with the number of qubits.

The Dicke states are coeigenstates of the collective spin operators. Each qubit is described by a Paul matrix σ. For N qubits, we define the collective spin operator $\mathbf{J}$ as $\mathbf{J} = \sum_{i=1}^{N} \sigma/2$. The Dicke state $|N/2, n/2\rangle$ is defined as a coeigenstate of the operators $\mathbf{J}^2 \equiv J_x^2 + J_y^2 + J_z^2$ and J_z, with the eigenvalues $N(N+2)/4$ and $n/2$ ($n = -N/2, -N/2+1, \ldots, N/2$), respectively. The Dicke states can be conveniently generated in experiments without the need of separate addressing [5–7,14]. Except for the trivial case of $n = \pm N$, the Dicke states are multipartite entangled states with interesting applications in both precision measurements and quantum information [2–5,14,15].

To construct an entanglement detection criterion in the vicinity of Dicke states, we note that the variances of the collective spin operators J_x, J_y, J_z have very special properties for these states. The variance of J_z is minimized (ideally it should be zero), while the variances of J_x, J_y are maximized under the constraint of $\langle J_z\rangle$. So, to detect entanglement, we should construct an inequality to bound the variances of J_x, J_y with the variance of J_z for any separable states or insufficiently entangled states, and at the same time this inequality should be violated by the states sufficiently close to a Dicke state.

For a composite system of N qubits (distinguishable or indistinguishable), we note that its density operator ρ can always be written into the following form if ρ does not contain genuine N-particle entanglement [16]:

$$\rho = \sum_{\mu} p_{\mu} \rho_{\mu}, \tag{1}$$

with $p_{\mu} \geq 0$, $\sum_{\mu} p_{\mu} = 1$, and

$$\rho_{\mu} = \rho_{1\mu} \otimes \rho_{2\mu} \otimes \cdots \otimes \rho_{k_u \mu}, \tag{2}$$

where $\rho_{i\mu}$ $(i = 1, 2, \ldots, k_u)$ represents an arbitrary component state of $m_{i\mu}$ $(1 \leq m_{i\mu} < N)$ qubits with $\sum_{i=1}^{k_u} m_{i\mu} = N$. In other words, for each component μ, the N qubits are divided into k_u groups with $m_{i\mu}$ qubits for the ith group, and the component state ρ_{μ} is a tensor product of the states for each group. For a fixed component μ, each qubit uniquely belongs to one group; however, for different μ, the group division of the qubits can be different. If all $m_{i\mu} = 1$ (and corresponding $k_u = N$), ρ reduces to a separable state. If the maximum of $m_{i\mu}$ is m_0, we conclude that the state ρ has no genuine $(m_0 + 1)$-qubit entanglement [16]. With a smaller m_0, the entanglement depth of the state gets reduced.

We now show that for any states in the form Eqs. (1) and (2), the variance of the collective spin operators are severely bounded, while this bound is violated by the Dicke states. For each group division μ of N qubits, the total collective spin operators $\mathbf{J}$ can be written as $\mathbf{J} = \sum_{i=1}^{k_{\mu}} \mathbf{J}_i$, where $\mathbf{J}_i = \sum_{j=1}^{m_{i\mu}} \sigma_j/2$ is the collective spin operator for $m_{i\mu}$ qubits in the ith group. Through addition of the angular momenta, we know the maximum spin of $\mathbf{J}_i$ is $m_{i\mu}/2$, so the moments of $J_{\alpha i}$ $(\alpha = x, y, z)$ are bounded by

$$\langle J_{\alpha i}^2 \rangle \leq m_{i\mu}^2/4, \quad \text{and} \quad \langle \mathbf{J}_i^2 \rangle \leq m_{i\mu}(m_{i\mu} + 2)/4. \tag{3}$$

Under state ρ, we have $\langle J_x^2 \rangle = \sum_{\mu} p_{\mu} \langle J_x^2 \rangle_{\mu}$ and

$$\langle J_x^2 \rangle_{\mu} = \sum_{i_1, i_2} \langle J_{xi_1} \rangle_{\mu} \langle J_{xi_2} \rangle_{\mu} + \sum_i \langle (\Delta J_{xi})^2 \rangle_{\mu}. \tag{4}$$

Using the uncertainty relation $\langle (\Delta J_{yi})^2 \rangle_{\mu} \langle (\Delta J_{zi})^2 \rangle_{\mu} \geq \langle J_{xi} \rangle_{\mu}^2/4$, we can bound the term $\sum_{i_1, i_2} \langle J_{xi_1} \rangle_{\mu} \langle J_{xi_2} \rangle_{\mu}$ as

$$\begin{aligned} \sum_{i_1, i_2} \langle J_{xi_1} \rangle_{\mu} \langle J_{xi_2} \rangle_{\mu} &\leq \sum_{i_1, i_2} 4\sqrt{\langle (\Delta J_{yi_1})^2 \rangle_{\mu} \langle (\Delta J_{zi_1})^2 \rangle_{\mu} \langle (\Delta J_{yi_2})^2 \rangle_{\mu} \langle (\Delta J_{zi_2})^2 \rangle_{\mu}} \\ &\leq \sum_{i_1, i_2} 2[\langle (\Delta J_{yi_1})^2 \rangle_{\mu} \langle (\Delta J_{zi_2})^2 \rangle_{\mu} + \langle (\Delta J_{yi_2})^2 \rangle_{\mu} \langle (\Delta J_{zi_1})^2 \rangle_{\mu}] = 4\langle (\Delta J_z)^2 \rangle_{\mu} \sum_i \langle (\Delta J_{yi})^2 \rangle_{\mu}, \end{aligned} \tag{5}$$

where we have used the relation $\langle (\Delta J_z)^2 \rangle_{\mu} = \sum_i \langle (\Delta J_{zi})^2 \rangle_{\mu}$ for the state in the form of Eqs. (1) and (2). Combining Eqs. (4) and (5), we get

$$\langle J_x^2 \rangle \leq \sum_{\mu, i} p_{\mu} [\langle (\Delta J_{xi})^2 \rangle_{\mu} + 4\langle (\Delta J_z)^2 \rangle_{\mu} \langle (\Delta J_{yi})^2 \rangle_{\mu}]. \tag{6}$$

Using the relation $\langle (\Delta J_{\alpha i})^2 \rangle_{\mu} \leq \langle J_{\alpha i}^2 \rangle_{\mu} \leq m_{i\mu}^2/4$ (see Eq. (3) and $\langle (\Delta J_z)^2 \rangle \geq \sum_{\mu} p_{\mu} \langle (\Delta J_z)^2 \rangle_{\mu}$, we can bound $\langle J_x^2 \rangle$ by

$$\langle J_x^2 \rangle \leq [1 + 4\langle (\Delta J_z)^2 \rangle] \max_{\{m_{i\mu}\}} \left(\sum_{i=1}^{k_u} m_{i\mu}^2/4 \right), \tag{7}$$

where the maximum is taken over all the group division $\{m_{i\mu}\}$ ($m_{i\mu}$ are positive integers) of the N qubits with the constraint of $\sum_{i=1}^{k_u} m_{i\mu} = N$ and $m_{i\mu} \leq m_0$. The maximum value is obtained by choosing $k_u = \lceil N/m_0 \rceil$ ($\lceil N/m_0 \rceil$ denotes the smallest integer no less than N/m_0), $m_{1\mu} = N - m_0(k_u - 1)$, and all the other $m_{i\mu} = m_0$ $(i = 2, \ldots, k_u)$. Correspondingly, Eq. (7) reduces to

$$\langle J_x^2 \rangle \leq [1 + 4\langle (\Delta J_z)^2 \rangle] m_0 N/4, \tag{8}$$

where we have used the relation $m_{1\mu}^2 + m_0^2(k_u - 1) \leq m_0[m_{1\mu} + m_0(k_u - 1)] = m_0 N$. So, for any states without genuine $(m_0 + 1)$-qubit entanglement, the moment $\langle J_x^2 \rangle$ (and similarly also $\langle J_y^2 \rangle$) will be bounded by the inequality (8). When $m_0 \geq 2$, we can derive a stronger bound. Note that $\langle J_y^2 \rangle$ satisfies an inequality similar to Eq. (6), but with the indices x and y exchanged. If we add up the inequalities for $\langle J_x^2 \rangle$ and $\langle J_y^2 \rangle$, and use the relation $\langle (\Delta J_{xi})^2 \rangle_{\mu} + \langle (\Delta J_{yi})^2 \rangle_{\mu} \leq \langle \mathbf{J}_i^2 \rangle \leq m_{i\mu}(m_{i\mu} + 2)/4$ [see Eq. (3)], we obtain

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq [1 + 4\langle (\Delta J_z)^2 \rangle] N(m_0 + 2)/4. \tag{9}$$

We can use violation of the inequality (8) with $m_0 = 1$ to prove entanglement of the system in experiments and then use the following criterion to quantify its entanglement depth:

Criterion 1.—We can experimentally measure the following quantity ξ through detection of the collective spin operator $\mathbf{J}$:

$$\xi = \frac{\langle J_x^2 \rangle + \langle J_y^2 \rangle}{N(1/4 + \langle (\Delta J_z)^2 \rangle)} - 1. \tag{10}$$

If $\xi > m$, it is confirmed that the system has genuine m-qubit entanglement.

Criterion 1 is most strong for detection of the entanglement depth in the vicinity of the Dicke state $|N/2, 0\rangle$. For the state $|N/2, 0\rangle$, we have $\langle J_x^2 \rangle = \langle J_y^2 \rangle = N(N + 2)/8$ and $\langle (\Delta J_z)^2 \rangle = 0$, so $\xi = N + 1 > N$, and from measurement of ξ, we can confirm that all the qubits are in a genuine N-qubit entangled state (i.e., the entanglement depth is N). The criterion becomes weaker for the Dicke states $|N/2, n/2\rangle$ with increasing $|n|$. For the state $|N/2, n/2\rangle$, the moments of J_x and J_y are bounded by $\langle J_x^2 \rangle + \langle J_y^2 \rangle = \langle \mathbf{J}^2 \rangle - \langle J_z^2 \rangle = N(N + 2)/4 - n^2/4$. Criterion 1 does not take into account this bound from a finite $\langle J_z\rangle$. To derive a stronger detection criterion for the Dicke states $|N/2, n/2\rangle$, we start from Eq. (6) and a similar bound for $\langle J_y^2\rangle$. When we add up the inequalities for $\langle J_x^2\rangle$ and $\langle J_y^2\rangle$ both in the form of Eq. (6), we need to find a better bound for $\langle(\Delta J_{xi})^2\rangle_\mu + \langle(\Delta J_{yi})^2\rangle_\mu$ under a finite $\langle J_z\rangle$. Using the relation $\langle(\Delta J_{xi})^2\rangle_\mu + \langle(\Delta J_{yi})^2\rangle_\mu \leq \langle \mathbf{J}_i^2\rangle_\mu - \langle J_{zi}^2\rangle_\mu$ and $\langle J_z^2\rangle_\mu = \langle(\sum_{i=1}^{k_u} J_{zi})^2\rangle_\mu \leq k_u \sum_i \langle J_{zi}^2\rangle_\mu$, we obtain

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq \sum_\mu p_\mu[1 + 4\langle(\Delta J_z)^2\rangle_\mu] \times \left[\sum_i m_{i\mu}(m_{i\mu}+2)/4 - \langle J_z^2\rangle_\mu/k_u\right]. \quad (11)$$

To bound the right side of Eq. (11), we consider the twofold average $\sum_\mu p_\mu \langle(\Delta J_z)^2\rangle_\mu \langle J_z^2\rangle_\mu = \langle\langle(\Delta J_z)^2\rangle_\mu \langle J_z^2\rangle_\mu\rangle$, where $\langle\cdots\rangle$ denotes the average over μ with the weight function p_μ. For any two variables A and B, we know their average satisfies the following property:

$$\langle AB\rangle = \langle A\rangle\langle B\rangle + \langle \Delta A \Delta B\rangle \geq \langle A\rangle\langle B\rangle - \sqrt{\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle}. \quad (12)$$

Taking A and B as $\langle J_z^2\rangle_\mu$ and $\langle(\Delta J_z)^2\rangle_\mu$, respectively, we have

$$-\langle\langle(\Delta J_z)^2\rangle_\mu\langle J_z^2\rangle_\mu\rangle \leq -\langle J_z^2\rangle\langle\langle(\Delta J_z)^2\rangle_\mu\rangle + \langle(\Delta J_z^2)^2\rangle(1+2\alpha), \quad (13)$$

where $\langle(\Delta J_z^2)^2\rangle \equiv \langle J_z^4\rangle - \langle J_z^2\rangle^2$ and

$$\alpha \equiv \sqrt{(\langle J_z^4\rangle - \langle J_z\rangle^4)/(\langle J_z^4\rangle - \langle J_z^2\rangle^2)}, \quad (14)$$

which is typically close to 1. In deriving Eq. (13), we have used $\langle\langle J_z^2\rangle_\mu^2\rangle \leq \langle\langle J_z^4\rangle_\mu\rangle = \langle J_z^4\rangle$ and

$$\begin{aligned}&\langle\langle(\Delta J_z)^2\rangle_\mu^2\rangle - \langle\langle(\Delta J_z)^2\rangle_\mu\rangle^2 \\ &= \langle\langle J_z^2\rangle_\mu^2\rangle - \langle J_z^2\rangle^2 - 2[\langle\langle J_z^2\rangle_\mu\langle J_z\rangle_\mu^2\rangle - \langle J_z^2\rangle\langle\langle J_z\rangle_\mu^2\rangle] \\ &\leq \langle(\Delta J_z^2)^2\rangle + 2\sqrt{\langle(\Delta J_z^2)^2\rangle[\langle J_z^4\rangle - \langle J_z\rangle^4]}. \quad (15)\end{aligned}$$

In the last line of Eq. (15), we use again the property in Eq. (12). Substituting Eq. (13) into Eq. (11), we finally obtain the following bound for any state in the form of Eqs. (1) and (2)

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq [1 + 4\langle(\Delta J_z)^2\rangle] \times \max_{\{m_{i\mu}\}}\left[\sum_i m_{i\mu}(m_{i\mu}+2)/4 - \chi/k_u\right], \quad (16)$$

where χ is defined by

$$\chi = \langle J_z^2\rangle - [1/4 + \langle(\Delta J_z)^2\rangle]^{-1}\langle(\Delta J_z^2)^2\rangle(1+2\alpha). \quad (17)$$

The parameter χ is determined experimentally by measuring the operator J_z, and its value is basically given by the first term $\langle J_z^2\rangle$, with small correction from the fluctuation of J_z^2 when the real state deviates from the Dicke state (the latter has $\langle(\Delta J_z^2)^2\rangle = 0$). Summarizing the result, we arrive at the following criterion.

Criterion 2.—We can experimentally measure the values of ξ and χ [defined by Eqs. (10) and (17)] through detection of the collective spin operator $\mathbf{J}$. The system has genuine m-qubit entanglement if

$$\xi > f(m,\chi) \equiv \frac{4}{N}\max_{\{m_{i\mu}\}}\left(\sum_{i=1}^{k_u} m_{i\mu}(m_{i\mu}+2)/4 - \chi/k_u\right) - 1, \quad (18)$$

where the maximum is taken under the constraint of $m_{i\mu} \leq m-1$ and $\sum_{i=1}^{k_u} m_{i\mu} = N$.

With a known χ, it is typically easy to calculate the function $f(m,\chi)$. For instance, for the state $|N/2, n/2\rangle$, $\chi \approx n^2/4$, and we find $f(m,\chi) \approx m - (m-1)n^2/N^2$ for the simple case when $m-1$ divides N and $(m-1)n^2 < 2N^2$.

The noise in experiments will degrade the entanglement depth of the system. In the following, we discuss robustness of the detection criteria 1 and 2 under noise. First, we consider dephasing noise which is a major source of error in many experiments. The detection criteria 1 and 2 are very robust to the dephasing noise as dephasing brings no change to the eigenvalue of J_z and thus does not increase the variance $\langle(\Delta J_z)^2\rangle$. To see this clearly, let us estimate the value of ξ when the target states $|N/2, n/2\rangle$ in experiments are distorted by the dephasing noise with an error rate p for each individual qubit. Note that $|N/2, n/2\rangle$ is a big superposition state in the computational basis with an extremely large number (of the order of $2^N e^{-n^2/N}$ when $n \ll N$) of terms and all the superposition terms are eigenstates of J_z with eigenvalue $n/2$. The dephasing error degrades coherence between these terms but does not increase $\langle(\Delta J_z)^2\rangle$. For each superposition term, we know $\langle J_y^2\rangle = \langle J_x^2\rangle = \sum_{i=1}^N \langle(\sigma_{ix}/2)^2\rangle = N/4$. So, if coherence is completely gone ($p = 1$), ξ reduces to 1, and the state has no entanglement as expected. However, under incomplete dephasing, it is still possible to demonstrate a significant entanglement depth by measuring ξ in experiments even if the state fidelity becomes very small. With a dephasing error rate p for each qubit, the state fidelity typically goes down exponentially with pN for N qubits when $N \sim N - n \gg 1$. To estimate the value of ξ in this case, we note that with a probability

$$\binom{N}{i} p^i (1-p)^{N-i}$$

(according to the binormal distribution), i qubits are decohered among the N qubits, which contribute a value of $i/2$ to $\langle J_x^2\rangle + \langle J_y^2\rangle$. The remaining $N-i$ qubits still have coherence, which contribute a value of $(N-i)\times(N-i+2)/4 - \langle J_z^2\rangle_{N-i}$ to $\langle J_x^2\rangle + \langle J_y^2\rangle$. Since the N qubits are in the $J_z = n/2$ eigenstates, the mean value of $\langle J_z^2\rangle_{N-i}$

for the $N - i$ qubits is equal to $\langle (n/2 - J_z)^2 \rangle_i$ for the decohered i qubits. We have $\langle (n/2 - J_z)^2 \rangle_i = [n(N - i)/2N]^2 + \langle (\Delta J_z)^2 \rangle_i$, where $\langle (\Delta J_z)^2 \rangle_i \equiv \langle (J_z - ni/2N)^2 \rangle_i = \langle J_z^2 \rangle_i - \langle J_z \rangle_i^2$. For the decohered i qubits, if we neglect correlation between different qubits, $\langle (\Delta J_z)^2 \rangle_i \sim i(1 - n^2/N^2)/4$. However, the constraint of $J_z = n/2$ for the total N qubits further bounds the variance of $\langle (\Delta J_z)^2 \rangle_i$ as i increases and reduces it to zero when $i = N$. To take into account this constraint, we estimate $\langle (\Delta J_z)^2 \rangle_i$ roughly by $\langle (\Delta J_z)^2 \rangle_i \sim i(1 - n^2/N^2)(N - i)/4N$ with inclusion of a linear reduction factor $(N - i)/N$. The value of ξ is then estimated by

$$\xi \sim 4/N \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} \{ i/2 + [(N-i)(N-i+2)/4 - [n(N-i)/2N]^2 - i(1-n^2/N^2)(N-i)/4N] \} - 1 \sim N(1-p)^2(1-n^2/N^2) + 1.$$

By measuring ξ, it is possible to demonstrate a significant entanglement depth about $N(1-p)^2$ qubits, even if the state fidelity has become very small when $pN \gg 1$.

The detection criteria 1 and 2 are more sensitive to the bit-flip error as this type of error significantly increases $\langle (\Delta J_z)^2 \rangle$. With a bit-flip error rate p_b for each qubit, the variance of J_z is estimated by $\langle (\Delta J_z)^2 \rangle \sim N p_b (1 - p_b)$. We need $N p_b (1 - p_b) < 1/4$ to minimize change to ξ. For tens of qubits, we can tolerate a bit-flip error rate at a percent level to keep the qubits in a genuine multipartite entangled state. Alternatively, in the limit of large N with $N p_b (1 - p_b) \gg 1/4$, the value of ξ is estimated by $\xi \approx 1/[4 p_b (1 - p_b)] - 1$ for states in the vicinity of $|N/2, 0\rangle$ and by $\xi \approx (1 - n^2/N^2)/[4 p_b (1 - p_b)] - 1$ for states in the vicinity of $|N/2, n/2\rangle$. So, using criterion 1 for $|N/2, 0\rangle$ or criterion 2 for $|N/2, n/2\rangle$, with a percent of bit-flip error rate for each qubit, it is possible to demonstrate an entanglement depth of more than 20 qubits in experiments by measuring ξ.

In summary, we have proposed powerful detection criteria to experimentally demonstrate entanglement and quantify the entanglement depth for many-body systems in the vicinity of arbitrary Dicke states. The criteria are based on simple measurements of the collective spin operators and ready to be implemented in future experiments.

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