Ranking Mechanism Design for Price-setting Agents in E-commerce

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ABSTRACT

Ranking algorithms of e-commerce sites take the buyer's search query and information of the corresponding sellers' items as input, and output a ranking of sellers' items that maximizes sites' objectives. However, the conversion rate of each item (i.e., the probability of a completed transaction) not only depends on the ranking given by the site (which controls click-through rates), but also depends on the item price set by its seller(which controls the buyer's willingness to buy). As a result, a ranking algorithm is in fact a mechanism that deals with sellers who strategically set item prices.

An interesting fact about this setting, at least the *status quo* for the largest e-commerce sites such as Taobao, Amazon, and eBay, is that sellers are usually not given the option to report their private costs but can only communicate with the site by setting item prices. In terms of mechanism design, this is a setting where the designer is restricted to design a specific type of indirect mechanisms.

We follow the framework of implementing optimal direct mechanisms by indirect mechanisms to tackle this optimal indirect ranking mechanism design problem. We firstly define a related optimal direct ranking mechanism design setting and use Myerson's characterization to optimize in that setting. We then characterize the class of direct mechanisms which could be implemented by indirect mechanisms, and construct a mapping that maps the mechanisms designed in the previous direct setting to indirect mechanisms in the original setting where sellers are allowed only to set item prices. We show that, using this technique, one can obtain mechanisms in the indirect setting that maximize expected total trading volume. We then present the mechanism employed by one of the largest e-commerce websites currently, get a Bayesian Nash Equilibrium of it and obtain the gap of the volume of the site's mechanism and the optimal mechanism. Given real dataset from the site, we also simulate our optimal mechanism and the site's mechanism, and it shows that our mechanism outperforms the site's mechanism significantly.

KEYWORDS

Mechanism Design; Ranking; E-commerce

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1 INTRODUCTION

When a buyer types a search query on an e-commerce site, the site is called a ranking algorithm that sorts all the sellers' items that match the search query and displays the ranked list to the buyer. This ranking is crucial for the sellers, because higher rankings will lead to higher click-through rates, a key factor in sellers' profit. A standard ranking algorithm prefers sellers with higher reputations, the number of completed historical transactions and lower item prices. From a seller's point of view, on the one hand, she would like to set a lower price, that yields a good ranking and attracts more transactions that lead to better future rankings; on the other hand, she would like to raise the price for better profit. The current design of ranking algorithms on e-commerce sites makes these details very obscure and difficult for sellers to optimize her own utility. As a result, it is widely observed from data on these sites that the prices fluctuate frequently and new sellers have to set very low prices in order to attract more buyer impressions[1].

In this paper, we investigate the design of ranking algorithms from a mechanism design perspective. We model the problem as a variant of one-shot sponsored search auction problem and aim to incentivize the sellers to set prices appropriately in order to maximize the site's objectives, such as the total trading volume. For a related problem on how to prevent the sellers from manipulating reputation scores and the number of transactions, see a recent mechanism design approach proposed by [4].

As mentioned in the abstract, an interesting fact about the e-commerce setting, different from any auction setting, is that sellers are usually not given the option to report their private types, which are normally their costs for producing the items, but can only communicate with the site by setting item prices. This renders our problem an instance of indirect mechanism design.

Our work falls under the umbrella of *implementing optimal direct mechanisms by indirect mechanisms*. In the auction design literature, a line of work considers the problem where bidders are only allowed to report several discrete bid levels [2, 3, 18, 21], even though they have a continuous type space. [12] present a class of winners-pay-bid mechanisms where there exist simple and nontruthful equilibria in Internet advertising. As a result, the design problem also becomes indirect. Note that the set of prices sellers could post is not restricted in our setting, and the utility of each seller does not only rely on the allocation and the transfer, but also depends on the conversion rate that is decided by the action she chooses, which means that their characterizations of the class of direct mechanisms which can

[1] Back to 2015, there was a very famous price competition for a type of tea at Taobao and eventually the price of a box of tea, which was normally worth more than 20 dollars, lowered to 1 dollar.

be implemented by indirect mechanisms can not be applied in our setting.

In this paper, we aim to design indirect mechanisms that implement the site's objectives at their (nontruthful) Bayes Nash equilibria(BNE) [13, 20], called total trading volume, which most e-commerce websites care most. Our first contribution is that we construct a related direct mechanism design setting and use Myerson's characterization [19, 22] to optimize in that setting. We then construct a mapping that maps the mechanisms designed in the previous direct setting to indirect mechanisms in the original setting where sellers are allowed only to set item prices and characterize the class of direct mechanisms which could be implemented by indirect mechanisms. Our second contribution is that we obtain optimal mechanisms in this indirect setting in terms of the expected total trading volume and BNE of these mechanisms are simple functions. Our third contribution is that we obtain a simple BNE of the mechanism employed by one of the largest e-commerce websites in the world and the gap of volume between it and the optimal mechanism. The other contribution is that we simulate the optimal mechanism and compare it with the site's mechanism based on a dataset from the site, which shows our mechanism performs significantly better than the site's mechanism.

1.1 Related work

It is necessary to compare our setting with the well-studied sponsored search setting [7, 10, 17], in which the outcome of the mechanism is also rankings of slots and transfers of agents. However, besides the difference about the spaces of reporting, most of sponsored search literature focuses on objectives of revenue and social welfare, our setting is the first work that considers maximizing the expected total trading volume to the best of our knowledge. We introduce a direct mechanism problem which is related to our problem and show the convertibility of it to the traditional sponsored search auction.

We are not the first to study mechanism design in e-commerce and reputation sites. A line of work [8, 14–16] study how to incentivize the buyers to leave truthful feedbacks. [4], mentioned before, consider the problem that sellers manipulating their reputation scores by creating fake transactions and designing truthful mechanisms that maximize social welfare. [5] consider a multiple rounds version of our problem and tackle it by a approach of reinforcement mechanism design [6, 23]. [24] consider reputation systems with strategic buyers and sellers. [1] consider the setting that buyers and sellers trade through intermediaries.

2 THE SETTING

In a typical e-commerce setting, a ranking mechanism ranks m sellers over n slots, and each one sells a different item that matches the same buyer query. Each seller i has a private cost c_i for his item. The prior distribution of c_i is independently drawn from $F_i[0, h_i]$. Let $f_i(c_i)$ denote the probability density function of seller i, $f(c) = \prod_{i=1}^{m} f_i(c_i)$ denote the joint probability density of c and $f_{-i}(c_{-i})$ denote the joint probability density of c_{-i}. The buyer's valuation towards seller i'item is a uniform distribution on $[0, h_i]$. [2] Each seller i sets a take-it-or-leave-it price p_i for the buyer.

[2] We make the uniform valuation assumption for ease of presentation. The approach and analysis extend straightforwardly to any valuation.

A ranking mechanism f takes as input the posted prices p of all sellers, and outputs a ranking of these sellers in the result page for the buyer. Let x_{ij} denote the probability that seller i is ranked in j-th slot. The probability that the buyer clicks on seller i's item is denoted by α_{ij}[3].

Given a ranking x, the expected probability that the buyer clicks seller i's item is $q_i = \sum_{j=1}^{n} \alpha_{ij} x_{ij}$. A ranking x is feasible if it satisfies the following constraints

$$\forall i \forall j, 0 \leq x_{ij} \leq 1.$$
$$\forall j, \sum_{i=1}^{m} x_{ij} \leq 1.$$
$$\forall i, \sum_{j=1}^{n} x_{ij} \leq 1.$$

Upon receiving the price vector p set by all sellers, a mechanism must return a ranking $x(p)$ and a transfer $t(p)$. Let $x_{ij}(p)$ denote the probability that seller i is assigned to slot j and $t_i(p)$[4] denote the transfer seller i makes to the mechanism. Given a mechanism f and sellers' prices p, the utility of seller i with a type c_i is

$$u_i(c_i, p) = \sum_{j=1}^{n} \alpha_{ij} x_{ij}(p)(h_i - p_i)^+ (p_i - c_i)/h_i - t_i(p).$$

$(h_i - p_i)^+/h_i$ means $max \{0, h_i - p_i\}/h_i$, which denotes the conversion rate of seller i given a price p_i. That is, the utility of a seller equals the difference between the expected profit(the product of the probability that the buyer clicks seller i' item, the conversion rate, and revenue of a seller selling one item) and the money paid to the designer.

The description so far prevents us from designing a direct mechanism since the sellers are not allowed to report their private costs. As a result, we hope to design indirect mechanisms whose interim individual rational bayesian nash equilibria(IIR-BNE) yield desirable the expected total trading volume for the designer.

Definition 2.1. Given a mechanism f, an interim individual rational bayesian nash equilibrium (IIR-BNE) is a profile of strategies s that maps each seller's type to a price. Let $s_i(c_i)$ denote the strategy function of seller i, $s_{-i}(c_{-i})$ denote the posted prices of sellers except from seller i when sellers follow strategies s_{-i}. Let

$$U_i(c_i, p_i) = \int u_i(c_i, (p_i, s_{-i}(c_{-i}))) f_{-i}(c_{-i}) dc_{-i}$$

be the interim utility of a seller i with type c_i who sets a price p_i, when others follow the strategies s_{-i}. s is called an IIR-BNE if and only if each seller gets the maximum interim utility if she follows strategy s_i with others following the equilibrium and each seller gets non negative interim utility in this IIR-BNE:

$$\forall i \forall c_i \forall p_i, U_i(c_i, s_i(c_i)) \geq U_i(c_i, p_i).$$

[3] We assume the click-through rates is independent of posted prices of sellers, which is a standard assumption in sponsored search auction setting. The main results of this paper hold without this assumption trivially.

[4] A positive transfer means that the seller pays a commission fee to the site, otherwise the site reimburses the seller.

$$\forall i \forall c_i, U_i(c_i, s_i(c_i)) \geq 0.$$

Our objective is to find a feasible mechanism with at least an IIR-BNE that maximizes the expected total trading volume. As a mechanism may have multiple IIR-BNE, we define a mechanism's volume as the maximum expected total trading volume among all IIR-BNE of this mechanism.

Definition 2.2. Given a mechanism f with at least an IIR-BNE, let $\mathcal{S}$ denote the set of all IIR-BNE where the posted price for any seller is no less than its cost, then its volume is

$$max_{s\in\mathcal{S}} \int \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} x_{ij}(s(c)) (h_i - s_i(c_i))^+ s_i(c_i) f(c)/h_i dc.$$

The reason why we only consider those IIR-BNE where the posted price for any seller is no less than its cost is that, without this restriction, one can design a trivial mechanism as follows: rank these sellers in descending order of the expected trading volume, and pay these sellers sufficiently to ensure IIR. There is an unique BNE, $s_i(c_i) = h_i/2$, and the mechanism is optimal. Obviously this mechanism is not practical, the site needs to reimburse to sellers.

To make the analysis easier to follow, we firstly design optimal direct mechanisms in a related setting, then use these direct mechanisms to construct optimal indirect mechanisms.

3 A RELATED DIRECT MECHANISM DESIGN SETTING

In this section we present a related direct mechanism design setting, and design optimal mechanisms. The major difference between this setting and previous setting is that sellers report their costs directly to the mechanism, and the mechanism decides prices for sellers rather than letting sellers posting prices.

A direct mechanism g takes the reported costs of items c as input, outputs an allocation $x(c)$, a vector of prices $p(c)$ and a vector of transfers $t(c)$. Given a mechanism g and reported prices c', the utility of seller i with a type c_i reporting type c_i' is

$$u_i(c_i, c') = \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c')(h_i - p_i(c'))^+ (p_i(c') - c_i)/h_i - t_i(c').$$

There are some properties that a direct mechanism satisfies.

Definition 3.1. Feasibility. A mechanism is feasible if it satisfies the following constraints

$$\forall i \forall j \forall c', 0 \leq x_{ij}(c') \leq 1.$$

$$\forall j \forall c', \sum_{i=1}^{m} x_{ij}(c') \leq 1.$$

$$\forall i \forall c', \sum_{j=1}^{n} x_{ij}(c') \leq 1.$$

$$\forall i \forall c', p_i(c') \geq c_i'.$$

Definition 3.2. BIC(Bayesian incentive compatible).

For any seller, telling the truth will get the maximum interim utility with others reporting costs truthfully, i.e.

$$U_i(c_i, c_i') = \int u_i(c_i, (c_i', c_{-i})) f_{-i}(c_{-i}) dc_{-i},$$
$$\forall i \forall c_i \forall c_i', U_i(c_i, c_i) \geq U_i(c_i, c_i').$$

Definition 3.3. IIR(Interim individual rational).

For any seller, telling the truth will get non-negative interim utility when others report costs truthfully, i.e.

$$\forall i \forall c_i, U_i(c_i, c_i) \geq 0.$$

Our objective in this setting is to find an optimal mechanism with feasibility, BIC and IIR property that maximizes the volume.

Definition 3.4. Given a BIC and IIR mechanism g, g's volume is

$$\int \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} x_{ij}(c)(h_i - p_i(c))^+ p_i(c) f(c)/h_i dc.$$

We get necessary and sufficient conditions of BIC and IIR by applying Myseron's approach in [19].

THEOREM 3.5. *Let*

$$B_i(a_i) = \int \sum_{j=1}^{n} \alpha_{ij} x_{ij}(a_i, c_{-i}) (h_i - p_i(a_i, c_{-i}))^+ f_{-i}(c_{-i})/h_i dc_{-i},$$

i.e. the expected probability of the buyer purchasing seller i's item when he reports a_i and others report costs truthfully. A mechanism is BIC and IIR if and only if for all i, $B_i(c_i)$ is non-increasing on c_i and

$$\forall i \forall c_i, \frac{\mathrm{d}U_i(c_i, c_i)}{\mathrm{d}c_i} = -B_i(c_i). \tag{1}$$

$$\forall i, U_i(h_i, h_i) \geq 0. \tag{2}$$

PROOF. If a mechanism is BIC and IIR, the property (2) is directly from IIR. For all i and c_i, let $c_i' = c_i + \delta c$ ($\delta c > 0$). By BIC, we have that

$$U_i(c_i, c_i) \geq U_i(c_i, c_i'),$$
$$U_i(c_i', c_i') \geq U_i(c_i', c_i).$$

Let

$$A_i(c_i) = \int \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c)(h_i - p_i(c))^+ p_i(c) f_{-i}(c_{-i})/h_i dc_{-i},$$

$$T_i(c_i) = \int t_i(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}.$$

From above formulas,

$$A_i(c_i) - T_i(c_i) - B_i(c_i)c_i \geq A_i(c_i') - T_i(c_i') - B_i(c_i')c_i,$$
$$A_i(c_i') - T_i(c_i') - B_i(c_i')c_i' \geq A_i(c_i) - T_i(c_i) - B_i(c_i)c_i'.$$

Then we can get

$$-B_i(c_i) \leq \frac{U_i(c_i', c_i') - U_i(c_i, c_i)}{\delta c} \leq -B_i(c_i').$$

We get $B_i(c_i)$ is non-increasing on c_i, by taking the limit of this formula on δc in two sides, we can get the equation (1) and the equation (2).

On the other hand, If a mechanism g satisfies these properties, by the equation (1) and the equation (2), the interim utility of a seller telling the truth when others tell the truth is non-negative, thus g is IIR. BIC is equivalent to

$$\forall i \forall c_i \forall c_i^{'}, U_i(c_i, c_i) \geq U_i(c_i^{'}, c_i^{'}) + B(c_i^{'})(c_i^{'} - c_i).$$

As the mechanism satisfies the equation (1) and $B_i(c_i)$ is non-increasing on c_i, the above inequality holds and g is BIC. □

3.1 Optimal direct mechanism that maximizes volume

In this section, we present the optimal direct ranking mechanism (ODRM). By recalling the definition, any direct mechanism's volume is

$$\int \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c)(h_i - p_i(c))^+ p_i(c) f(c)/h_i dc.$$

We optimize volume by maximizing the total trading volume point-wise and construct a transfer rule that makes the mechanism BIC and IIR. By the definition of feasibility, for any seller i and any input type profile c, $p_i(c) \geq c_i$. In ODRM, each seller i's price is set as $h_i/2$ if $c_i \leq h_i/2$ otherwise the price is set as c_i to maximize $(h_i - p_i(c))^+ p_i(c)$. The allocation $x(c)$ of any input type profile c is calculated by this linear programming:

$$\begin{aligned} & max_{x(c)} \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c) \frac{(h_i - p_i(c))^+ p_i(c)}{h_i} \\ & \forall i \forall j, 0 \leq x_{ij}(c) \leq 1. \\ & \forall j, \sum_{i=1}^{m} x_{ij}(c) \leq 1. \\ & \forall i, \sum_{j=1}^{n} x_{ij}(c) \leq 1. \end{aligned} \tag{3}$$

Note that all $h_i, p_i(c_i)$ in linear programming (3) are constant, if we replace $(h_i - p_i(c))^+ p_i(c)/h_i$ to a new variable v_i, then (3) is indeed a problem in the sponsored search auction, whose goal is allocative efficiency[20]. As shown in [20], this linear programming is equivalent to the maximum-weight perfect matching in a bipartite graph, which is solvable in $O((m+n)^3)$ [9].

To ensure BIC and IIR, we let $U_i(h_i, h_i) = 0$ because $U_i(h_i, h_i) \geq 0$ and the mechanism pays the minimum expected money if it equals to 0, i.e.

$$\forall i \forall c_i, U_i(c_i, c_i) - \int_{c_i}^{h_i} B_i(a_i) da_i = 0. \tag{4}$$

Then by the definition of $U_i(c_i, c_i)$,

$$\begin{aligned} \int t_i(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} = & \int \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c_i, c_{-i}) \\ & (h_i - p_i(c))^+ (p_i(c_i, c_{-i}) - c_i) \\ & f_{-i}(c_{-i})/h_i dc_{-i} \\ & - \int_{c_i}^{h_i} B_i(a_i) da_i. \end{aligned}$$

By the definition of function B_i,

$$\begin{aligned} \int_{c_i}^{h_i} B_i(a_i) da_i = \int_{c_i}^{h_i} \int \sum_{j=1}^{n} \alpha_{ij} x_{ij}(a_i, c_{-i}) \\ (h_i - p_i(a_i, c_{-i}))^+ f_{-i}(c_{-i})/h_i dc_{-i} da_i. \end{aligned}$$

If we let

$$\begin{aligned} \forall i \forall c, t_i(c_i, c_{-i}) = & \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c_i, c_{-i})(h_i - p_i(c))^+ \\ & (p_i(c_i, c_{-i}) - c_i)/h_i - \int_{c_i}^{h_i} \sum_{j=1}^{n} \alpha_{ij} \\ & x_{ij}(a_i, c_{-i})(h_i - p_i(a_i, c_{-i}))^+ f_i(a_i)/h_i da_i, \end{aligned} \tag{5}$$

(4) holds.

Now we prove that ODRM is BIC and IIR. Firstly we prove that each seller's clicked probability decreases as reported cost increases by the following lemma.

LEMMA 3.6. *For any output ranking of ODRM with any input profile $c^{'}$, let $q_i(c^{'}) = \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c^{'})$, then*

$$\forall i \forall c_{-i}^{'} \forall c_i^{'} \forall c_i (c_i^{'} \geq c_i), q_i(c_i, c_{-i}^{'}) \geq q_i(c_i^{'}, c_{-i}^{'}).$$

PROOF. By contradiction, we assume there exist some $c_i, c_{-i}^{'}$ and $c_i^{'} \geq c_i$, such that $q_i(c_i, c_{-i}^{'}) < q_i(c_i^{'}, c_{-i}^{'})$. Let S_1 denote the set of feasible rankings $x^{'}$ that satisfies

$$\forall j, x_{ij}^{'} = x_{ij}(c_i, c_{-i}^{'})$$

and S_2 denote the set of feasible rankings $x^{'}$ that satisfies

$$\forall j, x_{ij}^{'} = x_{ij}(c_i^{'}, c_{-i}^{'}).$$

Let $w_i(c_i) = (h_i - p_i(c))^+ p_i(c)/h_i$, by the definition of the allocation,

$$\begin{aligned} & q_i(c_i, c_{-i}^{'}) w_i(c_i) + max_{x^{'} \in S_1} \sum_{k \neq i} \sum_{j=1}^{n} \alpha_{kj} x_{kj}^{'} w_k(c_k^{'}) \geq \\ & q_i(c_i^{'}, c_{-i}^{'}) w_i(c_i) + max_{x^{'} \in S_2} \sum_{k \neq i} \sum_{j=1}^{n} \alpha_{kj} x_{kj}^{'} w_k(c_k^{'}), \end{aligned}$$

and

$$\begin{aligned} q_i(c_i, c_{-i}^{'}) < q_i(c_i^{'}, c_{-i}^{'}) \\ w_i(c_i, c_{-i}^{'}) \geq w_i(c_i^{'}, c_{-i}^{'}), \end{aligned}$$

we get

$$q_i(c_i, c'_{-i})w_i(c'_i) + max_{x' \in S_1} \sum_{k \neq i} \sum_{j=1}^{n} \alpha_{kj} x'_{kj} w_k(c'_k) >$$

$$q_i(c'_i, c'_{-i})w_i(c'_i) + max_{x' \in S_2} \sum_{k \neq i} \sum_{j=1}^{n} \alpha_{kj} x'_{kj} w_k(c'_k).$$

That is, $x(c'_i, c'_{-i})$ is not the ranking with the maximum volume for input profile (c'_i, c'_{-i}), which contradicts the assumption. □

From these facts of each seller i's the expected clicked probability and $p_i(c_i)$ are non-increasing as c_i increases and $p_i(c_i)$ is larger than $h_i/2$, we obtain that function $B_i(c_i)$ is monotone non-increasing as c_i increases. As the transfer rule is defined by (5), ODRM satisfies (1) and (2). By Theorem 1, we come to the conclusion that this mechanism is BIC, IIR and feasible.

4 INDIRECT MECHANISMS

In this section we establish the connections between indirect mechanisms with at least an IIR-BNE in the original setting and IIR, BIC and direct mechanisms in the related setting. Then we use this connection to construct the optimal indirect ranking mechanism(OIRM) that implements ODRM. We begin transformations in one direction with the following lemma with the idea of revelation principle.

LEMMA 4.1. *For any indirect mechanism f with at least an IIR-BNE, let x denote the allocation function, t denote the transfer function of f. For any IIR-BNE s of f, we define a direct mechanism g with allocation funtion x', pricing fuction p and transfer function t' that satisfies*

$$\forall i \forall j \forall c, x'_{ij}(c) = x_{ij}(s(c))$$

$$\forall i \forall c, p'_i(c) = s_i(c_i)$$

$$\forall i \forall c, t'_i(c) = t_i(s(c)),$$

then g is IIR, BIC and for any input type profile, the allocation, the prices, and the transfers of f in the BNE are the same with those of g.

PROOF. It suffices to prove that mechanism g is IIR and BIC. For any seller i, Let $U_i(g, c_i, c'_i)$ denote the interim utility of seller i with type c_i reporting c'_i when others tell the truth in mechanism g, and $U_i(f, c_i, p_i)$ denote the interim utility of seller i with type c_i posting the price p_i when others follow the strategies s_{-i} in mechanism f. By the construction of g,

$$\forall i \forall c_i, U_i(g, c_i, c'_i) = U_i(f, c_i, s_i(c'_i)).$$

By the definition of s,

$$\forall i \forall c_i, U_i(f, c_i, s_i(c_i)) \geq U_i(f, c_i, s_i(c'_i)).$$

Then $\forall i \forall c_i, U_i(g, c_i, c_i) \geq U_i(g, c_i, c'_i)$. Thus mechanism g is BIC. Because s is an IIR-BNE,

$$\forall i \forall c_i, U_i(f, c_i, s_i(c_i)) \geq 0.$$

We have $\forall i \forall c_i, U_i(g, c_i, c_i) \geq 0$, making g is IIR. □

The above lemma states that the optimal indirect mechanism's volume is less than that of the optimal direct mechanism.

Conversely, we characterize the class of IIR, BIC and direct mechanisms which can be implemented by indirect mechanisms.

Definition 4.2. Given a direct mechanism g, we define function G_i and D_i for each seller i,

$$G_i(c_i) = \int \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c)(h_i - p_i(c))^+ (p_i(c) - c_i) \\ f_{-i}(c_{-i})dc_{-i} / \int \sum_{j=1}^{n} \alpha_{ij} x_{ij}(c) f_{-i}(c_{-i})dc_{-i}. \quad (6)$$

$$D_i(c_i) = \frac{h_i + c_i + \sqrt{(h_i - c_i)^2 - 4G_i(c_i)}}{2}. \quad (7)$$

A mechanism is invertible if

$$\forall c, c' (\forall i, D_i(c_i) = D_i(c'_i)), x(c') = x(c)$$

$$\forall c, c' (\forall i, D_i(c_i) = D_i(c'_i)), t(c') = t(c).$$

That is, an invertible mechanism should satisfy the property: for any c, the outcome of this mechanism on two inputs c and c' such that $D_i(c_i) = D_i(c'_i)$ for each seller i are the same. Note that this class contains many mechanisms, as a direct mechanism with monotone price function $p_i(c)$ which only depends on c_i[5] for each seller i is within this class.

For any IIR and BIC mechanism, we prove that a mechanism is invertible if and only if there exists an indirect mechanism that implements it.

LEMMA 4.3. *For any IIR ,BIC and direct mechanism g with the allocation function x, the pricing function p and the transfer function t, there exists an indirect mechanism f with the allocation function x', and the transfer function t' where there exists an IIR-BNE s that satisfies*

$$\forall i \forall j \forall c, x'_{ij}(s(c)) = x_{ij}(c)$$

$$\forall i \forall c_i, s_i(c_i) = D_i(c_i).$$

$$\forall i \forall c, t'_i(s(c)) = t_i(c).$$

and for any type profile, the allocation, the transfer, and interim utility of sellers of f and g are the same if and only if g is invertible.

PROOF. On the one hand, If g is invertible, it suffices to construct an indirect mechanism f that satisfies the above conditions. For each seller i, let $I_i([0, h_i])$ be the image of function $D_i(c_i)$ on $[0, h_i]$. Then for each seller i, we define an inverse function of D_i, C_i on the set $I_i([0, h_i])$, i.e. $\forall a \in I_i([0, h_i]), D_i(C_i(a)) = a$.[6] We define function

$$C(p) = (C_1(p_1), ..., C_m(p_m))$$

for any input vector p such that the price of any seller i is in the domain of C_i.

The allocation and the transfer are designed as follows:

$$\forall i \forall j \forall p, x'_{ij}(p) = \begin{cases} x_{ij}(C(p)), p_i \in I_i([0, h_i]) \\ 0, p_i \notin I_i([0, h_i]) \end{cases} \quad (8)$$

[5] $D_i(c_i) = p_i(c)$ and is monotone in this case.

[6] For any value $a \in I_i([0, h_i])$ with multiple $b \in [0, h_i]$ such that $D_i(b) = a$, we choose the smallest one as the value of $C_i(a)$.

$$\forall i \forall p, t_i^{'}(p) = \begin{cases} t_i(C(p)), p_i \in I_i([0, h_i]) \\ 0, p_i \notin I_i([0, h_i]) \end{cases} \quad (9)$$

If there is another seller j that posts a price that is not in the domain of C_j, we ignore and remove the seller. Now we prove that s is a BNE of the mechanism f. Let $U_i(g, c_i, c_i^{'})$ denote the interim utility of seller i with type c_i reporting $c_i^{'}$ in mechanism g, and $U_i(f, c_i, p_i)$ denote the interim utility of seller i with type c_i posting the price p_i in mechanism f. It suffices to prove that

$$\forall i \forall c_i \forall p_i, U_i(f, c_i, s_i(c_i)) \geq U_i(f, c_i, p_i). \quad (10)$$

By the construction of f,

$$U_i(f, c_i, s_i(c_i)) = \int (\sum_{j=1}^{n} \alpha_{ij} x_{ij}^{'}(D(c))(h_i - D_i(c_i))^+ (D_i(c_i) - c_i)/h_i - t_i^{'}(D(c)) f_{-i}(c_{-i}) dc_{-i}. \quad (11)$$

By the definition of interim utility,

$$U_i(g, c_i, c_i) = \int (\sum_{j=1}^{n} x_{ij}(c) \alpha_{ij} (h_i - p_i(c_i))^+ (p_i(c_i) - c_i)/h_i - t_i(c)) f_{-i}(c_{-i}) dc_{-i}. \quad (12)$$

As mechanism g is invertible,

$$x_{ij}^{'}(D(c)) = x_{ij}(c)$$
$$t_i^{'}(D(c)) = t_i(c). \quad (13)$$

Then by (6)(7)(11)(12)(13), we get

$$U_i(f, c_i, s_i(c_i)) = U_i(g, c_i, c_i).$$

Because mechanism g is IIR, $U_i(g, c_i, c_i) \geq 0$. If seller i posts a price p_i that is not in the domain of C_i, the interim utility he will get, $U_i(f, c_i, p_i)$, is 0. Otherwise

$$U_i(f, c_i, p_i) = U_i(f, c_i, s_i(C_i(p_i))) = U_i(g, c_i, C_i(p_i)). \quad (14)$$

As g is BIC, $U_i(g, c_i, c_i) \geq U_i(g, c_i, C_i(p_i))$. Thus (10) holds, and in the BNE each seller gets non-negative interim utility.

On the other hand, if there exists an indirect mechanism f that implements mechanism g, by the definition of f, we have

$$\forall c, x^{'}(D(c)) = x(c)$$
$$\forall c, t^{'}(D(c)) = t(c).$$

If there exists $c^{'}$ such that $D(c^{'}) = D(c)$, it is natural to get that $x(c) = x(c^{'})$ and $t(c) = t(c^{'})$.

□

4.1 Optimal indirect mechanism that maximizes volume

In ODRM, the value of $D_i(c_i)$ for any seller i is $h_i/2$ if the reported price is less than $h_i/2$, otherwise the value is c_i. ODRM is not invertible because a seller reporting cost c_i and $c_i^{'}$ such that $D_i(c_i) = D_i(c_i^{'}) \leq h_i/2$ may get a different fraction of the buyer impression and pay different money. We are not able to use the technique in the proof of Lemma 4.3 directly. Here we present a class of BIC, IIR, invertible and direct mechanisms which are parameterized by a constant $\eta(0 < \eta \leq 1)$. If we let η approach to 0, its volume approaches to the optimum.

MECHANISM η-OPTIMAL. *For each value of $\eta(0 < \eta \leq 1)$, the price function of each seller i is*

$$p_i(c_i, c_{-i}^{'}) = \begin{cases} \eta c_i + (1-\eta) h_i/2, c_i \leq h_i/2 \\ c_i, c_i > h_i/2 \end{cases},$$

the transfer function is the same as (5) and the allocation is calculated by (3).

THEOREM 4.4. *For any choice of $\eta \in (0, 1]$, mechanism η-optimal is BIC, IIR, invertible, and mechanism η-optimal's volume approaches to that of ODRM as η approaches to 0.*

PROOF. For each value of $\eta \in (0, 1]$, by definition, $q_i(c_i, c_{-i})$ decreases as the reported cost c_i increases and the posted price function is increasing as the reported cost increases. Thus the probability that the item of each seller is purchased is monotone decreasing with his reported cost. As the transfer rule of the mechanism satisfies the equation (5) and the mechanism satisfies (1) and (2), by Theorem 1, we get that the mechanism is IIR and BIC.

By definition, the function $D_i(c_i)$ is strictly monotone increasing with the reported cost, for any c, $C(D(c)) = c$, the mechanism is invertible.

For any type profile, the allocation of mechanism η-optimal and the prices mechanism η-optimal sets approaches to that of ODRM as η approaches to 0. By definition, mechanism η-optimal's volume approaches to that of ODRM as η approaches to 0. □

We can use (8) and (9) to acquire a class of indirect mechanisms called OIRM that maximize volume with at least an IIR-BNE parameterized by η, and its volume approaches to that of ODRM.

5 THE SITE'S MECHANISM

In this section, we present the mechanism applied by one of the largest websites in the world, denoted by $\mathcal{S}$, and we obtain a simple BNE of the mechanism with the assumption of i.i.d uniform distribution $U(0, 1)$ of costs.

MECHANISM $\mathcal{S}$. *The mechanism ranks the sellers according to the weighted volume $p_i(h_i - p_i)^+/h_i$ by descending order and does not use the transfer.*

By definition of $\mathcal{S}$, it does not use the transfer to ensure the incentive for sellers. Thus it's natural to ask whether there exists a simple BNE of the mechanism or not. We consider the case that items of sellers are the same and the cost distributions of all sellers are i.i.d drawn from the uniform distribution $U(0, 1)$ and there is one slot to be allocated. For the ease of notations, we let $\alpha_{11} = .. = \alpha_{m1} = 1$. Let s denote the BNE of $\mathcal{S}$. By LEMMA 4.1. We get a direct mechanism g that satisfies BIC and IIR. Applying the results of equation (1) in THEOREM 3.5, we get that for any seller i and any cost c_i,

$$\frac{\mathrm{d}s_i(c_i)}{\mathrm{d}c_i}(2s_i(c_i) - c_i - 1)Q_i(c_i) = \frac{\mathrm{d}Q_i(c_i)}{\mathrm{d}c_i}(1 - s_i(c_i))(s_i(c_i) - c_i), \quad (15)$$

where

$$Q_i(c_i) = \int x_{i1}(s_i(c_i), s_{-i}(c_{-i})) f_{-i}(c_{-i}) dc_{-i}.$$

Firstly we get some properties of the strategy function of each seller i.

LEMMA 5.1. *If s is a BNE of $\mathcal{S}$, for any seller i and any type $c_i \in [0,1]$,*

$$1/2 \le s_i(c_i) \le (1+c_i)/2.$$

PROOF. By the definition of the interim utility,

$$U_i(f, c_i, p_i) = \int x_{i1}(p_i, s_{-i}(c_{-i}))(1-p_i)^+(p_i - c_i)f_{-i}(c_{-i})dc_{-i}.$$

As $\mathcal{S}$ gives the slot to the seller with posted prices nearest to $1/2$, and the function $g(p_i) = (1-p_i)^+(p_i - c_i)$ attains larger value as p_i approaches to $(1+c_i)/2$.

Firstly we get that for any i and c_i, $s_i(c_i) \ge 1/2$, otherwise the seller posting a price $1/2$ will get more fraction of buyer impression, i.e. $x_{i1}(p_i, s_{-i}(c_{-i}))$, more revenue selling one item, i.e. $(1-p_i)^+(p_i - c_i)$ and get more interim utility because $1/2$ is closer to $1/2$ and $(1+c_i)/2$ than $s_i(c_i)$.

Secondly for any i and c_i, $s_i(c_i) \le (1+c_i)/2$, otherwise let $p_i = 1 + c_i - s_i(c_i)$, the seller posting the price p_i will get more fraction of buyer impression as p_i is closer to $1/2$ than $s_i(c_i)$ and the same revenue selling one item because the distance between p_i and $(1+c_i)/2$ is the same as the distance between $s_i(c_i)$ and $(1+c_i)/2$, and get more interim utility. □

To get the close form of function $Q_i(c_i)$, we assume that the strategy function of each seller is monotone non-decreasing and symmetric. By LEMMA 5.1, the posted price of the seller with the minimum cost is closest to $1/2$, which means that for any type profile c, $\mathcal{S}$ gives the slot to the seller with the minimum cost. Thus $Q_i(c_i)$ is the expected probability that c_i is the minimum cost among all sellers, i.e. $(1-c_i)^{m-1}$. By solving the differential equation (15), we get that

$$s_i(c_i) = \frac{mc_i + 1}{m+1}.$$

It is not hard to verify that these functions are symmetric and monotone non-decreasing. Combining this equation with LEMMA 5.1, we get the BNE of $\mathcal{S}$.

THEOREM 5.2. *Let*

$$s_i(c_i) = \begin{cases} 1/2 & 0 \le c_i < \frac{m-1}{2m} \\ \frac{mc_i+1}{m+1} & \frac{m-1}{2m} \le c_i \le 1 \end{cases},$$

then s is a BNE of $\mathcal{S}$.

PROOF. It suffices to prove that $s_i(c_i)$ is the best response of seller i with cost c_i when others follow s_{-i}. For any seller i, If others following the strategies s_{-i}, others's posted prices is $1/2$ with probability $\frac{m-1}{2m}$, and a uniform distribution $U(1/2, 1)$ with density $\frac{m+1}{m}$. We only need to consider the case that $p_i \ge 1/2$ by LEMMA 5.1, the seller's expected clicked probability is $((1-p_i)\frac{m+1}{m})^{m-1}$. The interim utility of seller i with cost c_i posting a price p_i is

$$U_i(f, c_i, p_i) = (1-p_i)(p_i - c_i)((1-p_i)\frac{m+1}{m})^{m-1}. \quad (16)$$

It is easy to verify that $U_i(f, c_i, p_i)$ attains the maximum value when $p_i = s_i(c_i)$. □

This result explains that why the site uses this mechanism and gives sellers a simple BNE strategy to follow. Also, we get the gap between the volume of OIRM mechanism and $\mathcal{S}$. We omit the proof due to lack of space.

LEMMA 5.3. *For any number m of sellers, we have*

$$Vol(OIRM) - Vol(f) \ge \frac{3m+2}{4(m+1)(m+2)}((\frac{m+1}{2m})^m - (\frac{1}{2})^m). \quad (17)$$

6 EXPERIMENTAL EVALUATION

Besides the theoretical analysis of $\mathcal{S}$ with i.i.d uniform distribution $U(0,1)$ of costs and uniform valuation of buyers, we simulate OIRM and compare its performance with $\mathcal{S}$ based on the trading dataset from one of the largest e-commerce sites in the world.

- **Dataset**
  The relational dataset contains a history of 9354 different items controlled by different sellers matching a certain keyword in 64 days. Each record in the dataset contains a daily record of the number of buyers clicks that an item i received $v_i(p_i)$, the item's number of transactions $n_i(p_i)$, and the price of the item p_i. We delete records that correspond to items that were not sold even once during 64 days and filter all items with price lower than 1 RMB[7]. Then we get a new relational dataset with 579 sellers. The click-through rates α of these sellers over different slots is provided by the site.
- **Estimate the valuation of buyer**
  By the assumption that the valuation of interested buyers over each item i is a uniform distribution $U[0, h_i]$, the conversion rate(the ratio between the number of transactions and the number of clicks of this item) given a price i is $(h_i - p_i)^+/h_i$. We then use linear interpolation to fit parameters h_i of each seller i.
- **Estimate the cost distributions**
  We estimate sellers' cost distributions according to the dataset of prices by the following two steps.
  1. First, we construct each seller's price distribution to be the uniform distribution over the dataset of the prices, called empirical distribution [11].
  2. Second, we assume that sellers' price distributions compose a Bayes Nash equilibrium (BNE) in $\mathcal{S}$. So for seller i with cost c_i and price p_i, the following first-order condition is satisfied:

  $$\frac{d(x^*(p_i)(p_i - c_i)(h_i - p_i)^+/h_i)}{dp_i} = 0,$$

  where $x^*(p_i)$ is the interim clicked probability, i.e.,

  $$x^*(p_i) = \int_{p_{-i}} \sum_{j=1}^{n} \alpha_{ij} x_{ij}(p_i, p_{-i}) dp_{-i}.$$

  Note that both $x^*(p_i)$ and its derivation can be inferred from the price distributions and the formula above is a linear equation of c_i. So the empirical distribution of c_i can be computed by solving the equation above.
- **Simulation and Results**
  Given buyer valuations and distributions of costs of each seller, we randomly sample $m(100 - 1000)$ sellers from 579 sellers, and simulate OIRM, $\mathcal{S}$ and uniform mechanism for 10000 times, then we calculate these mechanisms' volume.

[7] We calculate the sum of trading volume of these items and their effect on volume is negligible.

Uniform mechanism means each seller gets the same fraction of buyer imperssions, and pays nothing. So each seller i with c_i will post price $argmax_{p_i}(p_i - c_i)(h_i - p_i)^+/h_i$ to maximize his expected profit.

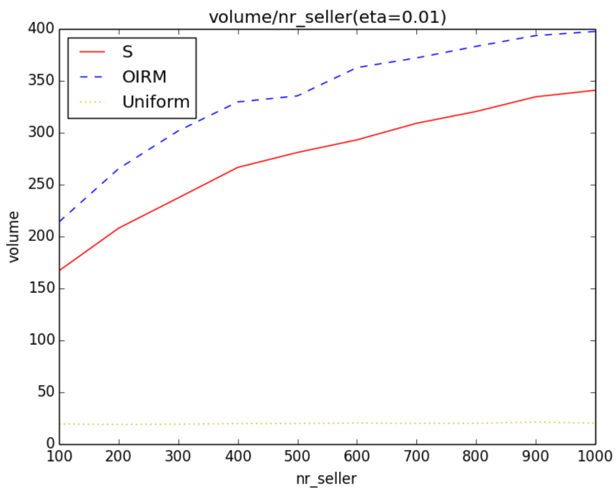

Figure 1: Volume per number of sellers

Figure 1 shows the volume of these mechanisms with the change of the sample size, it illustrates that OIRM outperforms $\mathcal{S}$ significantly and the volume of the uniform mechanism is inferior compared with other mechanisms. The volume of OIRM and $\mathcal{S}$ increases as the size of the sample increases while the uniform mechanism does not.
We also simulate OIRM with different parameters η and 100 sellers and calculate the volume and the expected total transfer of all sellers.

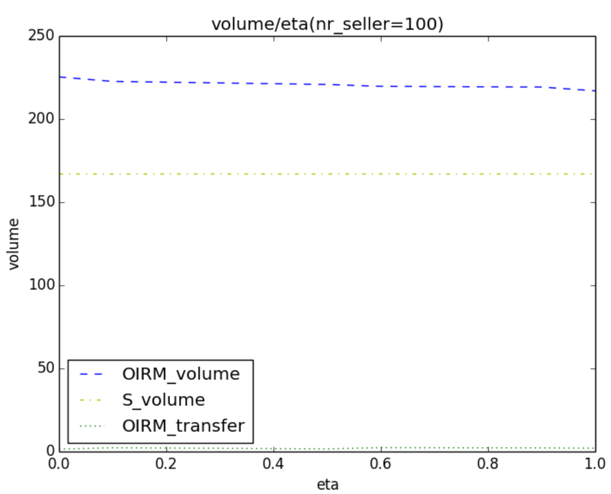

Figure 2: Volume and transfer per η

Figure 2 shows that the volume of OIRM decreases as η increases, but OIRM always performs better than $\mathcal{S}$ for any η. Also, the expected total transfer of OIRM is positive for any η, the designer needs not to pay any money.
We randomly sample 50 sellers and calculate the expected posted prices of each seller i in $\mathcal{S}$, OIRM and each seller's optimal price to maximize the expected trading volume given all buyer impressions, i.e.

$$argmax_{p_i} p_i(h_i - p_i)^+/h_i.$$

Note that both $\mathcal{S}$ and OIRM rank sellers by it, thus the performance of these mechanisms will be better if posted prices of sellers are closer to optimal prices. We sort these sellers by the expected trading volume given all buyer impressions. Figure 3 shows that the posted prices of OIRM and $\mathcal{S}$ are larger than optimal prices for each seller, and the posted prices of OIRM are closer to optimal prices compared with the posted prices of $\mathcal{S}$.

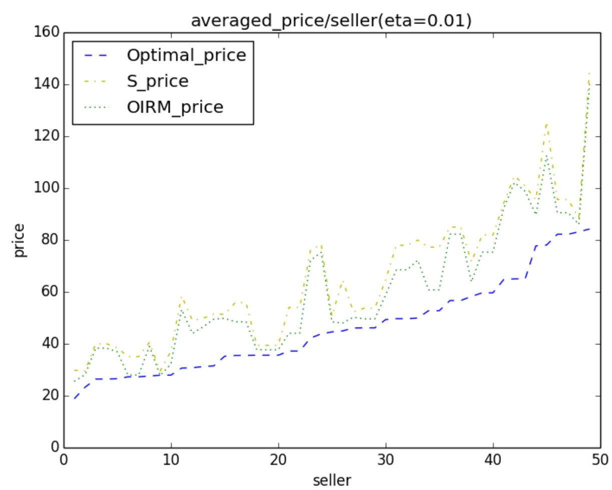

Figure 3: Prices of sellers

7 FUTURE WORK

In this paper, we assume that each seller sells an item and the probability that the buyer clicks on sellers' items is independent of posted prices. Future work could consider designing optimal indirect mechanisms in the setting where each seller has multiple items or the clicked probability depends on prices. We consider implementing OIRM in e-commerce a promising direction.

We get the BNE of the site's mechanism with the assumption that the distribution of costs is i.i.d uniform and we only need to allocate one slot. It's also important to solve the BNE of the mechanism in a more general setting.

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