
The Power of Predictions in Online Control

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Abstract

We study the impact of predictions in online Linear Quadratic Regulator control with both stochastic and adversarial disturbances in the dynamics. In both settings, we characterize the optimal policy and derive tight bounds on the minimum cost and dynamic regret. Perhaps surprisingly, our analysis shows that the conventional greedy MPC approach is a near-optimal policy in both stochastic and adversarial settings. Specifically, for length- T problems, MPC requires only $O(\log T)$ predictions to reach $O(1)$ dynamic regret, which matches (up to lower-order terms) our lower bound on the required prediction horizon for constant regret.

1 Introduction

This paper studies the effect of using predictions for online control in a linear dynamical system governed by $x_{t+1} = Ax_t + Bu_t + w_t$, where x_t , u_t , and w_t are the state, control, and disturbance (or exogenous input) respectively. At each time step t , the controller incurs a quadratic cost $c(x_t, u_t)$. Recently, considerable effort has been made to leverage and integrate ideas from learning, optimization and control theory to study the design of optimal controllers under various performance criteria, such as static regret [2, 3, 12, 13, 15, 20, 29], dynamic regret [16, 23] and competitive ratio [17, 28]. However, the study of online convergence when incorporating predictions has been largely absent.

Indeed, a key aspect of online control is considering the amount of available information when making decisions. Most recent studies focus on the basic setting where only past information, x_0, w_0, \dots, w_{t-1} , is available for u_t at every time step [2, 13, 15, 28]. However, this basic setting does not effectively characterize situations where we have accurate predictions, e.g., when $x_0, w_0, \dots, w_{t-1+k}$ are available at step t . These types of accurate predictions are often available in many applications, including robotics [8, 27], energy systems [30], and data center management [22]. Moreover, there are many practical algorithms that leverage predictions, such as the popular Model Predictive Control (MPC) [6–9, 18, 19].

While there has been increased interest in studying online guarantees for control with predictions, to our knowledge, there has been no such study for the case of a finite-time horizon with disturbances. Several previous works studied the economic MPC problem by analyzing the asymptotic performance without disturbances [6, 7, 18, 19]. Rosolia and Borrelli [25, 26] studied learning for MPC but focused on the episodic setting with asymptotic convergence guarantees. Li et al. [23] considered a linear system where finite predictions of costs are available, and analyzed the dynamic regret of their new algorithm; however, they neither consider disturbances nor study the more practically relevant MPC approach. Goel and Hassibi [16] characterized the offline optimal policy (i.e., with infinite predictions) and cost in LQR control with i.i.d. zero-mean stochastic disturbances, but those results do not apply to limited predictions or non-i.i.d. disturbances. Other prior works analyze the power of

predictions in online optimization [11, 24], but the connection to online control in dynamical systems is unclear.

From this literature, fundamental questions about online control with predictions have emerged:

1. *What are the cost-optimal and regret-minimizing policies when given k predictions? What are the corresponding cost and regret of these policies?*
2. *What is the marginal benefit from each additional prediction used by the policy, and how many predictions are needed to achieve (near-)optimal performance?*
3. *How well does MPC with k predictions perform compared to cost-optimal and regret-minimizing policies?*

Main contributions. We systematically address each of the questions above in the context of LQR systems with general stochastic and adversarial disturbances in the dynamics. In the stochastic case, we explicitly derive the cost-optimal and dynamic regret minimizing policies with k predictions. In both the stochastic and adversarial cases, we derive (mostly tight) upper bounds for the optimal cost and minimum dynamic regret given access to k predictions. We also show that the marginal benefit of an extra prediction exponentially decays as k increases. Additionally, for MPC specifically, we show that it has a bounded performance ratio against the cost-optimal policy in both stochastic and adversarial settings. We further show that MPC is near-optimal in terms of dynamic regret, and needs only $O(\log T)$ predictions to achieve $O(1)$ dynamic regret (the same order as is needed by the dynamic regret minimizing policy) in both settings.

We would like to emphasize the generality of the results. The model we consider is the general LQR setting with disturbance in the dynamics, where only the stabilizability of $[A, B]$ and $[A^\top, Q]$ is assumed [4]. Further, in the stochastic setting we consider general distributions, which are not necessarily i.i.d. or zero-mean. Additionally, our results compare to the *globally* optimal policies for cost and regret rather than compare to the optimal linear or static policy. Finally, our upper bounds are (almost) *tight*, i.e., there exist some systems such that the bounds are (nearly) reached, up to lower-order terms.

It is perhaps surprising that classic MPC, which is a simple greedy policy (up to the prediction horizon), is near-optimal even with adversarial disturbances in the dynamics. Our results thus highlight the power of predictions to reduce the need for algorithmic sophistication. In that sense, our results somewhat mirror recent developments in the study of exploration strategies in online LQR control with unknown dynamics $\{A, B\}$: after a decade’s research beginning with the work of Abbasi-Yadkori and Szepesvári [1], Simchowitz and Foster [29] recently show that naive exploration is optimal. Taken together with the result from [29], our paper provides additional evidence for the idea that the structure of LQR allows simple algorithmic ideas to be effective, which sheds light on key algorithmic principles and fundamental limits in continuous control.

2 Background and model

We consider the *Linear Quadratic Regulator (LQR)* optimal control problem with disturbances in the dynamics. In particular, we consider a linear system initialized with $x_0 \in \mathbb{R}^n$ and controlled by $u_t \in \mathbb{R}^d$, with dynamics

$$x_{t+1} = Ax_t + Bu_t + w_t \quad \text{and cost} \quad J = \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) + x_T^\top Q_f x_T,$$

where $T \geq 1$ is the total length of the control period. The goal of the controller is to minimize the cost given A, B, Q, R, Q_f, x_0 , and the characterization of the disturbance w_t . Throughout this paper, we use $\rho(\cdot)$ to denote the spectral radius of a matrix and $\|\cdot\|$ to denote the 2-norm of a vector or the spectral norm of a matrix.

We assume $Q, Q_f \succeq 0, R \succ 0$ and the pair (A, B) is *stabilizable*, i.e., there exists a matrix $K_0 \in \mathbb{R}^{d \times n}$ such that $\rho(A - BK_0) < 1$. Further, we assume the pair (A, Q) is *detectable*, i.e., (A^\top, Q) is stabilizable, to guarantee stability of the closed-loop. Note that detectability of (A, Q) is more general than $Q \succ 0$, i.e., $Q \succ 0$ implies (A, Q) is detectable. For w_t , in the stochastic case,

we assume $\{w_t\}_{t=0,1,\dots}$ are sampled from a joint distribution with bounded cross-correlation, i.e., $\mathbb{E}[w_t^\top w_{t'}] \leq m$ for any t, t' ; in the adversarial case, we assume w_t is picked from a bounded set Ω .

These are standard assumptions in the literature, e.g., [13, 15, 29] and it is worth noting that our notion of stochasticity is much more general than typically considered [10, 12, 13]. We also note that many important problems can be straightforwardly converted to our model — for example, input-disturbed systems and the Linear Quadratic (LQ) tracking problem [4].

Example: linear quadratic tracking. The standard quadratic tracking problem is defined with dynamics $x_{t+1} = Ax_t + Bu_t + \tilde{w}_t$ and cost function $J = \sum_{t=0}^{T-1} (x_{t+1} - d_{t+1})^\top Q (x_{t+1} - d_{t+1}) + u_t^\top Ru_t$, where $\{d_t\}_{t=1}^T$ is the desired trajectory to track. To map this to our model, let $\tilde{x}_t = x_t - d_t$. Then, we get $J = \sum_{t=0}^{T-1} \tilde{x}_{t+1}^\top Q \tilde{x}_{t+1} + u_t^\top Ru_t$ and $\tilde{x}_{t+1} = A\tilde{x}_t + Bu_t + w_t$, which is an LQR control problem with disturbance $w_t = \tilde{w}_t + Ad_t - d_{t+1}$ in the dynamics.

2.1 Predictions

In the classic model, at each step t , the controller decides u_t after observing w_{t-1} and x_t . In other words, u_t is a function of all the previous information: x_0, x_1, \dots, x_{t-1} and w_0, w_1, \dots, w_{t-1} , or equivalently, of $x_0, w_0, w_1, \dots, w_{t-1}$. We describe this scenario via the following *event sequence*:

$$x_0 \quad u_0 \quad w_0 \quad u_1 \quad w_1 \quad \cdots \quad u_{T-1} \quad w_{T-1},$$

where each u_t denotes the decision of a control policy, each w_t denote the observation of a disturbance, and each decision may depend on previous events.

However, in many real-world applications the controller may have some knowledge about future. In particular, at time step t , the controller may have *predictions* of immediate k future disturbances and make decision u_t based on $x_0, w_0, \dots, w_{t+k-1}$. In this case, the event sequence is given by:

$$x_0 \quad w_0 \quad w_1 \quad \cdots \quad w_{k-1} \quad u_0 \quad w_k \quad u_1 \quad w_{k+1} \quad \cdots \quad u_{T-k-1} \quad w_{T-1} \quad u_{T-k} \quad \cdots \quad u_{T-1}.$$

The existence of predictions is common in many applications such as disturbance estimation in robotics [27] and model predictive control (MPC) [9], which is a common approach for the LQ tracking problem. When given k predictions of d_t , the LQ tracking problem can be formulated as a LQR problem with $k - 1$ predictions of future disturbances. In this paper we assume all the predictions are *exact*, and leave inexact predictions [11, 28] as future work. This is common in the literature on online algorithms with predictions, e.g., [23, 24].

2.2 Disturbances

The characteristics of the disturbances have a fundamental impact on the optimal control policy and cost. We consider two types of disturbance: stochastic disturbances, which are drawn from a joint distribution (not necessarily i.i.d.), and adversarial disturbances, which are chosen by an adversary to maximize the overall control cost of the policy.

In the stochastic setting, we model the disturbance sequence $\{w_t\}_{t=0}^{T-1}$ as a discrete-time stochastic process with joint distribution \mathcal{W} which is known to the controller. Let $W_t = W_t(w_0, \dots, w_{t-1})$ be the conditional distribution of w_t given w_0, \dots, w_{t-1} . Then the cost of the optimal online policy with k predictions is given by:

$$STO_k^T = \mathbb{E}_{w_0 \sim W_0, \dots, w_{k-1} \sim W_{k-1}} \left(\min_{u_0} \left(\mathbb{E}_{w_k \sim W_k} \left(\cdots \min_{u_{T-k-1}} \left(\mathbb{E}_{w_{T-1} \sim W_{T-1}} \left(\min_{u_{T-k}, \dots, u_{T-1}} J \right) \right) \right) \right) \right).$$

Note that the cost $J = J(x_0, u_0, \dots, u_{T-1}, w_0, \dots, w_{T-1})$. Two extreme cases are noteworthy: $k = 0$ reduces to the classic case without prediction and $k = T$ reduces to the offline optimal.

In the adversarial setting, each disturbance w_t is selected by an adversary from a bounded set $\Omega \subseteq \mathbb{R}^n$ in order to maximize the cost. The controller has no information about the disturbance except that it is in Ω . Similar to the stochastic setting, we define:

$$ADV_k^T = \sup_{w_0, \dots, w_{k-1} \in \Omega} \left(\min_{u_0} \left(\sup_{w_k \in \Omega} \left(\cdots \min_{u_{T-k-1}} \left(\sup_{w_{T-1} \in \Omega} \left(\min_{u_{T-k}, \dots, u_{T-1}} J \right) \right) \right) \right) \right).$$

This can be viewed as online \mathcal{H}_∞ control [31] with predictions.

Algorithm 1: Model predictive control with k predictions

Parameter: $\{A, B, Q, R\}$ and $\tilde{Q}_f \in \mathbb{R}^{n \times n}$ **Input:** x_0, w_0, \dots, w_{k-1} 1 **for** $t = 0$ **to** $T - 1$ **do** **Input:** x_t, w_{t+k-1} // The controller now knows $x_0, \dots, x_t, w_0, \dots, w_{t+k-1}$ 2 $(u_t, \dots, u_{t+k-1}) = \arg \min_u \sum_{i=t}^{t+k-1} (x_i^\top Q x_i + u_i^\top R u_i) + x_{t+k}^\top \tilde{Q}_f x_{t+k}$ subject to
 $x_{i+1} = A x_i + B u_i + w_i$ for $i = t, \dots, t + k - 1$ **Output:** u_t

The average cost in an infinite horizon is particularly important in both control and learning communities to understand asymptotic behaviors. We use separate notation for it:

$$\text{STO}_k = \lim_{T \rightarrow \infty} \frac{1}{T} \text{STO}_k^T, \quad \text{ADV}_k = \lim_{T \rightarrow \infty} \frac{1}{T} \text{ADV}_k^T.$$

We emphasize that we do not have any constraints (like linearity) on the policy space, and both STO_k^T and ADV_k^T are globally optimal with the corresponding type of disturbance. This point is important in light of recent results that show that linear policies cannot make use of predictions at all [16, 28], i.e., the cost of the best linear policy with infinite predictions ($k = \infty$) is asymptotically equal to that with no predictions ($k = 0$) in the setting with i.i.d. zero-mean stochastic disturbances. In this paper, we explicitly derive the optimal policy for every $k > 0$, which is *nonlinear* in general.

2.3 Model predictive control

Model predictive control (MPC) is perhaps the most common control policy for situations where predictions are available. MPC is a greedy algorithm with a receding horizon based on all available current predictions. Algorithm 1 provides a formal definition, and we additionally refer the reader to the book [9] for a literature review on MPC. We adopt a conventional definition of MPC as an online optimal control problem with a finite-time horizon with dynamics constraints. Note that other prior work on MPC sometimes considers other input and state constraints [9].

MPC is a practical algorithm in many scenarios like robotics [8], energy system [30] and data center cooling [22]. The existing theoretical studies of MPC focus on asymptotic stability and performance [6, 7, 18, 19, 25]. To our knowledge, we provide the first general, dynamic regret guarantee for MPC in this paper.

In this paper, we study the performance of MPC in three different cases, where disturbances are i.i.d. zero-mean stochastic, generally stochastic, and adversarial, corresponding to Sections 3 to 5 respectively. We define the performance of MPC in the stochastic and adversarial settings as follows:

$$\begin{aligned} \text{MPCS}_k^T &= \mathbb{E}_{w_0, \dots, w_{T-1}} J^{\text{MPC}_k}, & \text{MPCS}_k &= \lim_{T \rightarrow \infty} \frac{1}{T} \text{MPCS}_k^T, \\ \text{MPCA}_k^T &= \sup_{w_0, \dots, w_{T-1}} J^{\text{MPC}_k}, & \text{MPCA}_k &= \lim_{T \rightarrow \infty} \frac{1}{T} \text{MPCA}_k^T, \end{aligned}$$

where J^{MPC_k} is the cost of MPC given a specific disturbance sequence, i.e., $J^{\text{MPC}_k}(w) = J(u, w)$ where for each t , $u_t = \phi(x_t, w_t, \dots, w_{t+k-1})$ and $\phi(\cdot)$ is the function that maps $x_t, w_t, \dots, w_{t+k-1}$ to the policy u_t , as defined in Algorithm 1. By definition, $\text{MPCS}_k \geq \text{STO}_k$ and $\text{MPCA}_k \geq \text{ADV}_k$ for every $k \geq 1$ since they use the same information but the latter ones are defined to be optimal.

2.4 Dynamic regret and the performance ratio

In this paper, we focus on two performance metrics, the *dynamic regret* and the *performance ratio*.

Dynamic regret. Regret is a standard metric in online learning and provides a bound on the cost difference between an online algorithm and the optimal static policy given complete information. We focus on the *dynamic* regret, which compares to the optimal dynamic offline policy, rather than the optimal static offline policy. Note that the optimal offline policy may be nonlinear. It is important to

consider nonlinear policies because recent results highlight that the optimal offline policy can have cost that is arbitrarily lower than the optimal linear policy in hindsight [16, 28].

More specifically, we compare the cost of an online algorithm with k predictions to that of the offline optimal (nonlinear) algorithm, i.e., one that has predictions of all disturbances. For MPC with k predictions, we define its dynamic regret in the stochastic and adversarial settings, respectively, as:

$$\begin{aligned} \text{Reg}^S(\text{MPC}_k) &= \mathbb{E}_{(w_0, \dots, w_{T-1}) \sim \mathcal{W}} \left(J^{\text{MPC}_k}(w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right), \\ \text{Reg}^A(\text{MPC}_k) &= \sup_{w_0, \dots, w_{T-1} \in \Omega} \left(J^{\text{MPC}_k}(w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right). \end{aligned}$$

As compared to (static) regret, dynamic regret does not have any restriction on the policies u'_0, \dots, u'_{T-1} used for comparison and thus differs from other notions of regret where u'_0, \dots, u'_{T-1} are limited in special cases. For example, in the classic form of regret, $u'_0 = \dots = u'_{T-1}$; and in the regret compared to the best offline linear controller [2, 12], $u'_t = -K^*x_t$.

In this work, we obtain both upper bounds and lower bounds on dynamic regret. For lower bounds, we define the minimum possible regret that an algorithm with k predictions can achieve (i.e., the regret of the algorithm that minimizes the regret):

$$\begin{aligned} \text{Reg}_k^{S*} &= \mathbb{E}_{w_0, \dots, w_{k-1}} \min_{u_0} \mathbb{E} \dots \min_{w_k} \mathbb{E}_{u_{T-k-1}} \min_{w_{T-1}} \mathbb{E}_{u_{T-k}, \dots, u_{T-1}} \left(J(u, w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right), \\ \text{Reg}_k^{A*} &= \sup_{w_0, \dots, w_{k-1}} \min_{u_0} \sup_{w_k} \dots \min_{u_{T-k-1}} \sup_{w_{T-1}} \min_{u_{T-k}, \dots, u_{T-1}} \left(J(u, w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right). \end{aligned}$$

Finally, we end our discussion of dynamic regret with a note highlighting an important contrast between stochastic and adversarial settings. In the stochastic setting,

$$\begin{aligned} \text{Reg}_k^{S*} &= \mathbb{E}_{w_0, \dots, w_{k-1}} \min_{u_0} \mathbb{E} \dots \min_{w_k} \mathbb{E}_{u_{T-k-1}} \mathbb{E}_{w_{T-1}} \left(\min_{u_{T-k}, \dots, u_{T-1}} J(u, w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right) \\ &= \mathbb{E}_{w_0, \dots, w_{k-1}} \min_{u_0} \mathbb{E} \dots \min_{w_k} \mathbb{E}_{u_{T-k-1}} \mathbb{E}_{w_{T-1}} \min_{u_{T-k}, \dots, u_{T-1}} J(u, w) - \mathbb{E}_{w_0, \dots, w_{T-1}} \min_{u'_0, \dots, u'_{T-1}} J(u', w) \\ &= \text{STO}_k^T - \text{STO}_T^T. \end{aligned}$$

This equality still holds if we take $\arg \min$ instead of \min and thus the regret-optimal policy is the same as the cost-optimal policy. However, in the adversarial case, a similar reasoning gives an inequality: $\text{Reg}_k^{A*} \geq \text{ADV}_k^T - \text{ADV}_T^T$, and correspondingly, the regret-optimal and cost-optimal policies can be different. Similarly, for MPC, we have $\text{Reg}^S(\text{MPC}_k) = \text{MPCS}_k^T - \text{STO}_T^T$ while $\text{Reg}^A(\text{MPC}_k) \geq \text{MPCA}_k^T - \text{ADV}_T^T$.

Performance ratio. The second metric we study is a new metric that we term the *performance ratio*. It characterizes the ratio of the cost of an online algorithm with k predictions to the cost of the optimal online algorithm using k predictions. Thus, it gives a way of comparing to a weaker benchmark than regret – one that has the same amount of information as the algorithm. Note that it is related to, but different than, the competitive ratio in this context. Formally, the performance ratio of the MPC algorithm in stochastic and adversarial settings, respectively, is defined as:

$$\text{PR}^S(\text{MPC}_k) = \frac{\text{MPCS}_k}{\text{STO}_k}, \quad \text{PR}^A(\text{MPC}_k) = \frac{\text{MPCA}_k}{\text{ADV}_k}.$$

While the dynamic regret indicates whether the algorithm can match the optimal offline policy (which has complete information), the performance ratio measures whether the algorithm is using the information available to it in as efficient a manner as possible. Thus, the contrast between the two separates the efficiency of the algorithm from the inefficiency created by the lack of information about future disturbances.

Finally, one may wonder if there are connections between dynamic regret and performance ratio. As might be expected, in both the stochastic and adversarial settings, the performance ratio of an online policy with k predictions provides a lower bound of its dynamic regret:

$$\text{PR}^S(\text{MPC}_k) - 1 \leq \frac{\text{MPCS}_k}{\text{STO}_\infty} - 1 = \frac{\text{MPCS}_k - \text{STO}_\infty}{\text{STO}_\infty} = \frac{1}{\text{STO}_\infty} \lim_{T \rightarrow \infty} \frac{1}{T} \text{Reg}^S(\text{MPC}_k),$$

$$PR^A(\text{MPC}_k) - 1 \leq \frac{\text{MPCA}_k}{\text{ADV}_\infty} - 1 = \frac{\text{MPCA}_k - \text{ADV}_\infty}{\text{ADV}_\infty} \leq \frac{1}{\text{ADV}_\infty} \lim_{T \rightarrow \infty} \frac{1}{T} \text{Reg}^A(\text{MPC}_k).$$

3 Zero-mean i.i.d. disturbances

We begin our analysis with the simplest of the three settings we consider: the disturbances w_t are independent and identically distributed with zero mean. Though i.i.d. zero-mean is a limited setting, it is still complex enough to study predictions and the first results characterizing the optimal policy with predictions appeared only recently [15, 16], focusing only on the optimal policy when $k \rightarrow \infty$.

Before delving into our results, we first recap the classic *Infinite Horizon Linear Quadratic Stochastic Regulator* [4, 5], i.e., the case when $k = 0$:

Proposition 3.1 (Anderson and Moore [5]). *Let w_t be i.i.d. with zero mean and covariance matrix W . Then, the optimal control policy corresponding to STO_0 is given by:*

$$u_t = -(R + B^\top PB)^{-1} B^\top PAx_t =: -Kx_t,$$

where P is the solution of discrete-time algebraic Riccati equation (DARE)

$$P = Q + A^\top PA - A^\top PB(R + B^\top PB)^{-1} B^\top PA. \quad (1)$$

The corresponding closed-loop dynamics $A - BK$ is exponentially stable, i.e., $\rho(A - BK) < 1$. Further, the optimal cost is given by $\text{STO}_0 = \text{Tr}\{PW\}$.

This result has been extensively studied in optimal control theory [4, 21] as well as in reinforcement learning [13, 14, 29]. We want to emphasize two important properties of the optimal policy $u_t = -Kx_t$. First, the policy is *linear* in the state x_t . In contrast, we show later that the optimal policy when $k \neq 0$ is, in general, *nonlinear*. Second, under the assumptions of our model, this policy is *exponentially stable*, i.e., $\rho(A - BK) < 1$. We leverage this to show the power of predictions later in the paper.

Optimal policy. Let $F = A - BK$ and $\lambda = \frac{1+\rho(F)}{2} < 1$. From Gelfand's formula, there exists a constant $c(n)$ such that $\|F^k\| \leq c(n)\lambda^k$ for all $k \geq 1$.

Theorem 3.2. *Let w_t be i.i.d. with zero mean and covariance matrix W . Suppose the controller has $k \geq 1$ predictions. Then, the optimal control policy at each step t is given by:*

$$u_t = -(R + B^\top PB)^{-1} B^\top \left(PAx_t + \sum_{i=0}^{k-1} (A^\top - A^\top PH)^i Pw_{t+i} \right), \quad (2)$$

where P is the solution of DARE in Equation (1). The cost under this policy is:

$$\text{STO}_k = \text{Tr} \left\{ \left(P - \sum_{i=0}^{k-1} P(A - HPA)^i H(A^\top - A^\top PH)^i P \right) W \right\}, \quad (3)$$

where $H = B(R + B^\top PB)^{-1} B^\top$.

Following the approach developed in [15, 16], the proof is based on an analysis of quadratic cost-to-go functions in the form $V_t(x_t) = x_t^\top P_t x_t + v_t^\top x_t + q_t$. Note that $A - HPA = A - B(R + B^\top PB)^{-1} B^\top PA = A - BK = F$. Thus, the online optimal cost STO_k with k predictions approaches the offline optimal cost STO_∞ by an exponential rate. In other words, $\text{STO}_k/\text{STO}_\infty = 1 + O(\|F^k\|^2) = 1 + O(\lambda^{2k})$. Two extreme cases of our result are noteworthy. When $k = 0$, it reduces to the classic Proposition 3.1. When $k \rightarrow \infty$, it reduces to the offline optimal case derived by Goel and Hassibi [16].

Model predictive control. As might be expected, since the disturbances are i.i.d., future disturbances have no dependence on the current. As a result, MPC gives the *optimal* policy.

Theorem 3.3. *In Algorithm 1, let $\tilde{Q}_f = P$. Then, the MPC policy with k predictions is also given by Equation (2). Assuming i.i.d. disturbance with zero mean, the MPC policy is optimal.*

Due to the greedy nature, MPC does not utilize any properties of the disturbance, so the first part in Theorem 3.3 holds not only for i.i.d. disturbance, but also other types of disturbance considered in the later sections, i.e., MPC policy with k predictions is always given by Equation (2).

4 General stochastic disturbances

In this section, we consider a general form of stochastic disturbance, more general than typically considered in this context [10, 12, 13]. Suppose the disturbance sequence $\{w_t\}_{t=0,1,2,\dots}$ is sampled from a joint distribution \mathcal{W} such that the trace of the cross-correlation of each pair is uniformly bounded, i.e., there exist $m > 0$ such that for all $t, t' \geq 1$, $\mathbb{E}[w_t^\top w_{t'}] \leq m$.

Optimal policy. In the case of general stochastic disturbances, we cannot obtain as clean a form for STO_k as in the i.i.d. case in Section 3. However, the marginal benefit of having an extra prediction decays with the same (exponential) rate and the optimal policy is similar to that in Section 3, but with some additional terms that characterize the expected future disturbances given the current information.

Theorem 4.1. *The optimal control policy with general stochastic disturbance is given by:*

$$u_t = -(R + B^\top P B)^{-1} B^\top \left(P A x_t + \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} + \sum_{i=k}^{\infty} F^{\top i} P \mu_{t+i|t+k-1} \right), \quad (4)$$

where $\mu_{t'|t} = \mathbb{E}[w_{t'} | w_0, \dots, w_t]$. Under this policy, the marginal benefit of obtaining an extra prediction decays exponentially fast in the existing number k of predictions. Formally, for $k \geq 1$,

$$\text{STO}_k - \text{STO}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k}).^1$$

This proof leverages a novel difference analysis of cost-to-go functions. Note that for some distributions, STO_k may approach STO_∞ much faster than exponential rate. It is even possible that $\text{STO}_k = \text{STO}_\infty$ for finite k , as we show in Example 4.2 below. On the other hand, there are scenarios where STO_k approaches STO_∞ in an exactly exponential manner, as we show in Example 4.3 below.

Example 4.2. *Define the joint distribution \mathcal{W} such that with probability $1/2$, all $w_t = w$, and otherwise all $w_t = -w$. In this case, one prediction is equivalent to infinite predictions since it is enough to distinguish these two scenarios with only w_0 . As a result, $\text{STO}_1 = \text{STO}_\infty$.*

Example 4.3. *Suppose the system is 1-d ($n = d = 1$) and the disturbance is i.i.d. with zero mean, i.e., the setting of Section 3. Then, according to Equation (3), as long as F, P, H, W are non-zero,*

$$\text{STO}_k - \text{STO}_\infty = \sum_{i=k}^{\infty} F^{2i} P^2 H W = \Theta(F^{2k}).$$

Model predictive control. The comparison between the MPC policy in Equation (2) and the optimal policy in Equation (4) reveals that MPC is a *truncation* of the optimal policy and is no longer optimal because MPC is a greedy policy without considering future dependence on current information. Nevertheless, it is still a near-optimal policy, as characterized by the following results.

Theorem 4.4. $\text{MPCS}_k - \text{MPCS}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k})$. Moreover, in Example 4.3, $\text{MPCS}_k - \text{MPCS}_{k+1} = \Theta(\|F^k\|^2)$.

In other words, the marginal benefit for the MPC algorithm of an extra prediction decays exponentially fast, paralleling the result for optimal policy in Equation (4). Theorem 4.4 implies that MPC has a bounded performance ratio, which converges to 1 with an exponential rate in the number of available predictions. Formally:

Corollary 4.5. $PR^S(\text{MPC}_k) = \frac{\text{MPCS}_k}{\text{STO}_k} \leq \frac{\text{MPCS}_k}{\text{STO}_\infty} = \frac{\text{MPCS}_k}{\text{MPCS}_\infty} = 1 + O(\|F^k\|^2) = 1 + O(\lambda^{2k})$. Moreover, in Example 4.2, we have $PR^S(\text{MPC}_k) = 1 + \Theta(\|F^k\|^2)$.

Besides, the dynamic regret of MPC (nearly) matches the order of the optimal dynamic regret.

Theorem 4.6 (Main result). $Reg^S(\text{MPC}_k) = \text{MPCS}_k^T - \text{STO}_T^T = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$, where the second term results from the difference between finite/infinite horizons.

¹We say that $f(k) = O(g(k))$ if $\exists C > 0, \forall k \geq 1, |f(k)| \leq C g(k)$; $\Omega()$ is similar except that the last “ \leq ” is replaced by “ \geq ”; $\Theta()$ means both $O()$ and $\Omega()$. This is stronger than the standard definition where $f(k) = O(g(k))$ if $\exists C > 0, k^* > 0, \forall k \geq k^*, |f(k)| \leq C g(k)$.

Theorem 4.7. *The optimal dynamic regret $Reg_k^{S^*} = STO_k^T - STO_T^T = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$ and there exist A, B, Q, R, Q_f, x_0 , and \mathcal{W} such that $Reg_k^{S^*} = \Theta(\|F^k\|^2 (T - k))$.*

Note that, in the stochastic case, the regret-optimal policy is the same as the cost-optimal policy, i.e., the policy for STO_k^T is the same as $Reg_k^{S^*}$.

5 Adversarial disturbances

We now move from stochastic to adversarial disturbances. In this section, the disturbances are chosen from a bounded set $\Omega \subseteq \mathbb{R}^n$ by an adversary in order to maximize the controller's cost. Maintaining small regret is more challenging in adversarial models than in stochastic ones, so one may expect weaker bounds. Perhaps surprisingly, we obtain bounds with the same order.

Optimal policy. In the adversarial setting, the cost of the optimal policy, defined with a sequence of min's and sup's, is the equilibrium value of a two-player zero-sum game. In general, it is impossible to give an analytical expression of either ADV_k or the corresponding optimal policy. However, we prove a result that is structurally similar to the results from the stochastic setting, highlighting the exponential improvement from predictions.

Theorem 5.1. *For $k \geq 1$, $ADV_k - ADV_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k})$.*

Similarly to Example 4.2 for the stochastic case, in the adversarial setting, the optimal cost with k predictions may approach the offline optimal cost (under infinite predictions) much faster than exponential rate, and it is possible that $ADV_k = ADV_\infty$ for finite k , as shown in Example 5.2.

Example 5.2. *Let $A = B = Q = R = 1$ and $\Omega = [-1, 1]$. In this case, one prediction is enough to leverage the full power of prediction. Formally, we have $ADV_1 = ADV_\infty = 1$. In other words, for all $k \geq 1$, $ADV_k = 1$. The optimal control policy (as $T \rightarrow \infty$) is a piecewise function:*

$$u^*(x, w) = \begin{cases} -(x + w) & , -1 \leq x + w \leq 1 \\ -(x + w) + \frac{3-\sqrt{5}}{2}(x + w - 1) & , x + w > 1 \\ -(x + w) + \frac{3-\sqrt{5}}{2}(x + w + 1) & , x + w < -1 \end{cases}.$$

The proof leverages two different cost-to-go functions for the min player and the sup player.

Note that the optimal policy could be much more complex. Unlike Example 5.2, where the optimal policy is piecewise linear with only 3 pieces, for other values of A, B, Q, R , this function may have many more pieces.

Model predictive control. Under adversarial disturbances, MPC is suboptimal, e.g., in Example 5.2. However, its performance ratio and dynamic regret bounds turn out to be the same as those in the stochastic setting.

Theorem 5.3. $MPCA_k - MPCA_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k})$.

Corollary 5.4. *For $k \geq 1$, $PR^A(MPC_k) = \frac{MPCA_k}{ADV_k} \leq \frac{MPCA_k}{ADV_\infty} = \frac{MPCA_k}{MPCA_\infty} = 1 + O(\|F^k\|^2) = 1 + O(\lambda^{2k})$.*

This highlights that MPC has a bounded performance ratio, which converges to 1 with exponential rate. Additionally, MPC has the same order of dynamic regret as the stochastic case:

Theorem 5.5 (Main result). $Reg^A(MPC_k) = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$.

This dynamic regret is linear in the horizon T if we fix the number of predictions. However, if k is a super-constant function of T — an increasing function of T that is not upper-bounded by a constant — then the regret is sub-linear. Furthermore, if we let $k = \frac{\log T}{2 \log(1/\lambda)}$, then $Reg^A(MPC_k) = O(1)$. In other words, we can get constant regret with $O(\log T)$ predictions, even with adversarial disturbances. Finally, as implied by the following result, the $O(\log T)$ horizon cannot be improved since even the regret minimizing algorithm needs the same order of predictions to reach constant regret.

Theorem 5.6. $Reg_k^{A^*} = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$. *Moreover, there exist A, B, Q, R, Q_f, x_0 , and Ω such that $Reg_k^{A^*} = \Omega(\|F^k\|^2 (T - k))$.*²

² $\Omega(\cdot)$ is the growth order notation and has nothing to do with the bounded set Ω .

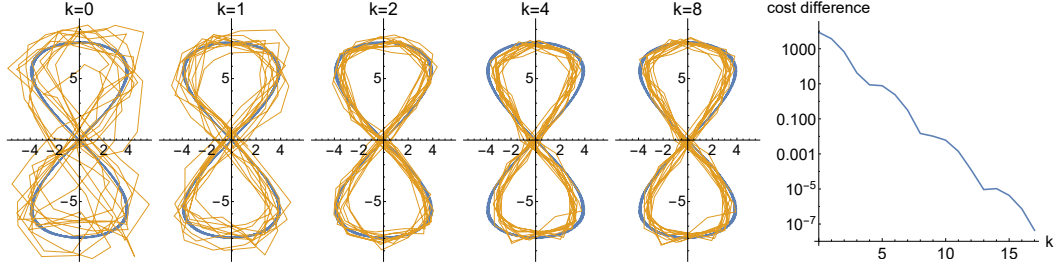


Figure 1: The power of predictions in online tracking. The left four figures show the desired trajectory (blue) and the actual trajectories (orange). The rightmost figure shows the cost difference (regret) between MPC using k predictions and the offline optimal policy. Note that the y-axis of the rightmost figure is in log-scale.

6 Numerical experiments

To illustrate our theoretical results, we test MPC with different numbers of predictions in a Linear Quadratic (LQ) tracking problem, where the desired trajectory is given by:

$$d_t = \begin{bmatrix} 8 \sin(t/3) \cos(t/3) \\ 8 \sin(t/3) \end{bmatrix}.$$

We consider following double integrator dynamics:

$$p_{t+1} = p_t + v_t + h_t, \quad v_{t+1} = v_t + u_t + \eta_t,$$

where $p_t \in \mathbb{R}^2$ is the position, v_t is the velocity, u_t is the control, and $h_t, \eta_t \sim \mathcal{U}[-1, 1]^2$ are i.i.d. noises. The objective is to minimize

$$\sum_{t=0}^{T-1} \|p_t - d_t\|^2 + \|u_t\|^2,$$

where we let $T = 200$. This problem can be converted to the standard LQR with disturbance w_t by letting $x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$ and $\tilde{w}_t = \begin{bmatrix} h_t \\ \eta_t \end{bmatrix}$ and then using the reduction in the LQ tracking example in Section 2. Note that after the reduction, the disturbances are the combination of a deterministic trajectory and i.i.d. noises, which corresponds to the case discussed in Section 4.

Figure 1 shows the tracking results with MPC using different numbers of predictions. We see that the regret exponentially decreases as the number of predictions increases, which is consistent with our theoretical results.

7 Concluding remarks

We conclude with several open problems and potential future research directions. Our results highlight the power of predictions and show that, given predictions, a simple greedy policy (MPC) is near-optimal for LQR control with disturbances in the dynamics, in terms of dynamic regret. Building on our results, it will be interesting to understand if MPC has a constant competitive ratio in this setting. In a different but related setting, Chen et al. [11] show for negative results on the competitive ratio so the answer is unclear at this point. Additionally, in this paper predictions are assumed to be perfect. Of course, in real applications predictions are noisy and are derived based on historical data. An important extension will be to understand how the analysis and results in this paper can extend to models with imperfect predictions learned from history, such as done in related models [11, 28]. Finally, real-world MPC problems often require non-linear dynamics and/or constraints, and the learning-theoretic study of such settings remains largely unexplored.

Broader Impact

Linear quadratic control is a common and powerful model with a variety of commercial and industrial applications, e.g., in robotics, chemical process control, and energy systems. This paper provides new

fundamental insights about the role of predictions in online linear quadratic control with disturbances and provides the first finite time performance guarantees for the most commonly used policy in the linear quadratic setting, model predictive control (MPC).

The guarantees provided by the theoretical analysis in this paper offer the potential for ensuring safety and robustness in industry applications where predictions are common and MPC is used. However, like many other theoretical contributions, this paper’s results are limited to its assumptions, e.g., linear system and fixed system parameters $\{A, B, Q, R\}$. The performance of MPC and the fundamental limits in other scenarios, e.g., nonlinear dynamics or time-variant $\{A, B, Q, R\}$, are still open research problems.

We see no ethical concerns related to the results in this paper.

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A Proofs of Section 3

In all proofs in this paper, for a sequence $x = (x_0, x_1, \dots, x_n)$, we use $x_{a:b}$ to denote its consecutive subsequence $(x_a, x_{a+1}, \dots, x_b)$.

A.1 Proof of Theorem 3.2

Let w_t be i.i.d. with zero mean and covariance matrix W . Suppose the controller has $k \geq 1$ predictions. Then, the optimal control policy at each step t is given by:

$$u_t = -(R + B^\top PB)^{-1} B^\top \left(PAx_t + \sum_{i=0}^{k-1} (A^\top - A^\top PH)^i P w_{t+i} \right), \quad (2)$$

where P is the solution of DARE in Equation (1). The cost under this policy is:

$$\text{STO}_k = \text{Tr} \left\{ \left(P - \sum_{i=0}^{k-1} P(A - HPA)^i H(A^\top - A^\top PH)^i P \right) W \right\}, \quad (3)$$

where $H = B(R + B^\top PB)^{-1} B^\top$.

Proof. Our proof technique closely follows that in Section 4.1 of [16]. To begin, note that the definition of STO_k^T has a structure of repeating min's and \mathbb{E} 's. We use dynamic programming to compute the value iteratively. In particular, we apply backward induction to solve the optimal cost-to-go functions, from time step T to the initial state. Given state x_t and predictions w_t, \dots, w_{t+k-1} , we define the cost-to-go function:

$$\begin{aligned} V_t(x_t; w_{t:t+k-1}) &:= \min_{u_t} \mathbb{E} \min_{w_{t+k}} \dots \mathbb{E} \min_{w_{T-1} u_{T-k}, \dots, u_{T-1}} \sum_{i=t}^{T-1} (x_i^\top Q x_i + u_i^\top R u_i) + x_T^\top Q_f x_T \quad (5) \\ &= x_t^\top Q x_t + \min_{u_t} \left(u_t^\top R u_t + \mathbb{E}_{w_{t+k}} [V_{t+1}(Ax_t + Bu_t + w_t; w_{t+1:t+k})] \right) \end{aligned}$$

with $V_T(x_T; \dots) = x_T^\top Q_f x_T$. Note that $\mathbb{E}_{w_{t+k}}$ has no effect for $t \geq T - k$. This function measures the expected overall control cost from a given state to the end, assuming the controller makes the optimal decision at each time.

We will show by backward induction that for every $t = 0, \dots, T$, $V_t(x_t; w_{t:t+k-1}) = x_t^\top P_t x_t + v_t^\top x_t + q_t$, where P_t, v_t, q_t are coefficients that may depend on $w_{t:t+k-1}$. This is clearly true for $t = T$. Suppose this is true at $t + 1$. Then,

$$\begin{aligned} &V_t(x; w_{t:t+k-1}) \\ &= x^\top Q x + \min_u \left(u^\top R u + (Ax + Bu + w_t)^\top P_{t+1} (Ax + Bu + w_t) \right. \\ &\quad \left. + \mathbb{E}_{w_{t+k}} [v_{t+1}]^\top (Ax + Bu + w_t) + \mathbb{E}_{w_{t+k}} [q_{t+1}] \right) \\ &= x^\top Q x + (Ax + w_t)^\top P_{t+1} (Ax + w_t) + \mathbb{E}_{w_{t+k}} [v_{t+1}]^\top (Ax + w_t) + \mathbb{E}_{w_{t+k}} [q_{t+1}] \\ &\quad + \min_u \left(u^\top (R + B^\top P_{t+1} B) u + u^\top B^\top \left(2P_{t+1} Ax + 2P_{t+1} w_t + \mathbb{E}_{w_{t+k}} [v_{t+1}] \right) \right). \end{aligned}$$

The optimal u is obtained by setting the derivative to be zero:

$$u^* = -(R + B^\top P_{t+1} B)^{-1} B^\top \left(P_{t+1} Ax + P_{t+1} w_t + \frac{1}{2} \mathbb{E}_{w_{t+k}} [v_{t+1}] \right). \quad (6)$$

Let $H_t = B(R + B^\top P_{t+1} B)^{-1} B^\top$. Plugging u^* back into V_t , we have

$$\begin{aligned} &V_t(x; w_{t:t+k-1}) \\ &= x^\top Q x + (Ax + w_t)^\top P_{t+1} (Ax + w_t) + \mathbb{E}_{w_{t+k}} [v_{t+1}]^\top (Ax + w_t) + \mathbb{E}_{w_{t+k}} [q_{t+1}] \end{aligned}$$

$$\begin{aligned}
& - \left(P_{t+1}Ax + P_{t+1}w_t + \frac{1}{2} \mathbb{E}_{w_{t+k}} [v_{t+1}] \right)^\top H_t \left(P_{t+1}Ax + P_{t+1}w_t + \frac{1}{2} \mathbb{E}_{w_{t+k}} [v_{t+1}] \right) \\
= & x^\top \left(Q + A^\top P_{t+1}A - A^\top P_{t+1}H_t P_{t+1}A \right) x \\
& + x^\top \left((A^\top - A^\top P_{t+1}H_t) \mathbb{E}_{w_{t+k}} [v_{t+1}] + 2(A^\top - A^\top P_{t+1}H_t)P_{t+1}w_t \right) \\
& + w_t^\top (P_{t+1} - P_{t+1}H_t P_{t+1})w_t + w_t^\top (I - P_{t+1}H_t) \mathbb{E}_{w_{t+k}} [v_{t+1}] \\
& - \frac{1}{4} \mathbb{E}_{w_{t+k}} [v_{t+1}]^\top H_t \mathbb{E}_{w_{t+k}} [v_{t+1}] + \mathbb{E}_{w_{t+k}} [q_{t+1}].
\end{aligned}$$

Thus, the recursive formulae, which parallel [16], are given by:

$$P_t = Q + A^\top P_{t+1}A - A^\top P_{t+1}H_t P_{t+1}A, \quad (7a)$$

$$v_t = (A^\top - A^\top P_{t+1}H_t) \mathbb{E}_{w_{t+k}} [v_{t+1}] + 2(A^\top - A^\top P_{t+1}H_t)P_{t+1}w_t, \quad (7b)$$

$$\begin{aligned}
q_t = & w_t^\top (P_{t+1} - P_{t+1}H_t P_{t+1})w_t + w_t^\top (I - P_{t+1}H_t) \mathbb{E}_{w_{t+k}} [v_{t+1}] \\
& - \frac{1}{4} \mathbb{E}_{w_{t+k}} [v_{t+1}]^\top H_t \mathbb{E}_{w_{t+k}} [v_{t+1}] + \mathbb{E}_{w_{t+k}} [q_{t+1}].
\end{aligned} \quad (7c)$$

As $T - t \rightarrow \infty$, P_t and H_t converge to P and H respectively, where P is the solution of discrete-time algebraic Riccati equation (DARE) $P = Q + A^\top P A - A^\top P H P A$, and $H = B(R + B^\top P B)^{-1} B^\top$. Note that $v_T = 0$ and $q_T = 0$. Then,

$$v_t = 2 \sum_{i=0}^{k-1} (A^\top - A^\top P H)^{i+1} P w_{t+i}, \quad (8)$$

$$q_t = w_t^\top (P - P H P) w_t + w_t^\top (I - P H) \mathbb{E}_{w_{t+k}} [v_{t+1}] - \frac{1}{4} \mathbb{E}_{w_{t+k}} [v_{t+1}]^\top H \mathbb{E}_{w_{t+k}} [v_{t+1}] + \mathbb{E}_{w_{t+k}} [q_{t+1}], \quad (9)$$

$$\mathbb{E}_{w_{t+k}} [v_{t+1}] = 2 \sum_{i=1}^{k-1} (A^\top - A^\top P H)^i P w_{t+i}. \quad (10)$$

Taking the expectation of q_t over all randomness, namely w_0, w_1, w_2, \dots , we have

$$\begin{aligned}
\mathbb{E}[q_t] & = \text{Tr}\{(P - P H P)W\} - \sum_{i=1}^{k-1} \text{Tr}\{P(A - H P A)^i H (A^\top - A^\top P H)^i P W\} + \mathbb{E}[q_{t+1}] \\
& = \text{Tr}\left\{ \left(P - \sum_{i=0}^{k-1} P(A - H P A)^i H (A^\top - A^\top P H)^i P \right) W \right\} + \mathbb{E}[q_{t+1}],
\end{aligned} \quad (11)$$

where in the first equality we use $\mathbb{E}[w_t] = 0$ and the independence of the disturbances. Thus, as $T \rightarrow \infty$, in each time step, a constant cost is incurred and the average cost STO_k is exactly this value.

$$\begin{aligned}
\text{STO}_k & = \lim_{T \rightarrow \infty} \frac{1}{T} \text{STO}_k^T = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[V_0(x_0; w_{0:k-1})] = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[q_0] \\
& = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[q_t] - \mathbb{E}[q_{t+1}] = \text{Tr}\left\{ \left(P - \sum_{i=0}^{k-1} P(A - H P A)^i H (A^\top - A^\top P H)^i P \right) W \right\}.
\end{aligned}$$

The explicit form of the optimal control policy is obtained by combining Equations (6) and (10). \square

A.2 Proof of Theorem 3.3

In Algorithm 1, let $\tilde{Q}_f = P$. Then, the MPC policy with k predictions is also given by Equation (2). Assuming i.i.d. disturbance with zero mean, the MPC policy is optimal.

Proof. Due to the greedy nature, MPC policy is given by the solution of a length- k optimal control problem, given deterministic w_t, \dots, w_{t+k-1} . In other words, we want to derive the optimal policy (u_t, \dots, u_{t+k-1}) that minimizes

$$\sum_{i=t}^{t+k-1} (x_i^\top Q x_i + u_i^\top R u_i) + x_{t+k}^\top P x_{t+k},$$

where $x_{i+1} = Ax_i + Bu_i + w_i$, given $x_t, w_t, \dots, w_{t+k-1}$. Define the cost-to-go function at time i given $x_i, w_i, \dots, w_{t+k-1}$:

$$\begin{aligned} V_i(x_i; w_{i:t+k-1}) &= \min_{u_{i:t+k-1}} \sum_{j=i}^{t+k-1} (x_j^\top Q x_j + u_j^\top R u_j) + x_{t+k}^\top P x_{t+k} \\ &= x_i^\top Q x_i + \min_{u_i} (u_i^\top R u_i + V_{i+1}(Ax_i + Bu_i + w_i; w_{i+1:t+k-1})). \end{aligned}$$

Note that $V_{t+k}(x_{t+k}) = x_{t+k}^\top P x_{t+k}$. Similar to the proof of Theorem 3.2, we can inductively show that $V_i(x_i; w_{i:t+k-1}) = x_i^\top P x_i + v_i^\top x_i + q_i$ for some v_i and q_i . Note that the second-degree coefficient no longer depends on the index i as in the previous proof because we start from P , the solution of DARE. We then have the followings equations that parallel with Equations (6) and (8):

$$\begin{aligned} v_i &= 2 \sum_{j=0}^{t+k-i-1} F^{\top j+1} P w_{i+j}, \\ u_i^* &= -(R + B^\top P B)^{-1} B^\top \left(P A x_i + P w_i + \frac{1}{2} v_{i+1} \right) \\ &= -(R + B^\top P B)^{-1} B^\top \left(P A x_i + \sum_{j=0}^{t+k-i-1} F^{\top j} P w_{i+j} \right). \end{aligned}$$

The case $i = t$ gives:

$$u_t^* = -(R + B^\top P B)^{-1} B^\top \left(P A x_t + \sum_{j=0}^{k-1} F^{\top j} P w_{t+j} \right),$$

which is the MPC policy at time step t , and is same as Equation (2). \square

B Proofs of Section 4

B.1 Proof of Theorem 4.1

The optimal control policy with general stochastic disturbance is given by:

$$u_t = -(R + B^\top P B)^{-1} B^\top \left(P A x_t + \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} + \sum_{i=k}^{\infty} F^{\top i} P \mu_{t+i|t+k-1} \right), \quad (4)$$

where $\mu_{t'|t} = \mathbb{E}[w_{t'} | w_0, \dots, w_t]$. Under this policy, the marginal benefit of obtaining an extra prediction decays exponentially fast in the existing number k of predictions. Formally, for $k \geq 1$,

$$\text{STO}_k - \text{STO}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k}).$$

Proof. Similar to the proof of Theorem 3.2, we assume

$$V_t(x_t; w_{0:t+k-1}) = x_t^\top P_t x_t + x_t^\top v_t + q_t,$$

where V_t has a similar definition as in Equation (5) but may further depend on w_0, \dots, w_{t-1} because the disturbance sequence is no longer Markovian. In this case, P_t, v_t and q_t still satisfy the recursive

forms in Equation (7). However, the expected values of w_t and v_t are different since we have a more general distribution now. Let $T - t \rightarrow \infty$, $\mu_{t'|t} = \mathbb{E}[w_{t'} | w_0, \dots, w_t]$ and $F = A - HPA$. Then,

$$v_t^k = 2 \sum_{i=0}^{k-1} F^{\top i+1} P w_{t+i} + 2 \sum_{i=k}^{\infty} F^{\top i+1} P \mu_{t+i|t+k-1}, \quad (12)$$

$$q_t^k = w_t^\top (P - PHP) w_t + w_t^\top (I - PH) \mathbb{E}_{w_{t+k}} [v_{t+1}^k] - \frac{1}{4} \mathbb{E}_{w_{t+k}} [v_{t+1}^k]^\top H \mathbb{E}_{w_{t+k}} [v_{t+1}^k] + \mathbb{E}_{w_{t+k}} [q_{t+1}^k],$$

where the superscript k denotes the number of predictions.

The optimal policy in this case has the same form as Equation (6). Plugging Equation (12) into it, we obtain the optimal policy in the theorem.

Further,

$$\mathbb{E}[q_t^k - q_t^{k+1}] = \mathbb{E} \left[w_t^\top (I - PH) \left(\mathbb{E}_{w_{t+k}} [v_{t+1}^k] - \mathbb{E}_{w_{t+k+1}} [v_{t+1}^{k+1}] \right) \right] \quad (13a)$$

$$+ \frac{1}{4} \mathbb{E} \left[\mathbb{E}_{w_{t+k+1}} [v_{t+1}^{k+1}]^\top H \mathbb{E}_{w_{t+k+1}} [v_{t+1}^{k+1}] - \mathbb{E}_{w_{t+k}} [v_{t+1}^k]^\top H \mathbb{E}_{w_{t+k}} [v_{t+1}^k] \right] \quad (13b)$$

$$+ \mathbb{E}[q_{t+1}^k - q_{t+1}^{k+1}], \quad (13c)$$

where the expectation \mathbb{E} is taken over all randomness. Part (13a) is zero because

$$\mathbb{E}_{w_{t+k}} [v_{t+1}^k] = \mathbb{E}_{w_{t+k}, w_{t+k+1}} [v_{t+1}^{k+1}].$$

$$\begin{aligned} \text{Part (13b)} &= \frac{1}{4} \mathbb{E}_{w_{t+k}} \left[\left(\mathbb{E}_{w_{t+k+1}} [v_{t+1}^{k+1}] - \mathbb{E}_{w_{t+k}} [v_{t+1}^k] \right)^\top H \left(\mathbb{E}_{w_{t+k+1}} [v_{t+1}^{k+1}] - \mathbb{E}_{w_{t+k}} [v_{t+1}^k] \right) \right] \\ &= \mathbb{E}_{w_{t+k}} [z_{k,t}^\top H z_{k,t}], \end{aligned}$$

where

$$z_{k,t} = F^{\top k} P (w_{t+k} - \mu_{t+k|t+k-1}) + \sum_{i=k+1}^{\infty} F^{\top i} P (\mu_{t+i|t+k} - \mu_{t+i|t+k-1}).$$

Note that $z_{k,t} = F^{\top} z_{k-1,t+1} = F^{\top k} z_{0,t+k}$. Thus,

$$\begin{aligned} \text{STO}_k - \text{STO}_{k+1} &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[q_0^k - q_0^{k+1}] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[z_{k,t}^\top H z_{k,t}] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[z_{0,t+k}^\top F^k H F^{\top k} z_{0,t+k}] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \text{Tr} \left\{ F^k H F^{\top k} \mathbb{E}[z_{0,t+k} z_{0,t+k}^\top] \right\} \\ &\leq \|F^k\|^2 \|H\| \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \text{Tr} \mathbb{E}[z_{0,t+k} z_{0,t+k}^\top] \end{aligned}$$

where in the last line we use the fact that if A is symmetric, then $\text{Tr}\{AB\} \leq \lambda_{\max}(A) \text{Tr}\{B\}$.

Finally we just need to show the last item $\text{Tr} \mathbb{E}[z_{0,t+k} z_{0,t+k}^\top]$ is uniformly bounded for all t . This is straightforward because the cross-correlation of each disturbance pair is uniformly bounded, i.e., there exists $m > 0$ such that for all $t, t' \geq 1$, $\mathbb{E}[w_t^\top w_{t'}] \leq m$.

$$\text{Tr} \mathbb{E}[z_{0,t} z_{0,t}^\top] = \sum_{i,j=0}^{\infty} \text{Tr} \mathbb{E} \left[P F^i F^{\top j} P (\mu_{t+j|t} - \mu_{t+j|t-1})(\mu_{t+i|t} - \mu_{t+i|t-1})^\top \right]$$

$$\begin{aligned}
&= \sum_{i,j=0}^{\infty} \text{Tr} \left\{ P F^i F^{\top j} P \mathbb{E} \left[\mu_{t+j|t} \mu_{t+i|t}^{\top} - \mu_{t+j|t-1} \mu_{t+i|t-1}^{\top} \right] \right\} \\
&\leq \sum_{i,j=0}^{\infty} \|F^i\| \|F^j\| \|P\|^2 \mathbb{E} [w_{t+j}^{\top} w_{t+i} - w_{t+j}^{\top} w_{t+i}] \\
&\leq \sum_{i,j=0}^{\infty} c \lambda^i c \lambda^j \|P\|^2 2m = 2 \frac{c^2}{(1-\lambda)^2} \|P\|^2 m
\end{aligned}$$

for some constant c from Gelfand's formula. Thus $\text{Tr} \mathbb{E} [z_{0,t} z_{0,t}^{\top}]$ is bounded by a constant independent of t . Thus,

$$\text{STO}_k - \text{STO}_{k+1} = O(\|F^k\|^2).$$

□

B.2 Proof of Theorem 4.4

$\text{MPCS}_k - \text{MPCS}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k})$. Moreover, in Example 4.3, $\text{MPCS}_k - \text{MPCS}_{k+1} = \Theta(\|F^k\|^2)$.

Proof. To recursively calculate the value of J^{MPC_k} , we define:

$$\begin{aligned}
V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) &= \sum_{i=t}^{T-1} (x_i^{\top} Q x_i + u_i^{\top} R u_i) + x_T^{\top} Q_f x_T \\
&= x_t^{\top} Q x_t + u_t^{\top} R u_t + V_{t+1}(A x_t + B u_t + w_t; w_{0:t+k})
\end{aligned}$$

as the cost-to-go function with MPC as the policy, i.e., u_t is the control at time step t from the MPC policy with k predictions. Similar to the previous proofs, we assume $V_t^{\text{MPC}_k}(x) = x^{\top} P_t x + x^{\top} v_t + q_t$ (which turns out to be correct by induction) and $T - t \rightarrow \infty$ so that $P_t = P$. Then,

$$\begin{aligned}
V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) &= x_t^{\top} Q x_t + u_t^{\top} R u_t + (A x_t + B u_t + w_t)^{\top} P (A x_t + B u_t + w_t) \\
&\quad + (A x_t + B u_t + w_t)^{\top} v_{t+1} + q_{t+1} \\
&= u_t^{\top} (R + B^{\top} P B) u_t + 2 u_t^{\top} B^{\top} (P A x_t + P w_t + v_{t+1}/2) \\
&\quad + x_t^{\top} Q x_t + (A x_t + w_t)^{\top} P (A x_t + w_t) + (A x_t + w_t)^{\top} v_{t+1} + q_{t+1}.
\end{aligned} \tag{14}$$

Let $F = A - H P A$. Plugging in the formula of u_t in Theorem 3.3, we have

$$\begin{aligned}
V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) &= \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right)^{\top} H \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right) \\
&\quad - \left(P A x_t + P w_t + \frac{1}{2} v_{t+1} \right)^{\top} H \left(P A x_t + P w_t + \frac{1}{2} v_{t+1} \right) \\
&\quad + x_t^{\top} Q x_t + (A x_t + w_t)^{\top} P (A x_t + w_t) + (A x_t + w_t)^{\top} v_{t+1} + q_{t+1} \\
&= x_t^{\top} (Q + A^{\top} P A - A^{\top} P H P A) x_t + x_t^{\top} (F^{\top} v_{t+1} + 2 F^{\top} P w_t) \\
&\quad + \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right)^{\top} H \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right) \\
&\quad - \left(P w_t + \frac{1}{2} v_{t+1} \right)^{\top} H \left(P w_t + \frac{1}{2} v_{t+1} \right) + w_t^{\top} P w_t + w_t^{\top} v_{t+1} + q_{t+1} \\
&= x_t^{\top} P x_t + x_t^{\top} v_t + q_t.
\end{aligned}$$

Thus,

$$v_t = F^{\top} v_{t+1} + 2 F^{\top} P w_t = 2 \sum_{i=0}^{\infty} F^{\top i+1} P w_{t+i}.$$

Then, we can plug v_{t+1} into q_t :

$$\begin{aligned}
q_t &= q_{t+1} + \left(\sum_{i=k}^{\infty} F^{\top i} P w_{t+i} \right)^{\top} H \left(\sum_{i=k}^{\infty} F^{\top i} P w_{t+i} \right) \\
&\quad - \left(\sum_{i=0}^{\infty} F^{\top i} P w_{t+i} \right)^{\top} H \left(\sum_{i=0}^{\infty} F^{\top i} P w_{t+i} \right) + w_t^{\top} P w_t + 2w_t^{\top} \left(\sum_{i=1}^{\infty} F^{\top i} P w_{t+i} \right). \quad (15)
\end{aligned}$$

Note that Equation (15) is for MPC with k predictions. With the disturbance sequence $\{w_t\}$ fixed, we can compare the per-step cost of MPC with k predictions and that with $k+1$ predictions:

$$\begin{aligned}
q_t^k - q_t^{k+1} &= q_{t+1}^k - q_{t+1}^{k+1} + \left(\sum_{i=k}^{\infty} F^{\top i} P w_{t+i} \right)^{\top} H \left(\sum_{i=k}^{\infty} F^{\top i} P w_{t+i} \right) \\
&\quad - \left(\sum_{i=k+1}^{\infty} F^{\top i} P w_{t+i} \right)^{\top} H \left(\sum_{i=k+1}^{\infty} F^{\top i} P w_{t+i} \right) \\
&= q_{t+1}^k - q_{t+1}^{k+1} + w_{t+k}^{\top} P F^k H F^{\top k} \left(P w_{t+k} + 2 \sum_{i=1}^{\infty} F^{\top i} P w_{t+i+k} \right). \quad (16)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})] &= \mathbb{E} \left[w_{t+k}^{\top} P F^k H F^{\top k} \left(P w_{t+k} + 2 \sum_{i=1}^{\infty} F^{\top i} P w_{t+i+k} \right) \right] \\
&= \text{Tr} \left\{ P F^k H F^{\top k} \left(P \mathbb{E}[w_{t+k} w_{t+k}^{\top}] + 2 \sum_{i=1}^{\infty} F^{\top i} P \mathbb{E}[w_{t+i+k} w_{t+k}^{\top}] \right) \right\} \\
&= \text{Tr} \left\{ P F^k H F^{\top k} Z_{k,t} \right\},
\end{aligned}$$

where $Z_{k,t} = P \mathbb{E}[w_{t+k} w_{t+k}^{\top}] + 2 \sum_{i=1}^{\infty} F^{\top i} P \mathbb{E}[w_{t+i+k} w_{t+k}^{\top}]$. Note that $Z_{k,t} = Z_{k-1,t+1}$.

$$\begin{aligned}
\text{MPCS}_k - \text{MPCS}_{k+1} &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[q_0^k - q_0^{k+1}] \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \text{Tr} \left\{ P F^k H F^{\top k} Z_{k,t} \right\} \\
&\leq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \|P\| \|H\| \|F^k\|^2 \text{Tr}\{Z_{k,t}\},
\end{aligned}$$

where in the last line we use the fact that if A is symmetric, then $\text{Tr}\{AB\} \leq \|A\| \text{Tr}\{B\}$. Similarly to the last part in the proof of Theorem 4.1, now we just need to show the last term $\text{Tr}\{Z_{k,t}\}$ is uniformly bounded for all t . Again, this is because the cross-correlation of each disturbance pair is uniformly bounded.

$$\begin{aligned}
\text{Tr}\{Z_{k,t}\} &\leq \|P\| \text{Tr} \mathbb{E}[w_{t+k} w_{t+k}^{\top}] + 2 \sum_{i=1}^{\infty} \|P\| \|F^i\| \mathbb{E} \left[\sum_j \sigma_j(w_{t+i+k} w_{t+k}^{\top}) \right] \\
&\leq \|P\| m + 2 \sum_{i=1}^{\infty} c \lambda^i \|P\| m = \|P\| m + 2c \frac{\lambda}{1-\lambda} \|P\| m
\end{aligned}$$

where c is some constant, and in the first line, we use the fact that $\text{Tr}\{AB\} \leq \|A\| \sum_j \sigma_j(B)$ with $\sigma_j(\cdot)$ denoting the j -th singular value. Thus, $\text{Tr}\{Z_{k,t}\}$ is uniformly bounded. Therefore, $\text{MPCS}_k - \text{MPCS}_{k+1} = O(\|F^k\|^2)$. \square

B.3 Proof of Theorem 4.6

$Reg^S(\text{MPC}_k) = \text{MPCS}_k^T - \text{STO}_T^T = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$, where the second term results from the difference between finite/infinite horizons.

Proof. To calculate the dynamic regret, we cannot simply let $T - t \rightarrow \infty$ as we did before Equation (14) in the proof of Theorem 4.4 and instead need to handle the expressions in a more delicate manner. In particular, we need to rigorously analyze the impact of finite horizon. Let $\Delta_t = P_t - P$.

$$\begin{aligned} & V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) \\ &= u_t^\top (R + B^\top P_{t+1} B) u_t + 2u_t^\top B^\top (P_{t+1} A x_t + P_{t+1} w_t + v_{t+1}/2) \\ &\quad + x_t^\top Q x_t + (A x_t + w_t)^\top P_{t+1} (A x_t + w_t) + (A x_t + w_t)^\top v_{t+1} + q_{t+1} \\ &= u_t^\top (R + B^\top P B) u_t + 2u_t^\top B^\top (P A x_t + P w_t + v_{t+1}/2) \\ &\quad + x_t^\top Q x_t + (A x_t + w_t)^\top P (A x_t + w_t) + (A x_t + w_t)^\top v_{t+1} + q_{t+1} \\ &\quad + u_t^\top B^\top \Delta_{t+1} B u_t + 2u_t^\top B^\top \Delta_{t+1} (A x_t + w_t) + (A x_t + w_t)^\top \Delta_{t+1} (A x_t + w_t). \end{aligned}$$

Plugging in the MPC policy as in Theorem 3.3, we have:

$$\begin{aligned} & V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) \\ &= x_t^\top (Q + A^\top P A - A^\top P H P A) x_t + x_t^\top (F^\top v_{t+1} + 2F^\top P w_t) \\ &\quad + \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right)^\top H \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right) \\ &\quad - \left(P w_t + \frac{1}{2} v_{t+1} \right)^\top H \left(P w_t + \frac{1}{2} v_{t+1} \right) + w_t^\top P w_t + w_t^\top v_{t+1} + q_{t+1} \\ &\quad + \left(F x_t + w_t - \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} \right)^\top \Delta_{t+1} \left(F x_t + w_t - \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} \right) \\ &= x_t^\top (Q + A^\top P A - A^\top P H P A + F^\top \Delta_{t+1} F) x_t \\ &\quad + x_t^\top \left(F^\top v_{t+1} + 2F^\top P w_t + 2F^\top \Delta_{t+1} \left(w_t - \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} \right) \right) \\ &\quad + \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right)^\top H \left(\frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{\top i} P w_{t+i} \right) \\ &\quad - \left(P w_t + \frac{1}{2} v_{t+1} \right)^\top H \left(P w_t + \frac{1}{2} v_{t+1} \right) + w_t^\top P w_t + w_t^\top v_{t+1} + q_{t+1} \\ &\quad + \left(w_t - \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} \right)^\top \Delta_{t+1} \left(w_t - \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} \right) \end{aligned}$$

Comparing this with the induction hypothesis $V_t^{\text{MPC}_k} = x_t^\top (P + \Delta_t) x_t + x_t^\top v_t + q_t$, we obtain the recursive formulae for Δ_t, v_t, q_t .

$$\Delta_t = F^\top \Delta_{t+1} F = F^{\top T-t} \Delta_T F^{T-t} = F^{\top T-t} (Q_f - P) F^{T-t}.$$

This implies that P_t converges to P exponentially fast, i.e., $\|\Delta_t\| = O(\|F^{T-t}\|^2) = O(\lambda^{2(T-t)})$.

$$\begin{aligned} v_t &= F^\top v_{t+1} + 2F^\top P w_t + 2F^\top \Delta_{t+1} \left(w_t - \sum_{i=0}^{k-1} F^{\top i} P w_{t+i} \right) \\ &= 2 \sum_{j=0}^{T-t-1} \left(F^{\top j+1} P w_{t+j} + F^{\top j+1} \Delta_{t+j+1} \left(w_{t+j} - \sum_{i=0}^{k-1} F^{\top i} P w_{t+j+i} \right) \right) \end{aligned}$$

$$= 2 \sum_{i=0}^{T-t-1} F^{\top i+1} P w_{t+i} + 2 \sum_{j=0}^{T-t-1} F^{\top j+1} \Delta_{t+j+1} \left(w_{t+j} - \sum_{i=0}^{k-1} F^{\top i} P w_{t+j+i} \right).$$

Denote the second term by $2d_t$. We have

$$\begin{aligned} d_t &= \sum_{j=0}^{T-t-1} F^{\top j+1} \Delta_{t+j+1} \left(w_{t+j} - \sum_{i=0}^{k-1} F^{\top i} P w_{t+j+i} \right) \\ &= \sum_{j=0}^{T-t-1} O(\lambda^j \lambda^{2(T-t-j)}) = O(\lambda^{T-t}). \end{aligned}$$

$$\begin{aligned} d_t^k - d_t^{k+1} &= \sum_{j=0}^{T-t-k-1} F^{\top j+1} \Delta_{t+j+1} F^{\top k} P w_{t+j+k} \\ &= \sum_{j=0}^{T-t-k-1} O(\lambda^j \lambda^{2(T-t-j)} \|F^k\|) = O(\lambda^{T-t+k} \|F^k\|). \end{aligned} \tag{17}$$

Finally, we have a formula for q_t that parallels Equation (15):

$$\begin{aligned} q_t &= q_{t+1} + \left(d_{t+1} + \sum_{i=k}^{T-t-1} F^{\top i} P w_{t+i} \right)^{\top} H \left(d_{t+1} + \sum_{i=k}^{T-t-1} F^{\top i} P w_{t+i} \right) \\ &\quad - \left(d_{t+1} + \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} \right)^{\top} H \left(d_{t+1} + \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} \right) \\ &\quad + w_t^{\top} P w_t + 2w_t^{\top} \left(d_{t+1} + \sum_{i=1}^{T-t-1} F^{\top i} P w_{t+i} \right). \end{aligned}$$

Taking the difference between k and $k+1$ predictions, we have

$$\begin{aligned} q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1}) &= (w_{t+k}^{\top} P F^k + (d_{t+1}^k - d_{t+1}^{k+1})^{\top}) H \left(d_{t+1}^k + d_{t+1}^{k+1} + F^{\top k} P w_{t+k} + 2 \sum_{i=1}^{T-t-k-1} F^{\top i+k} P w_{t+i+k} \right) \\ &= (w_{t+k}^{\top} P F^k + O(\lambda^{T-t} \|F^k\|)) H \left(O(\lambda^{T-t}) + F^{\top k} P w_{t+k} + 2 \sum_{i=1}^{T-t-k-1} F^{\top i+k} P w_{t+i+k} \right), \end{aligned} \tag{18}$$

and thus

$$\mathbb{E}[q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})] = O(\|F^k\|(\lambda^{T-t} + \|F^k\|)).$$

$$\begin{aligned} \mathbb{E}[q_0^k - q_0^T] &= \sum_{t=0}^{T-1} \mathbb{E}[q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})] \\ &= \sum_{t=0}^{T-1} O(\|F^k\|(\lambda^{T-t} + \|F^k\|)) \\ &= O(\|F^k\|^2 T + \|F^k\|). \end{aligned}$$

$$\mathbb{E}[v_0^k - v_0^T] = 2(d_0^k - d_0^T) = O(\lambda^{T+k} \|F^k\|).$$

$$\begin{aligned} \mathbb{E} J^{\text{MPC}_k} - \mathbb{E} J^{\text{MPC}_T} &= \mathbb{E}[V_0^k(x_0) - V_0^T(x_0)] \\ &= \mathbb{E}[x_0^{\top} (v_0^k - v_0^T) + (q_0^k + q_0^T)] \end{aligned}$$

$$= O(\|F^k\|^2 T + \|F^k\|). \quad (19)$$

By definition, J^{MPC_T} is the cost of MPC policy given all future disturbances before making any decisions. It almost equals to $\min_u J$, the optimal policy given all future disturbances, except that during optimization, MPC assumes the final-step cost to be $x_T^\top P x_T$ instead of $x_T^\top Q_f x_T$. This will incur at most constant extra cost, i.e.,

$$J^{\text{MPC}_T} - \min_u J = O(P - Q_f) = O(1). \quad (20)$$

By Equations (19) and (20),

$$\text{Reg}^S(\text{MPC}_k) = \mathbb{E} J^{\text{MPC}_k} - \mathbb{E} \min_u J = O(\|F^k\|^2 T + \|F^k\| + 1) = O(\|F^k\|^2 T + 1).$$

□

B.4 Proof of Theorem 4.7

The optimal dynamic regret $\text{Reg}_k^{S^*} = \text{STO}_k^T - \text{STO}_T^T = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$ and there exist A, B, Q, R, Q_f, x_0 , and \mathcal{W} such that $\text{Reg}_k^{S^*} = \Theta(\|F^k\|^2 (T - k))$.

Proof. The first part follows from Theorem 4.6 and that fact that $\text{Reg}_k^{S^*} \leq \text{Reg}^S(\text{MPC}_k)$.

The second part is shown by Example 4.3, i.e., suppose $n = d = 1$ and the disturbance are i.i.d. and zero-mean. Additionally, let $Q_f = P$ and $x_0 = 0$. In this case, MPC has not only the same policy but also the same cost as the optimal control policy. Also, $P_t = P$ for all t . To calculate the total cost, we follow the approach used in the proof of Theorem 3.2. Since T is finite now, we have a similar (to Equation (8)) but different form of v_t :

$$v_t = 2 \sum_{i=0}^{\min\{k-1, T-t-1\}} F^{\top i+1} P w_{t+i}.$$

Thus,

$$\begin{aligned} \mathbb{E}[q_t] &= \text{Tr} \left\{ \left(P - \sum_{i=0}^{\min\{k-1, T-t-1\}} P F^i H F^{\top i} P \right) W \right\} + \mathbb{E}[q_{t+1}]. \\ \mathbb{E}[q_0] &= \text{Tr} \left\{ \sum_{t=0}^{T-1} \left(P - \sum_{i=0}^{\min\{k-1, T-t-1\}} P F^i H F^{\top i} P \right) W \right\}. \end{aligned}$$

Let q_t^k denote q_t in the scenario of k predictions.

$$\begin{aligned} \text{Reg}^{S^*} &= \mathbb{E}[q_0^k - q_0^T] = \text{Tr} \left\{ \sum_{t=0}^{T-k-1} \sum_{i=k}^{T-t-1} P F^i H F^{\top i} P W \right\} \\ &\geq (T - k) \text{Tr} \left\{ P F^k H F^{\top k} P W \right\} = \Omega(\|F^k\|^2 (T - k)). \end{aligned}$$

On the other hand,

$$\text{Reg}^{S^*} = \mathbb{E}[q_0^k - q_0^T] \leq (T - k) \text{Tr} \left\{ \sum_{i=k}^{\infty} P F^i H F^{\top i} P W \right\} = O(\|F^k\|^2 (T - k)).$$

Therefore, $\text{Reg}^{S^*} = \Theta(\|F^k\|^2 (T - k))$. □

C Proofs of Section 5

C.1 Proof of Theorem 5.1

For $k \geq 1$, $\text{ADV}_k - \text{ADV}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k})$.

Proof. This proof is based on Theorem 5.3. It turns out that the behavior of the MPC policy and its cost is easier to analyze than the optimal one, especially in the adversarial setting.

$$\text{ADV}_k - \text{ADV}_{k+1} \leq \text{ADV}_k - \text{ADV}_\infty \leq \text{MPCA}_k - \text{ADV}_\infty = \sum_{i=k}^{\infty} \text{MPCA}_i - \text{MPCA}_{i+1}.$$

By Theorem 5.3,

$$\text{MPCA}_i - \text{MPCA}_{i+1} \leq O(\|F^i\|^2) \leq O(\|F^k\|^2 \|F^{i-k}\|^2) \leq O(\|F^k\|^2 \lambda^{2(i-k)}).$$

Thus,

$$\text{ADV}_k - \text{ADV}_{k+1} \leq O\left(\|F^k\|^2 \sum_{i=k}^{\infty} \lambda^{2(i-k)}\right) = O(\|F^k\|^2).$$

□

C.2 Proof of Example 5.2

Let $A = B = Q = R = 1$ and $\Omega = [-1, 1]$. In this case, one prediction is enough to leverage the full power of prediction. Formally, we have $\text{ADV}_1 = \text{ADV}_\infty = 1$. In other words, for all $k \geq 1$, $\text{ADV}_k = 1$. The optimal control policy (as $T \rightarrow \infty$) is a piecewise function:

$$u^*(x, w) = \begin{cases} -(x+w) & , -1 \leq x+w \leq 1 \\ -(x+w) + \frac{3-\sqrt{5}}{2}(x+w-1) & , x+w > 1 \\ -(x+w) + \frac{3-\sqrt{5}}{2}(x+w+1) & , x+w < -1 \end{cases}.$$

The proof leverages two different cost-to-go functions for the min player and the sup player.

Proof. We will show $\text{ADV}_1 = 1$ and $\text{ADV}_\infty = 1$ separately. The system dynamics is given by $x_{t+1} = x_t + u_t + w_t$ with $w_t \in [-1, 1]$ and

$$\text{ADV}_1^T = \max_{w_0} \min_{u_0} \cdots \max_{w_{T-1}} \min_{u_{T-1}} \sum_{t=0}^{T-1} (x_t^2 + u_t^2) + x_T^2.$$

We will calculate the results of each min and max by dynamical programming. In particular, we will define two cost-to-go functions for the min player and the max player respectively. Let $z_t = x_t + w_t$. Then, z_t can be regarded as the disturbed state. This is natural since the controller has one prediction and decides u_t after knowing w_t . Thus, the system dynamics can be split into two stages: $z_t = x_t + w_t$ and $x_{t+1} = z_t + u_t$. Let

$$\begin{aligned} f_t(z_t) &= \min_{u_t} \max_{w_{t+1}} \min_{u_{t+1}} \cdots \max_{w_{T-1}} \min_{u_{T-1}} \sum_{i=t}^{T-1} (u_i^2 + x_{i+1}^2) \\ &= \min_{u_t} (u_t^2 + (z_t + u_t)^2 + g_{t+1}(z_t + u_t)), \\ g_t(x_t) &= \max_{w_t} \min_{u_t} \cdots \max_{w_{T-1}} \min_{u_{T-1}} \sum_{i=t}^{T-1} (u_i^2 + x_{i+1}^2) \\ &= \max_{w_t} f_t(x_t + w_t). \end{aligned}$$

For $t = T - 1$, we have

$$\begin{aligned} f_{T-1}(z) &= \min_u u^2 + (z + u)^2 = \frac{z^2}{2}, \\ g_{T-1}(x) &= \max_w \frac{(x+w)^2}{2} = \frac{(|x|+1)^2}{2}. \end{aligned}$$

We will prove by backward induction that $g_t(x) = a_t x^2 + 2b_t |x| + c_t$ where a_t, b_t, c_t are some coefficients with $0 < b_t < 1$. Assuming this is true at t , we will show this is true at $t - 1$.

$$f_{t-1}(z) = \min_u (u^2 + (z + u)^2 + g_t(z + u))$$

$$\begin{aligned}
&= \min_y ((y-z)^2 + y^2 + g_t(y)) \\
&= \min_y ((y-z)^2 + y^2 + a_t y^2 + 2b_t |y| + c_t) \\
&= \min_y ((a_t + 2)y^2 - 2(z - b_t \text{sign}(y))y + z^2 + c_t),
\end{aligned}$$

where $y = z + u = x + w + u$ is the state after the control policy is applied. Let function $y(z)$ map from the disturbed old state to the new state. The optimal y is given by:

$$\begin{aligned}
y^*(z) &= \arg \min_y ((a_t + 2)y^2 - 2(z - b_t \text{sign}(y))y + z^2 + c_t) \\
&= \begin{cases} 0 & , -b_t \leq z \leq b_t \\ \frac{z - b_t \text{sign}(z)}{a_t + 2} & , \text{otherwise} \end{cases}. \tag{21}
\end{aligned}$$

Thus, for $z < -b_t$ or $z > b_t$, we have

$$\begin{aligned}
f_{t-1}(z) &= -\frac{(z - b_t \text{sign}(z))^2}{a_t + 2} + z^2 + c_t \\
&= -\frac{z^2 - 2b_t |z| + b_t^2}{a_t + 2} + z^2 + c_t \\
&= \frac{a_t + 1}{a_t + 2} z^2 + \frac{2b_t}{a_t + 2} |z| + c_t - \frac{b_t^2}{a_t + 2}.
\end{aligned}$$

For $z \in [-b_t, b_t]$, the value of $f_t(z)$ is not needed in the calculation of $g_t(x)$ because $0 < b_t < 1$ (induction hypothesis) and the adversary — who wants to maximize $f_t(z_t)$, a convex, even function — will never choose w_t such that $z_t = x_t + w_t \in (-1, 1)$ since w_t can be chosen from $[-1, 1]$.

$$\begin{aligned}
g_{t-1}(x) &= \max_w f_t(x+w) = f_t(x + \text{sign}(x)) \\
&= \frac{a_t + 1}{a_t + 2} (x^2 + 2|x| + 1) + \frac{2b_t}{a_t + 2} (|x| + 1) + c_t - \frac{b_t^2}{a_t + 2} \\
&= \frac{a_t + 1}{a_t + 2} x^2 + \frac{2(a_t + b_t + 1)}{a_t + 2} |x| + c_t + \frac{a_t + 1 + 2b_t - b_t^2}{a_t + 2} \\
&= a_{t-1} x^2 + 2b_{t-1} |x| + c_{t-1}.
\end{aligned}$$

Now, we have obtained the recursive formulae for a_t, b_t, c_t . The initial values are $a_{T-1} = b_{T-1} = c_{T-1} = 1/2$.

Let f_i be the i -th Fibonacci number with $f_0 = 0, f_1 = 1$. Then, $a_{T-i} = f_{i+1}/f_{i+2}$. As $i \rightarrow \infty$, $a_{T-i} \rightarrow \frac{\sqrt{5}-1}{2}$.

For b_t , we have $1 - b_{T-(i+1)} = (1 - b_{T-i})/(a_{T-i} + 2)$. When i is large, $1 - b_{T-i}$ approaches 0 but is always positive. Thus, b_{T-i} approaches 1 but is always less than 1.

For c_t , we have

$$c_{T-(i+1)} = c_{T-i} + 1 - \frac{(1 - b_{T-i})^2}{a_{T-i} + 2}$$

and thus $c_{T-(i+1)} - c_{T-i} \rightarrow 1$. Therefore, $\text{ADV}_1 = 1$.

The optimal control policy is obtained by plugging the above values back into Equation (21):

$$u^*(x, w) = -(x+w) + y^*(x+w) = -(x+w) + \begin{cases} 0 & , -1 \leq x+w \leq 1 \\ \frac{x+w - \text{sign}(x+w)}{\frac{\sqrt{5}+3}{2}} & , \text{otherwise} \end{cases}.$$

For ADV_∞ , we will show that $\text{STO}_\infty = 1$ at a specific disturbance sequence: $w_t = 1$ for all t . Because $\text{STO}_\infty \leq \text{ADV}_\infty \leq \text{ADV}_1 = 1$, we know that $\text{ADV}_\infty = 1$.

According to Equations (8) and (9) with $k \rightarrow \infty$,

$$\text{STO}_\infty = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} (2w_t \psi_t - Pw_t^2 - H\psi_t^2) \text{ with } \psi_t = \sum_{i=0}^{\infty} F^i Pw_{t+i}.$$

Solving the Riccati equation, we have $P = \frac{1+\sqrt{5}}{2}$, $H = F = \frac{3-\sqrt{5}}{2}$. When $w_t = 1$ for all t , $\text{STO}_\infty = 1$. \square

C.3 Proof of Theorem 5.3

$$\text{MPCA}_k - \text{MPCA}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k}).$$

Proof. Note that Equation (16) in the proof of Theorem 4.4 does not rely on the type of disturbance, i.e., Equation (16) holds for adversarial disturbance as well. Let $r = \sup_{w \in \Omega} \|w\|_2$.

$$\begin{aligned} q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1}) &= w_{t+k}^\top P F^k H F^{\top k} \left(P w_{t+k} + 2 \sum_{i=1}^{\infty} F^{\top i} P w_{t+i+k} \right) \\ &\leq \|w_{t+k}\| \|P\| \|H\| \|F^k\|^2 \left(\|P\| \|w_{t+k}\| + 2 \sum_{i=1}^{\infty} \|F^i\| \|P\| \|w_{t+i+k}\| \right) \\ &\leq \|F^k\|^2 \left(1 + 2 \sum_{i=1}^{\infty} \|F^i\| \right) \|H\| \|P\|^2 r^2 \\ &\leq \|F^k\|^2 \left(1 + 2 \frac{c\lambda}{1-\lambda} \right) \|H\| \|P\|^2 r^2 \end{aligned}$$

for some constant c .

$$\begin{aligned} \text{MPCA}_k - \text{MPCA}_{k+1} &= \lim_{T \rightarrow \infty} \frac{1}{T} (\max_w q_0^k - \max_w q_0^{k+1}) \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \max_w (q_0^k - q_0^{k+1}) \\ &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \max_w (q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})) \\ &\leq \|F^k\|^2 \left(1 + 2 \frac{c\lambda}{1-\lambda} \right) \|H\| \|P\|^2 r^2 = O(\|F^k\|^2). \end{aligned}$$

□

C.4 Proof of Theorem 5.5

$$\text{Reg}^A(\text{MPC}_k) = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1).$$

Proof. We follow the notations in the proof of Theorem 4.6. Equation (18) does not rely on the type of disturbance, so it holds for adversarial disturbance as well. By Equation (18) and the fact that w_t is bounded, we have

$$q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1}) = O(\|F^k\|(\lambda^{T-t} + \|F^k\|)),$$

where the constant in the Big-Oh notation does not depend on the disturbance sequence w . Thus,

$$\max_w (q_0^k - q_0^T) \leq \sum_{t=0}^{T-1} \max_w (q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})) = O(\|F^k\|^2 T + \|F^k\|).$$

By Equation (17) and the boundedness of w_t ,

$$\max_w (v_0^k - v_0^T) = 2 \max_w (d_0^k - d_0^T) = O(\lambda^{T+k} \|F^k\|).$$

$$\begin{aligned} \max_w (J^{\text{MPC}_k} - J^{\text{MPC}_T}) &= \max_w (V_0^k(x_0) - V_0^T(x_0)) \leq \max_w (x_0^\top (v_0^k - v_0^T)) + \max_w (q_0^k - q_0^T) \\ &= O(\|F^k\|^2 T + \|F^k\|). \end{aligned}$$

As Equation (20), $J^{\text{MPC}_T} - \min_u J = O(1)$. Thus,

$$\begin{aligned} \text{Reg}^A(\text{MPC}_k) &= \max_w (J^{\text{MPC}_k} - \min_u J) \leq \max_w (J^{\text{MPC}_k} - J^{\text{MPC}_T}) + \max_w (J^{\text{MPC}_T} - \min_u J) \\ &= O(\|F^k\|^2 T + \|F^k\| + 1) = O(\|F^k\|^2 T + 1). \end{aligned}$$

□

C.5 Proof of Theorem 5.6

$Reg_k^{A^*} = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$. Moreover, there exist A, B, Q, R, Q_f, x_0 , and Ω such that $Reg_k^{A^*} = \Omega(\|F^k\|^2(T - k))$.

Proof. The first part of the theorem follows from Theorem 5.5 and the fact that $Reg_k^{A^*} \leq Reg^A(\text{MPC}_k)$.

We reduce the second part of this theorem to the second part of Theorem 4.7. Since the proof of Theorem 4.7 works for any fixed distribution of w_t (with finite second moment), we can restrict that distribution to have bounded support. Denote this bounded support by Ω . Then, we have

$$\begin{aligned} Reg_k^{A^*} &= \sup_{w_0, \dots, w_{k-1}} \min_{u_0} \sup_{w_k} \dots \min_{u_{T-k-1}} \sup_{w_{T-1}} \min_{u_{T-k}, \dots, u_{T-1}} \left(J(u, w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right) \\ &\geq \mathbb{E}_{w_0, \dots, w_{k-1}} \min_{u_0} \mathbb{E}_{w_k} \dots \min_{u_{T-k-1}} \mathbb{E}_{w_{T-1}} \min_{u_{T-k}, \dots, u_{T-1}} \left(J(u, w) - \min_{u'_0, \dots, u'_{T-1}} J(u', w) \right) \\ &= Reg_k^{S^*} = \Theta(\|F^k\|^2(T - k)). \end{aligned}$$

□