Quantum Error-Correcting Codes
for Amplitude Damping

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Abstract—**Traditional quantum error-correcting codes are designed for the depolarizing channel modeled by generalized Pauli errors occurring with equal probability. Amplitude damping channels, in general, model the decay process of a multilevel atom or energy dissipation of a bosonic system at zero temperature. We discuss quantum error-correcting codes adapted to amplitude damping channels for higher dimensional systems (qudits). For multi-level atoms, we consider a natural kind of decay process, and for bosonic systems, we consider the qudit amplitude damping channel obtained by truncating the Fock basis of the bosonic modes to a certain maximum occupation number. We construct families of single-error-correcting quantum codes that can be used for both cases. Our codes have larger code dimensions than the previously known single-error-correcting codes of the same lengths.**

Index Terms—**amplitude damping channel, quantum codes**

I. INTRODUCTION

For a q-level quantum system with Hilbert space $\mathbb{C}^q$, called a qudit, the most general physical operations (or quantum channels) allowed by quantum mechanics are completely positive, trace preserving linear maps which can be represented in the following Kraus decomposition form $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$, where the matrices E_k are called Kraus operators of the quantum channel $\mathcal{N}$ and satisfy the trace-preserving condition $\sum_k E_k^\dagger E_k = \mathbf{1}$.

In designing error-correcting codes for sending n qudits through a channel $\mathcal{N}$, it is usually assumed that the errors to be corrected are completely random, with no knowledge other than that they affect different qudits independently [1], [2]. That is, the channel $\mathcal{N}$ is the depolarizing channel modeled by equal probability of generalized Pauli operators [3]–[5] $(X_q)^a (Z_q)^b$, for $a, b \in \{0, 1, \ldots, q-1\}$, where $X_q|s\rangle = |s+1 \mod q\rangle$ and $Z_q|s\rangle = \omega^s|s\rangle$, $\omega = \exp(2\pi i/q)$. When it is clear from the context, we may just write X and Z, dropping the index q.

However, if further information about the error process is available, more efficient codes can be designed. Indeed, in many physical systems, the noise is likely to be unbalanced between amplitude (X-type) errors and phase (Z-type) errors. Recently a lot of attention has been put into designing codes for this situation and into studying their fault tolerance prop-

erties [6]–[11]. All these results use error models described by Kraus operators that are generalized Pauli operators, but for which the X-type errors (i.e., non-diagonal Pauli matrices) happen with probability p_x which might be different from the probability p_z that Z-type errors (i.e., diagonal Pauli matrices) happen. Quantum channels affected by this kind of noise are called asymmetric channels.

A closer look at the real physical process of amplitude damping noise shows that one needs to go even further, beyond Kraus operators of Pauli type. To be more precise, for $q = 2$, the qubit amplitude damping (AD) channel is given by the Kraus operators [12]

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (1)$$

Since the error model of the qubit AD channel is not described by Pauli Kraus operators, the known techniques dealing with Pauli errors result in codes with non-optimal parameters. Several new techniques for the construction of codes adapted to this type of noise with non-Pauli Kraus operators, and the qubit AD channel in particular, have been developed [8], [12]–[15]. Systematic methods for constructing high performance single-error-correcting codes [13], [15] and multi-error-correcting codes [16] have been found.

In this paper, we discuss quantum code constructions for AD channels of general qudit systems. Unlike the qubit case, where the AD channel is unique, for qudit systems there are different AD channels associated with different physical systems. We will focus on two different models: multi-level atoms with a natural kind of decay process, and bosonic systems obtained by truncating the Fock basis of the bosonic modes to the maximum occupation number $q - 1$ for a single bosonic mode.

II. THE AMPLITUDE DAMPING CHANNEL

For two-level atoms, the decay process at zero temperature is described by the Kraus operators A_0, A_1 as given in Eq. (1). For multi-level atoms, there are different kinds of decay processes at zero temperature. One natural decay process is the cascade structure Ξ, where the decay process is governed

by the master equation [17], [18]

$$\frac{d\rho}{dt} =$$
$$\sum_{1 \leq i \leq q-1} k_i \left(2\sigma^-_{i-1,i} \rho \sigma^+_{i-1,i} - \sigma^+_{i-1,i} \sigma^-_{i-1,i} \rho - \rho \sigma^+_{i-1,i} \sigma^-_{i-1,i} \right). \tag{2}$$

Here $\{|i\rangle\}_{i=0}^{q-1}$ is a basis of the Hilbert space $\mathbb{C}^q$, and $\sigma^-_{i-1,i} = |i-1\rangle\langle i|$ and $\sigma^+_{i-1,i} = |i\rangle\langle i-1|$.

The solution to this master equation gives the Kraus expression

$$\Xi(\rho) = A_0 \rho A_0^\dagger + \sum_{0 \leq i < j \leq q-1} A_{ij} \rho A_{ij}^\dagger, \tag{3}$$

where $A_{ij} = \sqrt{\gamma_{ij}} |i\rangle\langle j|$ with positive coefficients γ_{ij}, and A_0 is a diagonal matrix with its elements given by $A_0^\dagger A_0 + \sum_{0 \leq i < j \leq q-1} A_{ij}^\dagger A_{ij} = I$. Furthermore, when the decay time t is small, γ_{ij} is of order t^ℓ for any $j = i + \ell$, $\ell > 0$. As a consequence, A_0 is of order t, and A_{ij} is of order $t^{\ell/2}$ for any $j = i + \ell$, $\ell > 0$. This is intuitively sound as for the cascade structure, the first order transition always happens from $|i+1\rangle$ to $|i\rangle$.

As an example, for three-level atoms, i.e., $q = 3$, we have

$$A_{01} = \sqrt{\gamma_{01}} |0\rangle\langle 1|, \qquad A_{12} = \sqrt{\gamma_{12}} |1\rangle\langle 2|,$$
$$A_{02} = \sqrt{\gamma_{02}} |0\rangle\langle 2|,$$
$$A_0 = |0\rangle\langle 0| + \sqrt{1-\gamma_{01}} |1\rangle\langle 1| + \sqrt{1-\gamma_{02}-\gamma_{12}} |2\rangle\langle 2|,$$

where

$$\gamma_{01} = 2k_2 t + O(t^2),$$
$$\gamma_{02} = 2(k_1 + k_2)t^2 + O(t^3),$$
$$\gamma_{12} = 2k_1 t + O(t^2),$$

for $k_1 \neq k_2$. The values of γ_{ij} are slightly different for $k_1 = k_2$, but the order of γ_{ij} in terms of t remains similar.

The channel $\mathcal{A}$ describing energy dissipation of a bosonic system at zero temperature is discussed in [12]. The Kraus operators are given by

$$A_k = \sum_{r=k}^{q-1} \sqrt{\binom{r}{k}} \sqrt{(1-\gamma)^{r-k}\gamma^k} |r-k\rangle\langle r|, \tag{4}$$

where $q - 1$ is the maximum occupation number of a single bosonic mode, and $k = 0, 1, \ldots, q-1$. The parameter γ is of first order in terms of the decay time t, i.e., $\gamma = ct + O(t^2)$. As a consequence, the non-identity part of A_0 is of order t, and A_k is of order $t^{k/2}$ for $1 \leq k \leq d-1$.

For instance, for the qubit case, i.e., $q = 2$, we have the qubit amplitude channel given by Eq. (1). For $q = 3$, we have

$$A_0 = |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1| + (1-\gamma)|2\rangle\langle 2|,$$
$$A_1 = \sqrt{\gamma} |0\rangle\langle 1| + \sqrt{2\gamma(1-\gamma)} |1\rangle\langle 2|,$$
$$\text{and} \quad A_2 = \gamma |0\rangle\langle 2|.$$

Note that for $q = 3$, the non-diagonal Kraus operators of the channel $\mathcal{A}$ for bosonic systems are linear combinations of the Kraus operators of the channel Ξ. Hence codes correcting errors of the channel Ξ are also codes for the channel $\mathcal{A}$.

III. ERROR CORRECTION CRITERIA

A quantum error-correcting code Q is a subspace of $(\mathbb{C}^q)^{\otimes n}$, the space of n qudits. For a K-dimensional code space spanned by the orthonormal basis $|\psi_i\rangle$, $i = 1, \ldots, K$, and a set of errors $\mathcal{E}$, there is a physical operation correcting all elements $E_k \in \mathcal{E}$ if the error correction conditions [19], [20] are satisfied:

$$\forall i, j, k, l : \langle \psi_i | E_k^\dagger E_l | \psi_j \rangle = \lambda_{kl} \delta_{ij}, \tag{5}$$

where λ_{kl} depends only on k and l. A code is said to be pure with respect to some set of errors $\mathcal{E}$ if $\lambda_{kl} = \delta_{kl}$.

For the AD channels Ξ and $\mathcal{A}$, if the decay time t is small, we would like to correct the leading order errors that occur during amplitude damping. Similar as for the qubit case [2, Section 8.7], in order to improve the fidelity of the transmission through the AD channel Ξ or $\mathcal{A}$ from $1 - O(t)$ to $1 - O(t^2)$, i.e., to correct a single error, it is sufficient to be able to detect one A_0 error and to correct one A_{ij} error with $j = i + 1$ for the channel Ξ (or to correct one A_i error with $i > 0$ for the channel $\mathcal{A}$).

Stabilizer codes are a large class of quantum codes which contain many good quantum codes [1], [2]. A stabilizer code with n qudits encoding k qudits has distance d if all errors of weight at most $d - 1$ can be detected or have no effect on Q, and we denote the parameters of Q by $[\![n, k, d]\!]_q$. Obviously a stabilizer code of distance 3 corrects a single AD error as it corrects an arbitrary single error.

Calderbank-Shor-Steane (CSS) codes [21], [22] are a subclass of the stabilizer codes. It has been shown that CSS codes can be used to construct codes for the binary AD channel [2, Section 8.7]. This has a direct generalization to the qudit case [23]. For single-error correcting codes for qudit AD channels, the following holds.

Theorem 1: An $[\![n, k]\!]_q$ CSS code Q with pure X-distance 3 and Z-distance 2 corrects a single AD error.

Proof: The diagonal error operator A_0 can be expanded in terms of the error operators Z_q^l, with the expansion coefficients of the operators Z_q^l, $l > 0$ being of first order in t. As the code Q is has Z-distance 2, all diagonal Kraus operators acting on a single qubit can be detected.

The diagonal of the other error operators A_{ij} or A_i is zero. They can be expanded in terms of operators $Z_q^l X_q^k$, $k \neq 0$, with the expansion coefficients being of order $\sqrt{t}$. We have to show that such an error can be corrected. Consider the erroneous states $Z_{(\alpha)}^l X_{(\alpha)}^k |\psi_i\rangle$ and $Z_{(\beta)}^{l'} X_{(\beta)}^{k'} |\psi_j\rangle$, where $M_{(\alpha)}$ denotes an operator acting at position α. The inner product between these two states is $\langle \psi_i | X_{(\alpha)}^{-k} Z_{(\alpha)}^{-l} Z_{(\beta)}^{l'} X_{(\beta)}^{k'} | \psi_j \rangle$. For $\alpha = \beta$ and $k = k'$, the X-parts cancel and we are left with a single Z-error that can be detected. For $l = l'$, we have pure X-errors of weight less than three, and Condition (5) holds as well. What is more, the erroneous states are superpositions of states from mutually orthogonal spaces. Hence (5) remains zero when introducing additional Z-errors $Z_{(\alpha)}^l$ and $Z_{(\beta)}^{l'}$. ∎

Corollary 2: If there exists an $[n, k+1, 3]_q$ linear code C containing the all-one-vector $\mathbf{1} \in C$, then there exists an $[\![n, k]\!]_q$ CSS code correcting a single AD error.

IV. CLASSICAL ASYMMETRIC CODES

We construct quantum codes correcting a single AD error. Codes for the qubit case have been presented in [13], [15]. Those codes are self-complementary, i.e., the basis states are of the form $|\psi_{\mathbf{u}}\rangle = \frac{1}{\sqrt{2}}(|\mathbf{u}\rangle + |\bar{\mathbf{u}}\rangle)$, where $\mathbf{u}$ is an n-bit string, $\bar{\mathbf{u}} = \mathbf{1} \oplus \mathbf{u}$, and $\mathbf{1}$ is the all-one-string.

For the non-binary case with $q > 2$, we consider a similar construction. Define $\bar{X} = X_q^{\otimes n}$, then the basis states are chosen as

$$|\psi_{\mathbf{u}}\rangle = \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} \bar{X}^l |\mathbf{u}\rangle. \tag{6}$$

For instance, for $q = 3$ and $n = 3$, we get $|\psi_{\mathbf{0}}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$.

The quantum code $\mathcal{Q}$ is then spanned by $\{|\psi_{\mathbf{u}}\rangle\}$, where $\mathbf{u} \in \tilde{C}$ is some length-n string ($\tilde{C}$ is a classical code of length n). The advantage of this construction is that the code automatically satisfies the error-detection condition for a single Z_q^l error ($l = 1, 2, \ldots, q - 1$), as the code is stabilized by $\bar{X}$. Now consider a classical code with codewords $C = \{\mathbf{u} + \alpha\mathbf{1} : \mathbf{u} \in \tilde{C}, \alpha = 0, \ldots, q-1\}$ and the corresponding quantum code spanned by $\{|\psi_{\mathbf{u}}\rangle : \mathbf{u} \in \tilde{C}\}$. The problem of correcting a single error for the qudit AD channels can then be reduced to finding certain classical codes.

The relevant classical channel is the classical asymmetric channel [24]. Let the alphabet be $\mathbb{Z}_q$ with the ordering $0 < 1 < 2 < \cdots < q - 1$. A channel is called asymmetric if any transmitted symbol a is received as $b \leq a$. The mostly studied asymmetric channel dates back to Varshamov [25], which can be described by the following asymmetric distance $\Delta(\mathbf{x}, \mathbf{y})$.

Definition 3 (see [26]): Let $B = \{0, 1, \ldots, q-1\} \subset \mathbb{Z}$. For $\mathbf{x}, \mathbf{y} \in B^n$, we define

1) $w(\mathbf{x}) := \sum_{i=1}^{n} x_i$.
2) $N(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} \max\{y_i - x_i, 0\}$.
3) $\Delta(\mathbf{x}, \mathbf{y}) := \max\{N(\mathbf{x}, \mathbf{y}), N(\mathbf{y}, \mathbf{x})\}$.

If $\mathbf{x}$ is sent and $\mathbf{y}$ is received, we say that $w(\mathbf{x} - \mathbf{y})$ errors have occurred.

Theorem 4 (see [26]): A code $C \in B^n$ corrects r errors of the asymmetrical channel if and only if $\Delta(\mathbf{x}, \mathbf{y}) > r$ for all $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \neq \mathbf{y}$.

Our goal is to link these classical asymmetric codes to quantum AD codes. As discussed above, we start from the following definition.

Definition 5: A classical code C over the alphabet B is called self-complementary if for any $\mathbf{x} \in C$, $\mathbf{1} \oplus \mathbf{x} \in C$.

For any self-complementary code C, there exists another code $\tilde{C}$ such that $C = \{\mathbf{u} + \alpha\mathbf{1} : \mathbf{u} \in \tilde{C}, \alpha = 0, \ldots q-1\}$ and $|C| = q|\tilde{C}|$. We may, for example, chose $\tilde{C}$ such that $u_1 = 0$ for $\mathbf{u} \in \tilde{C}$. This then corresponds to the quantum code $\mathcal{Q}$ spanned by $\{|\psi_{\mathbf{u}}\rangle : \mathbf{u} \in \tilde{C}\}$ as given in Eq. (6). Our main result is given by the following theorem.

Theorem 6: If C is a classical (linear or non-linear) self-complementary code correcting a single error with respect to Definition 3, then $\mathcal{Q}$ spanned by $\{|\psi_{\mathbf{u}}\rangle : \mathbf{u} \in \tilde{C}\}$ is a single-error-correcting code for the qudit AD channels Ξ and $\mathcal{A}$.

Proof: Let $E_{ij} = |i\rangle\langle j|$ with $i, j \in \{0, 1, \ldots, q - 1\}$ and $i < j$. For a small decay time t, in order to improve the fidelity of the transmission through the qudit AD channel $\mathcal{A}$ given by Eq. (4) from $1 - O(t)$ to $1 - O(t^2)$, it is sufficient to correct a single $E_{i,i+1}$-error and detect one Z_q^l-error for $l = 1, 2, \ldots, q - 1$. The self-complementary form of $|\psi_{\mathbf{u}}\rangle$ as given in Eq. (6) implies that $\bar{X}|\psi_{\mathbf{u}}\rangle = |\psi_{\mathbf{u}}\rangle$. In turn, this implies that $\langle\psi_{\mathbf{v}}|Z_q^l|\psi_{\mathbf{u}}\rangle = 0$ for any $\mathbf{u}, \mathbf{v}$ and $l = 1, 2, \ldots, q - 1$, i.e., the error-detection condition for a single Z_q^l error is fulfilled.

Next consider a single operator $E_{i,i+1}$. Every state of the quantum code is a linear combination of states $|\mathbf{c}\rangle$ with $\mathbf{c} \in C$. Applying the operator $E_{i,i+1}$ to $|\mathbf{c}\rangle$ corresponds to a single asymmetric error. As the classical code C corrects a single asymmetric error, the distance between any two codewords is at least two. Therefore, the supports (set of basis states with non-zero coefficient in the superposition) of the states $|\psi_{\mathbf{u}}\rangle$ and $E_{i,i+1}^{(\alpha)}|\psi_{\mathbf{v}}\rangle$ are disjoint for all positions α, where $E_{i,i+1}^{(\alpha)}$ denotes the operator $E_{i,i+1}$ acting at position α. Hence those states are mutually orthogonal. Finally note that for errors $E_{i,i+1}$ acting on the same position, the operator $E_{i,i+1}^{\dagger}E_{i,i+1}$ is diagonal and hence in the span of the operators Z_q^l, which can be detected. ∎

Corollary 7: If there exists an $(n, K, 3)_q$ self-complementary code C, then there exists an $((n, K/q))_q$ quantum code correcting a single AD error.

Such codes have, e.g., been studied in [27].

V. SINGLE-ERROR-CORRECTING AD CODES

We now use Theorem 6 to construct some families of good single-error-correcting AD codes. For this, we need to find some good self-complementary single-error-correcting classical asymmetric codes.

We will use the idea of generalized concatenation, which has been discussed in the context of constructing binary AD codes in [15], and in the context of constructing (classical) asymmetric codes in [28]. This method will allow us to construct good self-complementary asymmetric linear codes for the non-binary case, which will lead to good single-error-correcting quantum codes for AD channels.

A. Qutrit Codes

First, we consider the case of $q = 3$. For the generalized concatenation construction, we choose the outer code as some ternary classical code over the alphabet $\{\tilde{0}, \tilde{1}, \tilde{2}\}$. The inner codes are chosen as:

$$C_{\tilde{0}} = \{00, 11, 22\}, \ C_{\tilde{1}} = \{01, 12, 20\}, \ C_{\tilde{2}} = \{02, 10, 21\}. \tag{7}$$

Then we have the following construction.

Theorem 8: For n even, generalized concatenation with an outer $[n/2, k, 3]_3$ code results in an $[n, n/2 + k]_3$ self-complementary linear code C. This code leads to an $[\![n, n/2 + k - 1]\!]_3$ quantum stabilizer code $\mathcal{Q}$, correcting a single error for the channels Ξ and $\mathcal{A}$.

Proof: Note that $C_{\tilde{0}}$, $C_{\tilde{1}}$, and $C_{\tilde{2}}$ are all self-complementary codes correcting a single asymmetric error. Therefore, any outer ternary code will lead to a self-complementary ternary

code C, and hence a quantum code $\mathcal{Q}$. A single amplitude damping error induces only a single error with respect to $\tilde{0}$, $\tilde{1}$, $\tilde{2}$. As the outer ternary code has distance 3, such an error can be corrected. ∎

Note that with respect to the symbols $\tilde{0}, \tilde{1}, \tilde{2}$, the induced channel $\mathcal{R}_3$ is nothing but the ternary symmetric channel shown in FIG. 1.
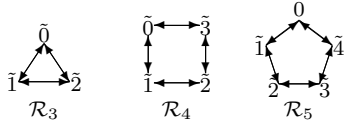

Fig. 1. The induced channel $\mathcal{R}_3$ for $q = 3$ (which is just the ternary symmetric channel), the induced channel $\mathcal{R}_4$ for $q = 4$, and the induced channel $\mathcal{R}_5$ for $q = 5$. The arrows indicate the possible transitions between symbols.

Example 9: For $n = 6$, take the outer code of length $n/2 = 3$ as $\{\tilde{0}\tilde{0}\tilde{0}, \tilde{1}\tilde{1}\tilde{1}, \tilde{2}\tilde{2}\tilde{2}\}$ with distance 3. Generalized concatenation yields a self-complementary ternary linear code of dimension 4. The corresponding quantum code $\mathcal{Q}$ encodes $6/2 + 1 - 1 = 3$ qutrits. Both the best corresponding single-error-correcting quantum code $[\![6, 2, 3]\!]_3$ as well as the best possible asymmetric CSS code $[\![6, 2, \{3, 2\}]\!]_3$ (see Corollary 2) encode only 2 qutrits.

As shown in Table I, for many lengths, the construction based on Theorem 8 outperforms both the quantum codes with distance 3, and the CSS codes of Corollary 2 (cf. [27]). The dimension of the asymmetric quantum codes (AQECC) is taken from [27].

TABLE I
DIMENSION OF SINGLE-ERROR-CORRECTING QUANTUM AD CODES FROM THE $GF(3^2)$ CONSTRUCTION WITH DISTANCE 3, THE CSS CONSTRUCTION, ASYMMETRIC QUANTUM CODES (AQECC), AND THE GENERALIZED CONCATENATION CONSTRUCTION (GC).

n	$GF(3^2)$	CSS	AQECC	GC
4	3^0	3^0	1	–
6	3^2	3^2	11	3^3
8	3^4	3^4	84	3^5
10	3^6	3^6	3^6	3^6
12	3^7	3^8	3^8	3^8
14	3^9	3^9	3^9	3^{10}
16	3^{11}	3^{11}	3^{11}	3^{12}

B. The Case $q > 3$

For $q = 4$, we choose the inner codes as

$$C_{\tilde{0}} = \{00, 11, 22, 33\}, \quad C_{\tilde{1}} = \{01, 12, 23, 30\},$$
$$C_{\tilde{2}} = \{02, 13, 20, 31\}, \quad C_{\tilde{3}} = \{03, 10, 21, 32\}. \quad (8)$$

Similar as in Theorem 8, an outer code with distance three yields a self-complementary code from which a quantum AD code can be derived. However, in this case, the induced channel for the outer code is no longer symmetric. A single damping error will, for example, never map a codeword of the inner code $C_{\tilde{0}}$ to a codeword of $C_{\tilde{2}}$. So on the level of the outer code, there are no transitions between $\tilde{0}$ and $\tilde{2}$, or between $\tilde{1}$ and $\tilde{3}$. The induced quaternary channel $\mathcal{R}_4$ is shown in FIG. 1, where we see that errors only happen between 'neighbors.'

The above constructions for $q = 3, 4$ have a direct generalization to general $q > 2$. For a given q, choose the outer

code as some code over the alphabet $\{\tilde{0}, \tilde{1}, \ldots, \widetilde{q-1}\}$. The q inner codes $C_{\tilde{0}}, C_{\tilde{1}}, \ldots, C_{\widetilde{q-1}}$ are the double-repetition code $C_{\tilde{0}} = \{00, 11, \ldots, (q-1)(q-1)\}$ and all its $q - 1$ cosets $C_{\tilde{i}} = C_{\tilde{0}} \oplus (0i)$, i.e., we apply the rule that $0i \in C_{\tilde{i}}$. It is straightforward to check that each inner code has asymmetric distance 2, hence corrects a single asymmetric error. Similar as in the case of $q = 4$, a single damping error will only drive transitions between $\tilde{i}, \tilde{j}$ for $\tilde{i} = \tilde{j} \pm \tilde{1}$. For instance, for $q = 5$, the induced channel $\mathcal{R}_5$ is shown in FIG. 1. In general, we will write the induced channel as $\mathcal{R}_q$ for outer codes over $\{\tilde{0}, \tilde{1}, \ldots, \widetilde{q-1}\}$.

Similar as Theorem 8, in general we have the following theorem.

Theorem 10: For n even, an outer $[n/2, k]_q$ code correcting a single error for the channel $\mathcal{R}_q$ leads to an $[n, n/2 + k]_q$ self-complementary linear code C and hence an $[\![n, n/2 + k - 1]\!]_q$ quantum code $\mathcal{Q}$, correcting a single error for the qudit AD channels Ξ and $\mathcal{A}$.

Note that the channel $\mathcal{R}_q$ is no longer a symmetric channel, so outer codes of Hamming distance 3 are no longer expected to give the best codes. It turns out, however, that single-error-correcting codes for the channel $\mathcal{R}_q$ are equivalent to single-symmetric-error correcting codes in Lee metric [29] (see also [30]), for which optimal linear codes are known (for more detailed discussion, see [28]).

VI. THE CASE OF ODD LENGTHS

The construction of AD codes for even lengths given in Sec. V based on generalized concatenation is relatively straightforward. The inner codes are just 1-codes of length 2 with q codewords and their cosets. In [28], codes of odd length were obtained using a mixed-alphabet code, treating one position differently. This does not directly translate to the situation considered here, as the resulting code has to be self-complementary.

Instead, we can directly search for q mutually disjoint inner codes of length 3 which are 1-codes.

For $q = 4$, consider the following $\mathbb{Z}_4$-linear code $C_{0'}$ of length 3 generated by $\{111, 002, 020\}$:

$$000\ 111\ 222\ 333 \qquad 002\ 113\ 220\ 331$$
$$020\ 131\ 202\ 313 \qquad 200\ 311\ 022\ 133. \quad (9)$$

The code $C_{0'}$ has asymmetric distance 2, as well as the three cosets $C_{1'} = C_{0'} + 001$, $C_{2'} = C_{0'} + 010$, and $C_{3'} = C_{0'} + 100$. Applying generalized concatenation to the outer code $\{\tilde{0}\tilde{0}0', \tilde{1}\tilde{1}1', \tilde{2}\tilde{2}2', \tilde{3}\tilde{3}3'\}$ and the inner codes of length 2 and 3 for the first two and the third position, respectively, yields a self-complementary 1-code $[7, 5]_4$. The corresponding quantum code has parameters $[\![7, 4]\!]_4$.

Note that the induced channel on the alphabet $\{0', 1', 2', 3'\}$ is no longer $\mathcal{R}_4$, but the symmetric channel over $\mathbb{Z}_4$. Therefore we have the following theorem for $q = 4$.

Theorem 11: For n odd, an outer $[(n-1)/2, k, 3]_4$ code leads to an $[n, (n+1)/2 + k]_4$ self-complementary linear 1-code C. The resulting quantum code $\mathcal{Q} = [\![n, (n-1)/2 + k]\!]_4$ corrects a single error for the qudit AD channels Ξ and $\mathcal{A}$.

Proof: The inner codes $C_{\tilde{i}}$ of length two as well as the inner codes $C_{i'}$ of length three are self-complementary 1-codes. Distance 3 of the outer code ensures that a single error mixing the inner codes can be corrected. For the outer code, we always take the last coordinate to be of type s', and all the other coordinates to be of type $\tilde{s}$, for $s = 0, \ldots, 3$. Therefore, for an outer $[(n-1)/2, k, 3]_4$ linear code, generalized concatenation yields an $[n, (n+1)/2 + k]_4$ self-complementary linear 1-code C, corresponding to an $[\![n, (n-1)/2 + k]\!]_4$ quantum code. ∎

We emphasize that the construction related to Theorem 11 is valid only for $q = 4$. For $q > 5$, however, the $\mathbb{Z}_q$-linear code C_0 generated by $\{111, 013\}$ and its q cosets are all self-complementary codes with asymmetric distance 2. For this, note that $\Delta(\mathbf{x}, \mathbf{y}) = 1$ if and only if, up to permutation, $\mathbf{x} - \mathbf{y} \in \{(1, 0, 0), (1, -1, 0)\}$. For $q > 5$, the code C_0 does not contain such a vector. Hence we obtain the analogue result as in Theorem 11 for $q > 5$. For $q = 3$ and $q = 5$, however, we cannot partition the trivial code $[3, 3]_q$ into q self-complementary codes $[3, 2]_q$ with asymmetric distance 2.

VII. Discussion

For correcting a single AD error, we use the error correction criteria that it is sufficient to detect one A_0 error and to correct one A_{ij} error with $j = i + 1$ for the channel Ξ (or one A_i error for the channel $\mathcal{A}$). Although this follows quite naturally from the analogy to the qubit case and can be generalized to correcting multiple AD errors, we leave the derivation of the more general criteria to the journal version of this paper.

For the decay process of multi-level atoms, we only considered the cascade structure Ξ, but there are other cases. For three-level atoms, for instance, the other two important types of AD channels are V-type and Λ-type [17], [18]. Although the code construction discussed for the AD channel Ξ does not directly apply to these other types of AD channels, the self-complementary construction helps to reduce the problem to finding some classical asymmetric codes.

It remains open how to construct good single-error-correcting AD codes based on generalized concatenation (or some other methods) for odd lengths when $q = 3$ or $q = 5$. It is also desired to find AD codes correcting multiple errors. These questions will be addressed in the journal version of this paper.

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References

[1] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*. Cambridge, England: Cambridge University Press, 2000.

[2] D. Gottesman, "Stabilizer codes and quantum error correction," Ph.D. dissertation, California Institute of Technology, 1997.

[3] D. Gottesman, A. Kitaev, and J. Preskill, "Encoding a qubit in an oscillator," *Physical Review A*, vol. 64, no. 1, p. 012310, July 2001.

[4] J. Patera and H. Zassenhaus, "The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type A$_{n-1}$," *Journal of Mathematical Physics*, vol. 29, pp. 665–673, Mar. 1988.

[5] E. Knill, "Non-binary unitary error bases and quantum codes," 1996, arXiv:quant-ph/9608048.

[6] P. Aliferis and J. Preskill, "Fault-tolerant quantum computation against biased noise," 2007, arXiv:0710.1301 [quant-ph].

[7] Z. W. E. Evans, A. M. Stephens, J. H. Cole, and L. C. L. Hollenberg, "Error correction optimisation in the presence of x/z asymmetry," 2007, arXiv:0709.3875 [quant-ph].

[8] A. S. Fletcher, P. W. Shor, and M. Z. Win, "Channel-adapted quantum error correction for the amplitude damping channel," *IEEE Transactions on Information Theory*, vol. 54, no. 12, pp. 5705–5718, 2008.

[9] L. Ioffe and M. Mezard, "Asymmetric quantum error-correcting codes," *Physical Review A*, vol. 75, no. 3, p. 032345, 2007.

[10] P. K. Sarvepalli, M. Roetteler, and A. Klappenecker, "Asymmetric quantum LDPC codes," in *Proceedings of the 2008 IEEE International Symposium on Information Theory*, 2008, pp. 305–309.

[11] L. Wang, K. Feng, S. Ling, and C. Xing, "Asymmetric quantum codes: Characterization and constructions," *IEEE Transactions on Information Theory*, vol. 56, no. 6, pp. 2938–2945, 2010.

[12] I. L. Chuang, D. W. Leung, and Y. Yamamoto, "Bosonic quantum codes for amplitude damping," *Physical Review A*, vol. 56, no. 2, pp. 1114–1125, 1997.

[13] R. Lang and P. W. Shor, "Nonadditive quantum error correcting codes adapted to the ampltitude damping channel," 2007, arXiv:0712.2586 [quant-ph].

[14] D. W. Leung, M. A. Nielsen, I. L. Chuang, and Y. Yamamoto, "Approximate quantum error correction can lead to better codes," *Physical Review A*, vol. 56, no. 4, pp. 2567–2573, 1997.

[15] P. W. Shor, G. Smith, J. A. Smolin, and B. Zeng, "High performance single-error-correcting quantum codes for the amplitude damping channel," *IEEE Transactions on Information Theory*, vol. 57, no. 10, pp. 7180–7188, 2011.

[16] R. Duan, M. Grassl, Z. Ji, and B. Zeng, "Multi-error-correcting amplitude damping codes," in *Proceedings of the 2010 IEEE International Symposium on Information Theory*, 2010, pp. 2672–2676.

[17] M. Scully and M. S. Zubairy, *Quantum Optics*. Cambridge, England: Cambridge University Press, 1997.

[18] R. R. Puri, *Mathematical Methods of Quantum Optics*. Springer Series in Optical Sciences: Springer, 2001.

[19] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed state entanglement and quantum error correction," *Physical Review A*, vol. 54, no. 5, pp. 3824–3851, 1996.

[20] E. Knill and R. Laflamme, "Theory of quantum error-correcting codes," *Physical Review A*, vol. 55, no. 2, pp. 900–911, 1997.

[21] A. R. Calderbank and P. W. Shor, "Good quantum error-correcting codes exist," *Physical Review A*, vol. 54, no. 2, pp. 1098–1106, 1996.

[22] A. Steane, "Multiple particle interference and quantum error correction," in *Proceedings of the Royal Society of London Series A*, vol. 452, 1996, pp. 2551–2577.

[23] M. F. Ezerman, S. Jitman, S. Ling, and D. V. Pasechnik, "CSS-like constructions of asymmetric quantum codes," *IEEE Transactions on Information Theory*, vol. 59, no. 10, pp. 6732–6754, 2013.

[24] T. Klove, *Codes for error detection*. Singapore: World Scientific Publishing Company, 2007.

[25] R. R. Varshamov, "Some features of linear codes that correct asymmetric errors," *Soviet Physics Doklady*, vol. 9, pp. 538–540, Jan. 1965.

[26] T. Klove, "Error correcting codes for the asymmetric channel," Department of Informatics, University of Bergen, Tech. Rep. 18-09-07-81, 1981.

[27] M. F. Ezerman and M. Grassl, "Asymmetric quantum codes detecting a single amplitude error," in *Proceedings of the 2013 IEEE International Symposium on Information Theory*, 2013, pp. 922–926.

[28] M. Grassl, P. Shor, G. Smith, J. Smolin, and B. Zeng, "New constructions of codes for asymmetric channels via concatenation," in *Proceedings of the 2012 IEEE International Symposium on Information Theory*, 2012, pp. 751–755.

[29] E. Berlekamp, *Algebraic Coding Theory*. New York: McGraw-Hill, 1968.

[30] T. Klove, B. Bose, and N. Elarief, "Systematic, single limited magnitude error correcting codes for flash memories," *IEEE Transactions on Information Theory*, vol. 57, no. 7, pp. 4477–4487, 2011.