

Probing topological order with Rényi entropyGábor B. Halász^{1,2} and Alioscia Hamma^{1,3}¹*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario, Canada N2L 2Y5*²*Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom*³*Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, People's Republic of China*

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We present an analytical study of the quantum phase transition between the topologically ordered toric-code-model ground state and the disordered spin-polarized state. The phase transition is induced by applying an external magnetic field, and the variation in topological order is detected via two nonlocal quantities: the Wilson loop and the topological Rényi entropy of order 2. By exploiting an equivalence with the transverse-field Ising model and considering two different variants of the problem, we investigate the field dependence of these quantities by means of an exact treatment in the exactly solvable variant and complementary perturbation theories around the limits of zero and infinite fields in both variants. We find strong evidence that the phase transition point between topological order and disorder is marked by a discontinuity in the topological Rényi entropy and that the two phases around the phase transition point are characterized by its different constant values. Our results therefore indicate that the topological Rényi entropy is a proper topological invariant: its allowed values are discrete and can be used to distinguish between different phases of matter.

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I. INTRODUCTION

There are states in quantum many-body physics that cannot be described in terms of local order parameters and the Landau paradigm of spontaneous symmetry breaking. These states exhibit a subtler kind of order called topological order [1]. Topologically ordered states include fractional quantum Hall liquids [2] and quantum spin liquids [3], which are at the forefront of research in condensed matter theory. Moreover, such states are of great interest in the field of quantum computation because one can encode quantum information in the topological degrees of freedom, and this way of encoding is intrinsically robust against decoherence [4,5].

Since topologically ordered states cannot be characterized by local order parameters, there has been an intense effort to find nonlocal quantities that can detect topological order in a wave function. A series of papers suggested that topological order can be detected through a component of quantum entanglement that contains a topological constraint. This constraint manifests itself as a universal negative correction to the boundary law for the entanglement entropy: the so-called topological entropy [6–9]. Recent works have showed that this component of quantum entanglement is indeed long ranged and so it cannot be destroyed by time evolution with a local Hamiltonian. Equivalently, the components corresponding to the long-range entanglement and the usual short-range entanglement are adiabatically disconnected [10].

These results suggest that the topological entropy is to some extent a nonlocal order parameter for topologically ordered phases. To make a more precise statement about the extent of its applicability, one needs to investigate its robustness against perturbations. If the topological entropy is to detect topologically ordered phases, it needs to be nonzero within all such phases. In other words, it should vanish only at quantum phase transitions to disordered phases. If it is to distinguish different topologically ordered phases from each other, it needs

to be constant within each phase. In other words, it should change only at quantum phase transitions [11].

Recent numerical studies on small systems have found evidence that the topological entropy takes discrete values [3] and changes only at quantum phase transitions [12]. On the other hand, analytic corrections to the topological entropy are extremely hard to obtain because one needs to consider the entanglement entropy of a many-body wave function [13]. Such a treatment for the topological entropy in the case of a finite correlation length can be found in Ref. [14]. Remarkably, it has been recently shown for an exactly solvable two-phase system that the topological entropy is constant within the entire topologically ordered phase [15].

Given the difficulties, it is important to find other entropic quantities that possess a topological component capable of detecting topological order. One potential candidate is the Rényi entropy of order α , which is a generalization of the usual (von Neumann) entanglement entropy. It is important that the Rényi entropy coincides with the entanglement entropy in the special case of $\alpha = 1$. It has also been shown that the ground-state Rényi entropies of different order α all contain the same topological component at the fixed points of nonchiral phases [8]. Such phases typically appear in string-net models [16] and quantum double models [17].

In this paper, we consider the Rényi entropy of order 2 and argue that it is a good probe of topological order because its topological component can change only at quantum phase transitions. In particular, we apply the concept of the topological Rényi entropy to the toric-code model (TCM) [4] in the presence of an external magnetic field. This model is, to paraphrase Goldenfeld [18], the *Drosophila* of topological order. Although it is a simple toy model, it contains all the elements that make topological order interesting: there is no local order parameter, there is a topology-dependent ground-state degeneracy that is robust against local perturbations, and there are excitations with anyonic particle statistics. Indeed, the

TCM is another beautiful example of the crucial role played by toy models in statistical mechanics.

To show that the topological Rényi entropy is a good probe of topological order, we demonstrate that the disordered and the topologically ordered phases of the TCM with external magnetic field are characterized by its distinct values. We also study the Wilson loop as a probe of both topological order and gauge structure. Concentrating on two different variants of the problem, we establish an exact treatment in the computationally simpler (exactly solvable) variant and supplement it with perturbation theories in both variants. The results obtained with the two methods for the two probing quantities in the two variants are highly consistent with each other.

II. GENERAL FORMALISM

We consider the TCM with an external magnetic field in the $+z$ direction. The system is an $N \times N$ square lattice with periodic boundary conditions, and $2N^2$ spins are located at the edges of the lattice [4]. In general, the spins on the horizontal (h) and the vertical (v) edges experience different magnetic fields: λ on the horizontal and $\kappa\lambda$ on the vertical edges ($\kappa > 0$). The Hamiltonian of the system takes the form

$$\hat{H} = - \sum_s \hat{A}_s - \sum_p \hat{B}_p - \lambda \sum_{i \in h} \hat{\sigma}_i^z - \kappa\lambda \sum_{i \in v} \hat{\sigma}_i^z, \quad (1)$$

$$\hat{A}_s \equiv \prod_{i \in s} \hat{\sigma}_i^x, \quad \hat{B}_p \equiv \prod_{i \in p} \hat{\sigma}_i^z,$$

where the indices s and p refer to stars and plaquettes on the lattice containing four spins each. For an illustration of this, see Fig. 1. Note that the four sums in Eq. (1) all contain N^2 terms, and that only $N^2 - 1$ star (plaquette) operators are independent because $\prod_s \hat{A}_s = \prod_p \hat{B}_p = 1$.

The TCM with zero external field ($\lambda = 0$) is exactly solvable because the stars \hat{A}_s and the plaquettes \hat{B}_p all commute with each other. The ground state is fourfold degenerate: there are four linearly independent states with $A_s \equiv \langle \hat{A}_s \rangle = +1$ and $B_p \equiv \langle \hat{B}_p \rangle = +1$ for all s and p . These degenerate ground

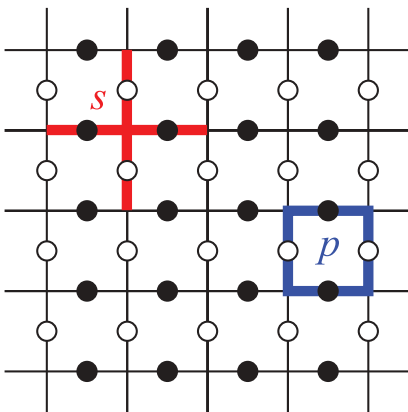


FIG. 1. (Color online) Illustration of the square lattice with the physical spins located at the horizontal (black circles) and the vertical (white circles) edges. Examples of a star (red cross labeled s) and a plaquette (blue square labeled p) are included.

states are distinguished by the topological quantum numbers $Z_1 = \pm 1$ and $Z_2 = \pm 1$, which are expectation values for products of $\hat{\sigma}_i^z$ operators along horizontal and vertical strings going round the lattice. The ground state with $Z_1 = Z_2 = +1$ can be written as

$$|0\rangle = \mathcal{N} \prod_s (1 + \hat{A}_s) |\uparrow\rangle, \quad (2)$$

where $\mathcal{N} = 1/\sqrt{2^{N^2+1}}$ is a normalization constant, and $|\uparrow\rangle$ denotes the completely polarized state with all spins pointing in the $+z$ direction ($\sigma_i^z = +1$ for all i).

The TCM with finite external field ($\lambda > 0$) is not exactly solvable because the magnetic fields $\hat{\sigma}_i^z$ do not commute with the stars \hat{A}_s . On the other hand, they commute with the plaquettes \hat{B}_p and the topological operators $\hat{Z}_{1,2}$; therefore there are four independent lowest-energy sectors with $Z_{1,2} = \pm 1$ and $B_p = +1$ ($\forall p$). In the rest of the paper, we consider the lowest-energy eigenstate $|\Omega(\lambda)\rangle$ within the $Z_1 = Z_2 = +1$ sector. This state becomes $|\uparrow\rangle$ in the limit of $\lambda \rightarrow \infty$ and $|0\rangle$ in the limit of $\lambda = 0$. Between the two limits, numerical studies reveal a quantum phase transition at a critical magnetic field $\lambda = \lambda_C$ [12,19]. Since $|\Omega(\lambda)\rangle$ is a ground state at the fixed points of both limiting phases, the adiabatic theorem guarantees that it is the unique ground state in the disordered phase at $\lambda > \lambda_C$ and one of the four degenerate ground states in the topologically ordered phase at $\lambda < \lambda_C$.

If we consider only the states with $Z_1 = Z_2 = +1$ and $B_p = +1$ ($\forall p$), the dimension of the effective Hilbert space is reduced from 2^{2N^2} to 2^{N^2-1} . The states within this reduced Hilbert space can be written as superpositions of loop configurations on the dual lattice: in each loop configuration, the spins on the loops have $\sigma_i^z = -1$ and the remaining spins have $\sigma_i^z = +1$. This implies that the reduced model is equivalent to a \mathbb{Z}_2 lattice gauge theory, and the phase transition at $\lambda = \lambda_C$ corresponds to a confinement-deconfinement transition [12,19]. Furthermore, since each loop configuration can be characterized by the values of the $N^2 - 1$ independent stars $A_s = \pm 1$, it is convenient to introduce a corresponding representation in which quasispins A_s are located at the stars [20]. This quasispin representation is particularly useful in the $\lambda \ll 1$ limit because $|\Omega(\lambda)\rangle$ is then close to $|0\rangle$ which is a product state with $A_s = +1$ ($\forall s$). Up to an irrelevant additive constant, the Hamiltonian in Eq. (1) becomes

$$\hat{H} = - \sum_s \hat{A}_s^z - \lambda \sum_{\langle s,s' \rangle \in h} \hat{A}_s^x \hat{A}_{s'}^x - \kappa\lambda \sum_{\langle s,s' \rangle \in v} \hat{A}_s^x \hat{A}_{s'}^x, \quad (3)$$

where $\langle s,s' \rangle$ means that the summation is over horizontal and vertical edges between nearest-neighbor stars s and s' . Note that $\hat{A}_s^z \equiv \hat{A}_s$ measures and \hat{A}_s^x switches the quantum number A_s ; therefore the quasispin operators \hat{A}_s^z and \hat{A}_s^x satisfy the standard spin commutation relations. In the quasispin representation of Eq. (3), the TCM with external magnetic field is equivalent to a two-dimensional (2D) transverse-field Ising model (TFIM) in which the coupling strengths on the horizontal and the vertical edges are different in general.

III. MEASURES OF TOPOLOGICAL ORDER

We aim to describe how the topological order in the ground state $|\Omega(\lambda)\rangle$ changes as a function of λ between the

topologically ordered limit at $\lambda = 0$ and the disordered limit at $\lambda \rightarrow \infty$. To quantify topological order in an analytically tractable manner, we consider two measures: the Wilson loop and the topological Rényi entropy.

A. General properties

The Wilson loop for a region R on the dual lattice is defined as the expectation value W_R of the operator

$$\hat{W}_R \equiv \prod_{i \in \partial R} \hat{\sigma}_i^x = \prod_{s \in R} \hat{A}_s^z, \quad (4)$$

where ∂R denotes the boundary of R . If the region R is macroscopic with linear dimension $D \gg 1$, the Wilson loop follows a perimeter law $W_R \propto \exp(-\beta D)$ in the presence of topological order and an area law $W_R \propto \exp(-\beta D^2)$ in the absence of topological order [12]. In this paper, we assume that the region R is a $D \times D$ square (see Fig. 2).

The topological Rényi entropy is based on the paradigm of quantum entanglement. The Rényi entropy of order α between two complementary subsystems A and $B \equiv \bar{A}$ reads

$$S_\alpha^{AB} \equiv \frac{1}{1-\alpha} \log_2 \text{Tr}[\hat{\rho}_A^\alpha] = \frac{1}{1-\alpha} \log_2 \text{Tr}[\hat{\rho}_B^\alpha], \quad (5)$$

where $\hat{\rho}_A$ and $\hat{\rho}_B$ are the reduced density operators for A and B . The topological contribution to the Rényi entropy can be extracted by taking a suitable linear combination of Rényi entropies that are calculated for different choices of the subsystems A and B [7]. In fact, the standard definition for the topological Rényi entropy of order α is

$$S_\alpha^T \equiv -S_\alpha^{(1)} + S_\alpha^{(2)} + S_\alpha^{(3)} - S_\alpha^{(4)}, \quad (6)$$

where $S_\alpha^{(m)} = S_\alpha^{AB}$ in the four cases (m) of partitioning the system shown in Fig. 3. The characteristic linear dimensions are the extension D and the thickness d of the subsystem A in all cases. To obtain a meaningful topological measure, these dimensions need to be macroscopic ($D > d \gg 1$).

The topological Rényi entropy S_α^T is nonzero if and only if the given state exhibits topological order [7,8]. For the TCM with external magnetic field, $S_\alpha^T = 0$ for the disordered ground state $|\uparrow\rangle$ and $S_\alpha^T = 2$ for the topologically ordered

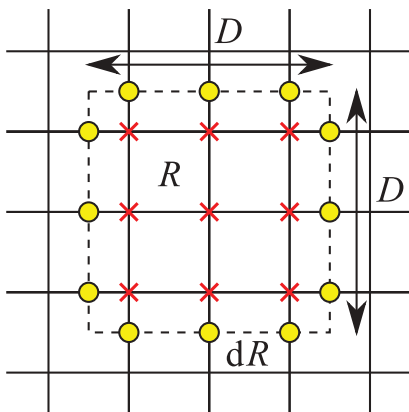


FIG. 2. (Color online) Illustration of the Wilson loop for a square region with $D = 3$. The region R contains D^2 stars (red crosses) and the boundary ∂R (dashed line) contains $4D$ spins (yellow circles).

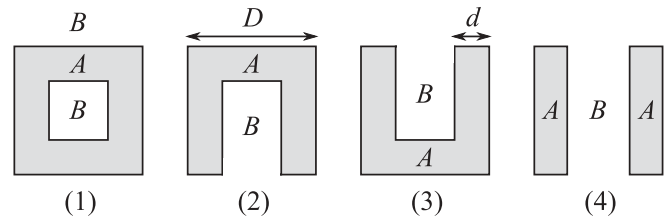


FIG. 3. Illustration of the subsystems A and B in the four cases (m) that are used to calculate the topological Rényi entropy. Each subsystem A has extension D and thickness d with $D > d \gg 1$.

ground state $|0\rangle$. In this paper, we demonstrate that S_α^T detects the presence of topological order in the entire topologically ordered phase at $\lambda < \lambda_C$. Note though that S_α^T is independent of α at the fixed point of a generic nonchiral phase [8]. Since the fixed points of different topologically ordered phases do not necessarily have distinct values of S_α^T , the topological Rényi entropy is unable to provide a complete characterization of a topologically ordered phase.

B. \mathbb{Z}_2 lattice gauge theory

Since the TCM is perturbed with external fields $\hat{\sigma}_i^z$ that commute with the plaquettes \hat{B}_p and the topological operators $\hat{Z}_{1,2}$, the ground state $|\Omega(\lambda)\rangle$ belongs to the lowest-energy sector with $Z_1 = Z_2 = +1$ and $B_p = +1$ ($\forall p$) for all values of λ . If we consider only this sector, the gauge structure constraint $\hat{B}_p|\Psi\rangle = |\Psi\rangle$ is enforced on all states $|\Psi\rangle$; therefore the model is equivalent to a \mathbb{Z}_2 lattice gauge theory.

An arbitrary state $|\Psi\rangle$ within the gauge theory can be expressed as a superposition of loop configurations. Each configuration is a finite set of closed loops on the dual lattice, and the spins on the loops are flipped with respect to the remaining ones. These properties motivate us to introduce a modified definition for the topological Rényi entropy in which the subsystem A is substituted by the boundary ∂A between A and B in each case (m) of partitioning the system. This means that $S_\alpha^{(m)} = S_\alpha^{\partial A, \bar{\partial A}} \equiv S_\alpha^{\partial A}$ in Eq. (6). Formally, we define C as the set of star operators acting on both subsystems A and B , and ∂A as the set of spins that are acted upon only by stars in C . For an illustration of this, see Fig. 4. The boundary ∂A is always a finite set of closed loops on the real lattice: the number of loops is $n = 2$ in the cases (1) and (4), while it is $n = 1$ in the cases (2) and (3). Since a loop on the real lattice and a loop on the dual lattice can intersect only at an even number of points, there are an even number of spins flipped on each loop of ∂A . This topological constraint ensures that the modified S_α^T has similar properties to the standard one. For example, it is still true that $S_\alpha^T = 0$ for $|\uparrow\rangle$ and $S_\alpha^T = 2$ for $|0\rangle$.

The calculations in the rest of the paper are immensely simplified by using the modified definition for S_α^T . Since the group generated by the star operators acting exclusively on the boundary subsystem ∂A contains only the identity, the reduced density matrix $\rho_{\partial A}$ is diagonal in the basis of the physical spins σ_i^z . Each diagonal element $(\rho_{\partial A})_{\Sigma\Sigma}$ gives the probability that $|\Psi\rangle$ realizes a given spin configuration $\{\Sigma_i^z = \pm 1\}$ in ∂A . Equivalently, if we choose a random loop configuration according to the probability distribution given by the

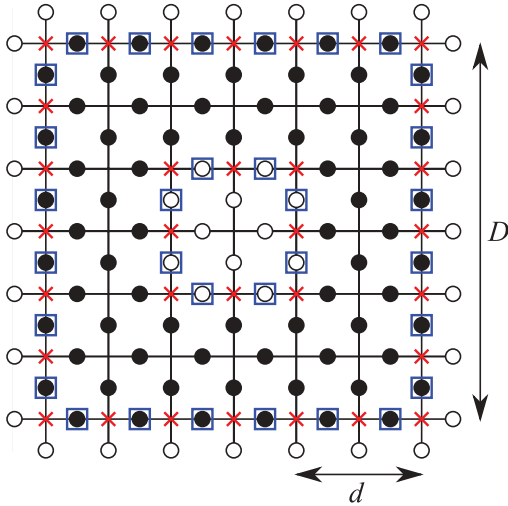


FIG. 4. (Color online) Illustration of the subsystems in case (1) with dimensions $D = 6$ and $d = 2$. Spins are either in subsystem A (black circles) or in subsystem B (white circles). Stars in the set C are marked by red crosses, and spins in the subsystem ∂A are marked by blue rectangles. The boundary contains $n = 2$ closed loops on the real lattice with a combined length $L = 32$.

state $|\Psi\rangle$, the probability of the spin configuration $\{\Sigma_i^z\}$ in ∂A is $P[\{\Sigma_i^z\}] = (\rho_{\partial A})_{\Sigma\Sigma}$. If we then choose two random loop configurations according to the same distribution, the probability of them having the same spin configuration in ∂A is

$$\mathcal{P} = \sum_{\Sigma} P[\{\Sigma_i^z\}]^2 = \sum_{\Sigma} (\rho_{\partial A})_{\Sigma\Sigma}^2 = \text{Tr}[\hat{\rho}_{\partial A}^2]. \quad (7)$$

This result motivates us to consider the topological Rényi entropy of order 2. In terms of the probabilities $\mathcal{P}^{(m)}$ in the four cases (m) of partitioning the system, this quantity takes the form $S_2^T = \log_2[\mathcal{P}^{(1)}\mathcal{P}^{(4)}/\mathcal{P}^{(2)}\mathcal{P}^{(3)}]$.

We can now develop an intuitive understanding of the phase transition by considering the two limiting cases. In the topologically ordered ground state at $\lambda \ll 1$, the spin loops are deconfined and all possible loop configurations are equally probable. This means that the allowed spin configurations in the subsystem ∂A also share the same probability: the inverse number of allowed spin configurations. It is important that the number of boundary loops is $n = 2$ in the cases (1) and (4), while it is $n = 1$ in the cases (2) and (3). The cases (1) and (4) are therefore more constrained and have fewer allowed spin configurations in ∂A . This implies that $\mathcal{P}^{(1)}, \mathcal{P}^{(4)} > \mathcal{P}^{(2)}, \mathcal{P}^{(3)}$ and $S_2^T > 0$. More precisely, since the constraint on each boundary loop reduces the number of allowed spin configurations by a factor of 2, the topological Rényi entropy is given by $S_2^T = n^{(1)} - n^{(2)} - n^{(3)} + n^{(4)} = 2$. In the disordered ground state at $\lambda \gg 1$, the spin loops are confined and only the loop configurations with small spin loops have significant probabilities. On the other hand, the small spin loops in these loop configurations correspond to local disturbances (nearby spin flips) in the spin configurations of the boundary subsystem ∂A . This means that the probability $\mathcal{P}^{(m)}$ in each case (m) can be written as a product over the small sections of the boundary loops; therefore $\log_2 \mathcal{P}^{(m)}$ is proportional to the length of the

boundary. Since the combined boundary length of the cases (1) and (4) is equal to the combined boundary length of the cases (2) and (3), the topological Rényi entropy vanishes: $S_2^T = \log_2[\mathcal{P}^{(1)}\mathcal{P}^{(4)}/\mathcal{P}^{(2)}\mathcal{P}^{(3)}] = 0$.

C. Formula for the Rényi entropy

Now we capitalize on the simplifications described above, and derive the Rényi entropy $S_2^{\partial A}$ for an arbitrary state $|\Psi\rangle$ within the gauge theory. In the most general case, ∂A consists of n closed loops on the real lattice, and the loops have a combined length L . This means that they contain L spins and L stars acting on these spins (see Fig. 4). Since there is a constraint on each loop due to the gauge structure, only $L - n$ spins are independent. If we label these spins with $1 \leq i \leq L - n$, the 2^{L-n} nonzero diagonal elements of $\rho_{\partial A}$ give the probabilities of $|\Psi\rangle$ realizing the 2^{L-n} respective spin configurations $\{\Sigma_i^z\}$. Since the projection operator onto the spin configuration $\{\Sigma_i^z\}$ is given by $2^{L-n} \prod_{i=1}^{L-n} (1 + \Sigma_i^z \hat{\sigma}_i^z)$, the corresponding diagonal element reads

$$(\rho_{\partial A})_{\Sigma\Sigma} = \frac{1}{2^{L-n}} \langle \Psi | \left[\prod_{i=1}^{L-n} (1 + \Sigma_i^z \hat{\sigma}_i^z) \right] | \Psi \rangle. \quad (8)$$

When expanding the product in Eq. (8) and summing the squares of the resulting expressions for $(\rho_{\partial A})_{\Sigma\Sigma}$, the cross-terms cancel each other, and we obtain

$$\text{Tr}[\hat{\rho}_{\partial A}^2] = \frac{1}{2^{L-n}} \sum_{\{q_i=0,1\}} \langle \Psi | \left[\prod_{i=1}^{L-n} (\hat{\sigma}_i^z)^{q_i} \right] | \Psi \rangle^2, \quad (9)$$

where the sum is over all the 2^{L-n} configurations $\{q_i = 0, 1\}$, and hence over all possible products of the $L - n$ independent spin operators $\hat{\sigma}_i^z$. If the edge occupied by the spin i connects the stars $s_{i,1}$ and $s_{i,2}$, the corresponding spin operator becomes $\hat{\sigma}_i^z = \hat{A}_{s_{i,1}}^x \hat{A}_{s_{i,2}}^x$. In terms of the quasispin operators \hat{A}_s^x , the Rényi entropy then takes the form

$$S_2^{\partial A} = (L - n) - \log_2 \sum_{\{q_i=0,1\}} \langle \Psi | \left[\prod_{i=1}^{L-n} (\hat{A}_{s_{i,1}}^x \hat{A}_{s_{i,2}}^x)^{q_i} \right] | \Psi \rangle^2. \quad (10)$$

This expression has an entirely precise notation, but it is cumbersome to use for calculating $S_2^{\partial A}$. To derive a more intuitive expression with a less precise notation, we expand the sum in Eq. (10) around the trivial configuration $\{q_i = 0\}$. Exploiting $(\hat{A}_s^x)^2 = 1$, the Rényi entropy then becomes

$$S_2^{\partial A} = (L - n) - \log_2 \left[1 + \sum_{s_1, s_2} \langle \Psi | \hat{A}_{s_1}^x \hat{A}_{s_2}^x | \Psi \rangle^2 + \sum_{s_1, s_2, s_3, s_4} \langle \Psi | \hat{A}_{s_1}^x \hat{A}_{s_2}^x \hat{A}_{s_3}^x \hat{A}_{s_4}^x | \Psi \rangle^2 + \dots \right], \quad (11)$$

where the sum inside the logarithm contains all 2^{L-n} possible products with an even number of quasispin operators \hat{A}_s^x chosen from each closed loop of the subsystem ∂A .

To understand how Eq. (11) works, we consider the two limiting ground states $|\uparrow\rangle$ and $|0\rangle$. In the first case, we have $\langle \uparrow | \hat{A}_{s_1}^x \hat{A}_{s_2}^x \dots \hat{A}_{s_n}^x | \uparrow \rangle = \langle \uparrow | \hat{\sigma}_{i_1}^z \hat{\sigma}_{i_2}^z \dots \hat{\sigma}_{i_n}^z | \uparrow \rangle = 1$ for

all expectation values because $|\uparrow\rangle$ has $\sigma_i^z = +1$ for all i . The sum inside the logarithm becomes 2^{L-n} , and the Rényi entropy $S_2^{\partial A}$ vanishes, as expected for a product state. In the second case, $|0\rangle$ has $A_s^z = +1$ for all s ; therefore $\langle 0|\hat{A}_{s_1}^x \hat{A}_{s_2}^x \cdots \hat{A}_{s_n}^x|0\rangle = 0$ for all expectation values. The only exception is the trivial one: $\langle 0|0\rangle = 1$. The sum inside the logarithm is 1, and the Rényi entropy is $S_2^{\partial A} = L - n$. When extracting the topological contribution, the terms proportional to L cancel because $L^{(1)} + L^{(4)} = L^{(2)} + L^{(3)}$ (see Fig. 3). On the other hand, the cases (1) and (4) have $n = 2$, while the cases (2) and (3) have $n = 1$; therefore the topological Rényi entropy is finite: $S_2^T = 2$.

IV. PHASE TRANSITION IN THE QUASI-1D CASE

In this section, we set $\kappa \ll 1$ in Eq. (1): this means that the spins on the vertical edges experience much smaller magnetic fields than those on the horizontal edges. The magnetic fields on the vertical edges become important only in the $\lambda \gg 1$ limit, and therefore the phase transition between topological order and disorder at $\lambda \sim 1$ occurs due to a competition between the star operators \hat{A}_s^z and the magnetic fields on the horizontal edges. When investigating this phase transition, the magnetic fields on the vertical edges can be neglected ($\kappa \rightarrow 0$), and the quasispin Hamiltonian in Eq. (3) becomes

$$\hat{H} = - \sum_s \hat{A}_s^z - \lambda \sum_{\langle s, s' \rangle \in h} \hat{A}_s^x \hat{A}_{s'}^x. \quad (12)$$

Since there are only Ising couplings on the horizontal edges, this Hamiltonian is the direct sum of N independent 1D TFIM copies along the horizontal chains of the lattice [21]. The system is therefore exactly solvable for all values of λ .

If we consider any of the independent 1D horizontal chains and label the N stars on the chain with $1 \leq l \leq N$, the Hamiltonian of the corresponding 1D TFIM reads

$$\hat{H} = - \sum_{l=1}^N (\hat{A}_l^z + \lambda \hat{A}_l^x \hat{A}_{l-1}^x), \quad (13)$$

where the periodic boundary conditions are taken into account by $A_0 \equiv A_N$. This Hamiltonian can be solved by following a standard procedure [22]. We first map the quasispins A_l to fermions via the Jordan-Wigner transformation

$$\hat{A}_l^z = 1 - 2c_l^\dagger c_l, \quad \hat{A}_l^- = (\hat{A}_l^+)^{\dagger} = c_l^\dagger e^{i\pi \sum_{j=1}^{l-1} c_j^\dagger c_j}, \quad (14)$$

where c_l^\dagger and c_l are standard fermionic creation and annihilation operators. The translational symmetry is then exploited by the Fourier transform

$$c_l = \frac{1}{\sqrt{N}} \sum_k e^{ikl} c_k, \quad (15)$$

where the sum is over the momenta $k_m = \pi(2m-1)/N$ with $1 \leq m \leq N$. We finally introduce new fermionic operators via the Bogoliubov transformation

$$c_k = \cos \theta_k \gamma_k + i \sin \theta_k \gamma_{-k}^\dagger, \quad (16)$$

and the Hamiltonian in Eq. (13) becomes

$$\hat{H} = \sum_k \Lambda_k (2\gamma_k^\dagger \gamma_k - 1), \quad (17)$$

where γ_k^\dagger and γ_k correspond to independent fermionic quasiparticles. The energies of these quasiparticles are proportional to $\Lambda_k = \sqrt{\epsilon_k^2 + \lambda^2 \sin^2 k}$ with $\epsilon_k \equiv 1 - \lambda \cos k$, and the mixing angle appearing in the Bogoliubov transformation is $\theta_k = \tan^{-1}[\lambda \sin k / (\epsilon_k + \Lambda_k)]$.

The ground state $|\Omega(\lambda)\rangle$ of the Hamiltonian in Eq. (12) is the direct product of N independent copies of the 1D ground state $|\Omega_0\rangle$. The 1D ground state is defined by $\gamma_k |\Omega_0\rangle = 0$ for all k ; therefore its two-operator expectation values in the position representation are given by

$$\begin{aligned} \langle c_l^\dagger c_{l'} \rangle &\equiv \langle \Omega_0 | c_l^\dagger c_{l'} | \Omega_0 \rangle = \frac{1}{N} \sum_{k, k'} e^{-ikl + ik'l'} \langle \Omega_0 | c_k^\dagger c_{k'} | \Omega_0 \rangle \\ &= \frac{1}{N} \sum_k e^{-ik(l-l')} \sin^2 \theta_k, \\ \langle c_l c_{l'}^\dagger \rangle &= \frac{1}{N} \sum_k e^{-ik(l-l')} \cos^2 \theta_k, \\ \langle c_l c_{l'} \rangle &= \frac{i}{N} \sum_k e^{-ik(l-l')} \sin \theta_k \cos \theta_k, \\ \langle c_l^\dagger c_{l'}^\dagger \rangle &= -\frac{i}{N} \sum_k e^{-ik(l-l')} \sin \theta_k \cos \theta_k. \end{aligned} \quad (18)$$

To calculate the Rényi entropy, we need to evaluate the quasispin expectation values appearing in Eq. (11). These expectation values are products of independent 1D expectation values $\langle \Omega_0 | \hat{A}_{l_1}^x \hat{A}_{l_2}^x \cdots \hat{A}_{l_{l-1}}^x \hat{A}_{l_l}^x | \Omega_0 \rangle$, where each pair of \hat{A}_l^\pm operators can be expressed in terms of the fermionic operators as

$$\hat{A}_l^x \hat{A}_{l'}^x = (c_l^\dagger + c_l) \prod_{j=l}^{l'-1} (1 - 2c_j^\dagger c_j) (c_{l'}^\dagger + c_{l'}). \quad (19)$$

Similarly, the quasispin expectation value in the Wilson loop for a $D \times D$ square region R becomes

$$W_R = \langle \Omega_0 | \prod_{l=1}^D \hat{A}_l^z | \Omega_0 \rangle^D = \langle \Omega_0 | \prod_{l=1}^D (1 - 2c_l^\dagger c_l) | \Omega_0 \rangle^D. \quad (20)$$

Using the identity $1 - 2c_l^\dagger c_l = (c_l^\dagger + c_l)(c_l^\dagger - c_l)$, the quasispin operator products appearing in both the Rényi entropy and the Wilson loop can then be written as simple products of $c_l^\dagger \pm c_l$ operators. On the other hand, the expectation values of these products can be reduced to the two-operator expectation values given in Eq. (18) by using Wick's theorem [23].

The exact dependence of the topological Rényi entropy on the magnetic field is plotted in Fig. 5. There are two phases: a topologically ordered phase at small λ and a disordered phase at large λ . These phases are separated by a clear phase transition at $\lambda = \lambda_C = 1$, which coincides with the well-known critical point of the 1D TFIM [22]. If we gradually increase λ , the topological Rényi entropy drops to zero around λ_C . This transition becomes sharper if we increase the system size N as well as the dimensions D and d of the subsystems; therefore we argue that S_2^T is discontinuous in the thermodynamic limit. The topological Rényi entropy is then constant in both limiting phases: the topologically

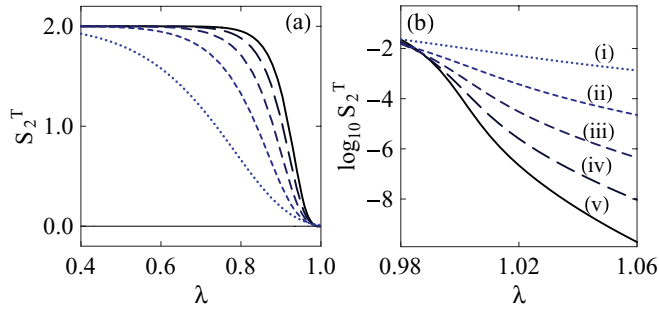


FIG. 5. (Color online) Topological Rényi entropy S_2^T as a function of magnetic field λ below (a) and above (b) the critical point at $\lambda_c = 1$. Five system sizes are plotted with $d = D/3$ for each: $N = 40$ and $D = 6$ (i); $N = 80$ and $D = 12$ (ii); $N = 120$ and $D = 18$ (iii); $N = 160$ and $D = 24$ (iv); $N = 200$ and $D = 30$ (v).

ordered phase at $\lambda < \lambda_c$ is characterized by $S_2^T = 2$, while the disordered phase at $\lambda > \lambda_c$ is characterized by $S_2^T = 0$.

The analogous exact behavior of the Wilson loop is illustrated in Fig. 6. In the topologically ordered phase at $\lambda < \lambda_c$, the reduced Wilson loop $W_0 \equiv W_R^{1/D}$ approaches a finite constant in the $D \rightarrow \infty$ limit. This implies $W_R \propto \exp(-\beta D)$ and the presence of topological order. In the disordered phase at $\lambda > \lambda_c$, W_0 decays exponentially with D . This implies $W_R \propto \exp(-\beta D^2)$ and the absence of topological order. By looking at the dependence $W_0(\lambda)$ for a sufficiently large value of D , we can establish that the critical point separating the two different behaviors is indeed at $\lambda_c = 1$. The results obtained for the topological Rényi entropy and the Wilson loop are therefore consistent with each other.

V. PHASE TRANSITION IN THE ACTUAL 2D CASE

In this section, we set $\kappa = 1$ in Eq. (1): this means that the spins on the horizontal and the vertical edges experience the same magnetic field. Up to an irrelevant additive constant, the quasispin Hamiltonian in Eq. (3) becomes

$$\hat{H} = \sum_s (1 - \hat{A}_s^z) - \lambda \sum_{(s,s')} \hat{A}_s^x \hat{A}_{s'}^x, \quad (21)$$

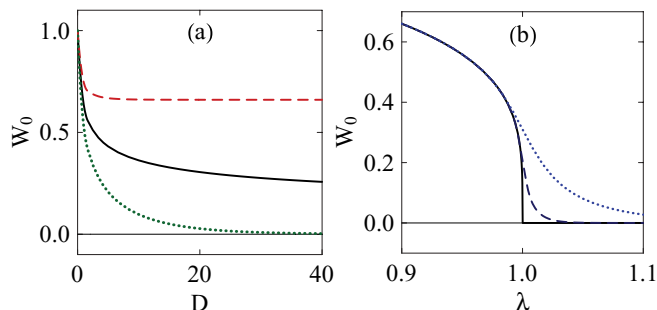


FIG. 6. (Color online) (a) Reduced Wilson loop W_0 as a function of region size D for $\lambda = 0.9 < \lambda_c$ (red dashed line), $\lambda = 1.0 = \lambda_c$ (black solid line), and $\lambda = 1.1 > \lambda_c$ (green dotted line). The system size is $N = 500$. (b) Reduced Wilson loop W_0 as a function of magnetic field λ for system sizes $N = 5D = 100$ (dotted line) and $N = 5D = 500$ (dashed line). The solid line is the result in Eq. (30) for the thermodynamic limit.

and the system is equivalent to the standard 2D TFIM. Since the Hamiltonian in Eq. (21) is not exactly solvable in general, we use perturbation theories around the exactly solvable limits at $\lambda = 0$ and $\lambda \rightarrow \infty$. The corresponding calculations are most efficiently performed by the method of perturbative continuous unitary transformations (PCUT). The general method is discussed in the literature [20,24] and we illustrate its use by the example of our particular problem.

A. Perturbation theory at small magnetic field

In the limit of $\lambda \ll 1$, it is useful to work in the quasispin representation because the unperturbed ground state $|0\rangle$ is then a product state. The perturbation theory is based on Eq. (21), where the second term is treated as a perturbation in the small parameter $\lambda \ll 1$. Using the PCUT procedure described in Appendix A 1, we obtain corrections to the Rényi entropy $S_2^{\partial A}$ for each case of partitioning in Fig. 3 and the Wilson loop W_R for a square region R . The Rényi entropy after the first three corrections reads

$$S_2^{\partial A} = (L - n) - \frac{L}{\ln 2} \left[\frac{\lambda^2}{4} + \frac{63\lambda^4}{64} + \frac{503\lambda^6}{96} + O(\lambda^8) \right] - \frac{K}{\ln 2} \left[\frac{27\lambda^4}{64} + \frac{737\lambda^6}{256} + O(\lambda^8) \right], \quad (22)$$

where the boundary ∂A contains n closed loops with a combined length L and a total number of K corners that are sufficiently far away from each other. The analogous expression for the Wilson loop after the first three corrections is

$$W_R = \exp \left\{ -L \left[\frac{\lambda^2}{8} + \frac{\lambda^4}{2} + \frac{7697\lambda^6}{3072} + O(\lambda^8) \right] + K \left[\frac{3\lambda^4}{32} + \frac{89\lambda^6}{128} + O(\lambda^8) \right] \right\}, \quad (23)$$

where the square region R has a boundary length $L = 4D$ and a corner number $K = 4$.

Since K is merely an $O(1)$ constant, the corrections inside the exponential of Eq. (23) are linearly proportional to the region dimension D . The Wilson loop has therefore a functional form $W_R \propto \exp(-\beta D)$ that shows the presence of topological order. Since the corrections to the Rényi entropy are all linearly proportional to either L or K , the corrections to the topological contribution proportional to n vanish. When calculating the topological Rényi entropy, the corrections proportional to L and K cancel because the combined values of L and K in the cases (1) and (4) match those in the cases (2) and (3) (see Fig. 3). The topological Rényi entropy is therefore constant up to the third correction: $S_2^T = 2 + O(\lambda^8)$ in the $\lambda \ll 1$ phase.

B. Perturbation theory at large magnetic field

In the limit of $\lambda \gg 1$, it is useful to return to the physical spin representation because the unperturbed ground state $|\uparrow\uparrow\rangle$ is then a product state. Up to an irrelevant additive constant and an overall multiplicative factor λ^{-1} , the 2D TFIM Hamiltonian

in Eq. (21) becomes

$$\hat{H} = \sum_i (1 - \hat{\sigma}_i^z) - \lambda^{-1} \sum_s \prod_{i \in s} \hat{\sigma}_i^x. \quad (24)$$

The perturbation theory is based on Eq. (24), where the second term is treated as a perturbation in the small parameter $\lambda^{-1} \ll 1$. Using the PCUT procedure described in Appendix A 2, the Rényi entropy after the first three corrections is

$$S_2^{\partial A} = \frac{L}{\ln 2} \left[\frac{\lambda^{-2}}{32} + \frac{\lambda^{-4}}{1024} + \frac{115\lambda^{-6}}{2\,359\,296} + O(\lambda^{-8}) \right] - \frac{K}{\ln 2} \left[\frac{35\lambda^{-6}}{4\,718\,592} + O(\lambda^{-8}) \right]. \quad (25)$$

Since the corrections to the Rényi entropy are all linearly proportional to either L or K , the corrections to the topological contribution proportional to n vanish. The topological Rényi entropy is therefore zero up to the third correction: $S_2^T = O(\lambda^{-8})$ in the $\lambda \gg 1$ phase.

To obtain a nonzero result for the Wilson loop expectation value $\langle \Omega(\lambda) | \hat{W}_R | \Omega(\lambda) \rangle$, we need to consider higher orders of perturbation theory. Since \hat{W}_R is a product of D^2 star operators \hat{A}_s^z , the first nonzero contribution to W_R appears at order $D^2/2$ in perturbation theory. At this order, \hat{W}_R links order $D^2/2$ states to each other; therefore $W_R \propto \lambda^{-D^2}$. This result can be rearranged into the form $W_R \propto \exp[-\ln(\lambda) D^2]$ which shows the absence of topological order.

C. Discussion of the phase transition

The results of the perturbation theories indicate two distinct phases around the limits $\lambda = 0$ and $\lambda \rightarrow \infty$. The phase at $\lambda \ll 1$ is topologically ordered because the topological Rényi entropy is nonzero and the Wilson loop follows a perimeter law: $W_R \propto \exp(-\beta D)$. Conversely, the phase at $\lambda \gg 1$ is disordered because the topological Rényi entropy is zero and the Wilson loop follows an area law: $W_R \propto \exp(-\beta D^2)$.

The topological distinctness implies at least one phase transition between the two limiting phases, and we argue that there can only be one phase transition. Recall that the TCM with external field is equivalent to the standard 2D TFIM when $\kappa = 1$. In particular, the quantities S_2^T and W_R that describe topological order can be expressed in terms of the 2D TFIM correlation functions. A phase transition is therefore possible only at the critical point of the 2D TFIM, which has been determined by various numerical methods [25] to be at $\lambda_C \approx 0.33$. This critical field is also consistent with previous numerical studies on the TCM with external field [12,19].

It is clear that the perturbation theories around the two limits need to break down at $\lambda = \lambda_C$. On the other hand, the results of the perturbation theories hold because the expansions in Eqs. (22), (23), and (25) have particular structures: they each contain two power series in λ that are proportional to the boundary length L and the corner number K . It is plausible that higher-order corrections preserve this form and only add further terms to the respective power series. Terms that are not linearly proportional to either L or K appear only when the order of the perturbation theory exceeds the dimensions D and d of the subsystems (regions). Since these dimensions are macroscopic in the thermodynamic limit, the perturbation theories can break down only at infinitely large orders. These

TABLE I. Estimates for the critical field λ_C obtained from the power series of Eqs. (22), (23), and (25).

Estimates for λ_C		$\sqrt{a_1/a_2}$	$\sqrt{a_2/a_3}$
Equation (22)	Series $\propto L$	0.504	0.433
	Series $\propto K$		0.383
Equation (23)	Series $\propto L$	0.500	0.447
	Series $\propto K$		0.367
Estimates for λ_C		$\sqrt{b_2/b_1}$	$\sqrt{b_3/b_2}$
Equation (25)	Series $\propto L$	0.177	0.223
	Series $\propto K$		

in turn become important at the radii of convergence where the series actually diverge. If we write the power series in Eqs. (22) and (23) as $\sum_{k=1}^{\infty} a_k \lambda^{2k}$ and those in Eq. (25) as $\sum_{k=1}^{\infty} b_k \lambda^{-2k}$, the critical field λ_C is given by

$$\lambda_C = \lim_{k \rightarrow \infty} \sqrt{\left| \frac{a_k}{a_{k+1}} \right|} = \lim_{k \rightarrow \infty} \sqrt{\left| \frac{b_{k+1}}{b_k} \right|}. \quad (26)$$

Although it is not possible to determine these limits from a finite-order perturbation theory, we can give estimates for the critical field by looking at the first couple of terms and calculating analogous quantities. The resulting estimates are summarized in Table I: they suggest $0.2 \lesssim \lambda_C \lesssim 0.5$. This range is fully consistent with $\lambda_C \approx 0.33$.

The most remarkable result of this section is that the topological Rényi entropy is constant in both limiting phases: $S_2^T = 2$ in the topologically ordered phase and $S_2^T = 0$ in the disordered phase. This happens because the perturbative corrections to $S_2^{\partial A}$ do not contain any topological contributions proportional to n in Eqs. (22) and (25). The topological Rényi entropy is therefore an exclusive function of the phase: it can change only if a phase transition takes place. We argue that S_2^T is a good probe of topological order with the potential to characterize topologically ordered phases.

VI. COMPARISON AND DISCUSSION

A. Perturbation theories in the quasi-1D case

Although the quasi-1D case is exactly solvable, it is instructive to treat it with perturbation theories as well: the results obtained this way are directly comparable with those in the actual 2D case. Using a modified version of the PCUT procedures described in the Appendix, we find analogous expressions to those in Eqs. (22), (23), and (25). Without including the detailed calculations, the Rényi entropies after the first three corrections in the two limiting regimes are

$$S_2^{\partial A} = (L - n) - \frac{L'}{\ln 2} \left[\frac{\lambda^2}{4} + \frac{7\lambda^4}{64} + \frac{5\lambda^6}{96} + O(\lambda^8) \right] + \frac{H'}{\ln 2} \left[\frac{5\lambda^4}{64} + \frac{3\lambda^6}{32} + O(\lambda^8) \right] \quad (\lambda \ll 1), \quad (27)$$

$$S_2^{\partial A} = \frac{L'}{\ln 2} \left[\frac{\lambda^{-2}}{8} + \frac{\lambda^{-4}}{32} + \frac{47\lambda^{-6}}{3072} + O(\lambda^{-8}) \right] + \frac{H'}{\ln 2} \left[\frac{\lambda^{-2}}{8} + \frac{7\lambda^{-4}}{128} + \frac{107\lambda^{-6}}{3072} + O(\lambda^{-8}) \right] \quad (\lambda \gg 1), \quad (28)$$

TABLE II. Estimates for the critical field λ_C obtained from the power series of Eqs. (27), (28), and (29).

Estimates for λ_C		$\sqrt{a_1/a_2}$	$\sqrt{a_2/a_3}$
Equation (27)	Series $\propto L'$	1.512	1.449
	Series $\propto H'$		0.913
Equation (29)	Series $\propto 1$	1.414	1.225
Estimates for λ_C		$\sqrt{b_2/b_1}$	$\sqrt{b_3/b_2}$
Equation (28)	Series $\propto L'$	0.500	0.700
	Series $\propto H'$	0.661	0.798

where L' is the combined horizontal length of the boundary ∂A , and H' is the number of horizontal sections with a nonzero length contributing to L' . Since the corrections to the Rényi entropies are all linearly proportional to either L' or H' , there are no topological corrections proportional to n . The topological Rényi entropy takes the constant value $S_2^T = 2 + O(\lambda^8)$ at $\lambda \ll 1$ and the constant value $S_2^T = O(\lambda^{-8})$ at $\lambda \gg 1$.

When taking into account the first three corrections, the reduced Wilson loop in the $\lambda \ll 1$ regime becomes

$$W_0 = \exp\left\{-\left[\frac{\lambda^2}{4} + \frac{\lambda^4}{8} + \frac{\lambda^6}{12} + O(\lambda^8)\right]\right\}, \quad (29)$$

which indicates $W_R \propto \exp(-\beta D)$ and the presence of topological order. In the $\lambda \gg 1$ regime, the first nonzero contribution to W_0 appears at order $D/2$ in perturbation theory. This contribution is $W_0 \propto \lambda^{-D} = \exp[-\ln(\lambda) D]$, which indicates $W_R \propto \exp(-\beta D^2)$ and the absence of topological order. Note that the power series inside the exponential of Eq. (29) suggests that W_0 takes the exact form

$$W_0 = \exp\left(-\sum_{k=1}^{\infty} \frac{\lambda^{2k}}{4k}\right) = (1 - \lambda^2)^{1/4} \quad (30)$$

in the thermodynamic limit. This result is consistent with the critical field $\lambda_C = 1$ obtained from the exact treatment.

The perturbative expansions in Eqs. (27), (28), and (29) each contain at least one power series in λ . The critical field λ_C marks the breakdown of the perturbation theories, and it is again related to the appropriate radii of convergence. The estimates obtained with the method of Sec. VC are summarized in Table II: they suggest $0.5 \lesssim \lambda_C \lesssim 1.5$. This range is fully consistent with $\lambda_C = 1$.

B. Comparison with the actual 2D case

When discussing the phase transition in the actual 2D case, we argued that it occurs at the critical point $\lambda_C \approx 0.33$ of the equivalent 2D TFIM and that the two limiting phases are characterized by different constant values of the topological Rényi entropy. The argument referred only to the perturbation theories and the equivalence with the 2D TFIM. On the other hand, the quasi-1D case is more versatile because an exact solution is available. The exact treatment of the quasi-1D case suggests a behavior that is entirely analogous to our claims for the actual 2D case: the phase transition occurs at the critical point $\lambda_C = 1$ of the equivalent 1D TFIM, and the topological Rényi entropy is constant in the two limiting phases.

A direct comparison between the respective perturbation theories also provides evidence that the 1D and the 2D systems are similar in terms of their phase transitions. The behaviors of the λ_C estimates and their relations to the actual λ_C are entirely analogous in the two cases. First, the estimates are all reasonably close to the actual λ_C . Second, the estimates converge towards λ_C as the order is increased. Third, the estimates from the $\lambda \ll 1$ series generally overestimate while those from the $\lambda \gg 1$ series underestimate λ_C . These similarities suggest that the phase transitions in the 1D and the 2D cases are analogous; therefore the conclusions drawn from the exact treatment in the quasi-1D case are applicable to the physically more interesting actual 2D case as well.

VII. CONCLUSIONS

In this paper, we investigated the quantum phase transition between the topologically ordered and the disordered phases of the TCM with external magnetic field. The variation in topological order was probed via S_2^T : the topological Rényi entropy of order 2. We determined the exact field dependence of S_2^T in the computationally simpler case (quasi-1D case) and established perturbation theories in the physically more interesting case (actual 2D case). It was demonstrated that S_2^T takes distinct values in the two phases and has a discontinuity at the quantum phase transition. We therefore argue that S_2^T is a good probe of topological order that can effectively characterize topologically ordered phases.

The equivalence between the quasi-1D case of our problem and the exactly solvable 1D TFIM is a quite remarkable tool for obtaining exact results. So far it has provided us with an exact treatment of the quasi-1D case and a corresponding exact $S_2^T(\lambda)$ dependence. In perspective, such an exact treatment also makes it possible to search for critical exponents that can reveal the topological character of the quantum phase transition. Moreover, the exact time dependence of the system far away from equilibrium can be studied, as for example, in the case of a quantum quench [26,27].

It is important to point out that the Hamiltonian in Eq. (1) preserves the \mathbb{Z}_2 gauge structure of the bare TCM for all values of the magnetic field λ . This gauge structure justifies the simplifying step of substituting the subsystem A by its boundary ∂A when calculating S_2^T (thin subsystem). Indeed, as long as the gauge structure is preserved by the perturbation, the ground state can be expressed as a superposition of loop configurations. For such a system, all the relevant topological constraints are necessarily connected to the subsystem boundary ∂A . For example, in our \mathbb{Z}_2 gauge theory, the topological constraint manifests itself in the fact that there are an even number of spins flipped on each boundary loop of ∂A . On the other hand, considering only the boundary is the essential simplification we need for deriving the Rényi entropy formula in Eq. (11), which in turn makes the exact treatment in the quasi-1D case and the perturbation theories in the actual 2D case possible. The gauge structure also explains why the topological Rényi entropy is conserved during a quantum quench with a gauge-preserving Hamiltonian [27].

For a more generic Hamiltonian, the \mathbb{Z}_2 gauge structure is broken. This means that the spin configurations with an odd

number of spins flipped on a boundary loop of ∂A are allowed; therefore the topological constraint is no longer connected to the subsystem boundary ∂A . Note that we can also achieve an effective gauge structure breaking by drawing the boundary loops of ∂A on the dual lattice rather than on the real lattice (see Fig. 4) because they can then intersect with the spin loops on the dual lattice at an arbitrary number of points. To recover the robustness of S_2^T in such a non-gauge-preserving case, one needs to calculate it by using the original subsystem A (thick subsystem). This complicates the situation because the reduced density matrix ρ_A is not diagonal and so Eq. (11) becomes invalid. However, we believe that if a generalization of the Rényi entropy formula is found, the results in this paper can be extended to the more generic non-gauge-preserving case as well. This further step is crucial for verifying the robustness of the topological Rényi entropy against generic perturbations and hence proving its applicability as a nonlocal order parameter for topologically ordered phases.

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APPENDIX: DETAILED DESCRIPTIONS OF THE PCUT CALCULATIONS IN THE ACTUAL 2D CASE

1. PCUT calculation at small magnetic field

When considering Eq. (21) in the $\lambda \ll 1$ limit, we can use the PCUT procedure to relate the eigenstates of the perturbed Hamiltonian \hat{H} with $\lambda > 0$ to those of the unperturbed Hamiltonian \hat{H}_0 with $\lambda = 0$. This method relies on the concept of elementary excitations. In the case of \hat{H}_0 , these excitations are flips of stars (quasispins) A_s^z with an energy cost of 2 for each. They appear pairwise when switching on the perturbation, and the perturbed Hamiltonian can be written as

$$\hat{H} = 2\hat{Q} + \hat{T}_{+2} + \hat{T}_0 + \hat{T}_{-2}, \quad (\text{A1})$$

where \hat{Q} counts the number of excitations, and \hat{T}_n is the component of the perturbation that changes the number of excitations by n . It can be verified that $[\hat{Q}, \hat{T}_n] = n\hat{T}_n$ and that $\hat{T}_n^\dagger = \hat{T}_{-n}$. The explicit forms of the terms in Eq. (A1) are

$$\begin{aligned} \hat{Q} &= \frac{1}{2} \sum_s (1 - \hat{A}_s^z), & \hat{T}_{+2} &= -\lambda \sum_{\langle s, s' \rangle} \hat{A}_s^- \hat{A}_{s'}^-, \\ \hat{T}_0 &= -\lambda \sum_{\langle s, s' \rangle} (\hat{A}_s^+ \hat{A}_{s'}^- + \hat{A}_s^- \hat{A}_{s'}^+), & \hat{T}_{-2} &= -\lambda \sum_{\langle s, s' \rangle} \hat{A}_s^+ \hat{A}_{s'}^+, \end{aligned} \quad (\text{A2})$$

where $\hat{A}_s^\pm = (\hat{A}_s^x \pm i\hat{A}_s^y)/2$ are the standard spin raising and lowering operators. In the basis of the \hat{H}_0 excitations, the

term \hat{Q} is diagonal, while the terms \hat{T}_n are nondiagonal. The application of the PCUT involves an iterative sequence of steps to construct a unitary basis transformation $\hat{U}(l)$ such that the transformed Hamiltonian $\hat{H}(l) = \hat{U}^\dagger(l)\hat{H}\hat{U}(l)$ changes continuously from \hat{H} at $l = 0$ to a block-diagonal form at $l \rightarrow \infty$. The blocks in the asymptotic form $\hat{H}' \equiv \hat{H}(\infty)$ correspond to subspaces of constant excitation number, and the excitations can be found by solving the blocks. Note that these excitations belong to the perturbed Hamiltonian \hat{H} ; therefore they are not the same as the original \hat{H}_0 excitations. To avoid confusion, we refer to them as quasiexcitations.

According to the standard procedure of the PCUT [20,24], we write $\hat{H}(l) = 2\hat{Q} + \hat{T}_{+2}(l) + \hat{T}_0(l) + \hat{T}_{-2}(l)$ as in Eq. (A1), and define $\hat{\eta}(l) \equiv \hat{T}_{+2}(l) - \hat{T}_{-2}(l)$. If we then require $\hat{U}(l)$ to satisfy the equation $\partial_l \hat{U}(l) = -\hat{U}(l)\hat{\eta}(l)$, it follows that $\partial_l \hat{H}(l) = [\hat{\eta}(l), \hat{H}(l)]$. In terms of the components $\hat{T}_n(l)$, this equation for $\hat{H}(l)$ becomes

$$\begin{aligned} \partial_l \hat{T}_0(l) &= 2[\hat{T}_{+2}(l), \hat{T}_{-2}(l)], \\ \partial_l \hat{T}_{+2}(l) &= -4\hat{T}_{+2}(l) + [\hat{T}_{+2}(l), \hat{T}_0(l)], \\ \partial_l \hat{T}_{-2}(l) &= -4\hat{T}_{-2}(l) + [\hat{T}_0(l), \hat{T}_{-2}(l)]. \end{aligned} \quad (\text{A3})$$

The last two equations show that $\hat{T}_{+2}(\infty) = \hat{T}_{-2}(\infty) = 0$, which is consistent with the block-diagonal form of $\hat{H}(\infty)$. To solve the equations for $\hat{U}(l)$ and $\hat{T}_n(l)$ iteratively, we write these quantities in a series as

$$\hat{U}(l) = \sum_{k=0}^{\infty} \hat{U}^{(k)}(l), \quad \hat{T}_n(l) = \sum_{k=1}^{\infty} \hat{T}_n^{(k)}(l). \quad (\text{A4})$$

The equations for $\hat{U}(l)$ and $\hat{T}_n(l)$ then take the forms

$$\begin{aligned} \partial_l \hat{U}^{(k)}(l) &= -\sum_{j=0}^{k-1} \hat{U}^{(j)}(l) \{ \hat{T}_{+2}^{(k-j)}(l) - \hat{T}_{-2}^{(k-j)}(l) \}, \\ \partial_l \hat{T}_0^{(k)}(l) &= 2 \sum_{j=1}^{k-1} [\hat{T}_{+2}^{(j)}(l), \hat{T}_{-2}^{(k-j)}(l)], \\ \partial_l \hat{T}_{+2}^{(k)}(l) &= -4\hat{T}_{+2}^{(k)}(l) + \sum_{j=1}^{k-1} [\hat{T}_{+2}^{(j)}(l), \hat{T}_0^{(k-j)}(l)], \\ \partial_l \hat{T}_{-2}^{(k)}(l) &= -4\hat{T}_{-2}^{(k)}(l) + \sum_{j=1}^{k-1} [\hat{T}_0^{(j)}(l), \hat{T}_{-2}^{(k-j)}(l)], \end{aligned} \quad (\text{A5})$$

and the corresponding starting conditions at $l = 0$ become

$$\begin{aligned} \hat{U}^{(k)}(0) &= \begin{cases} 1 & (k = 0), \\ 0 & (k \geq 1), \end{cases} \\ \hat{T}_n^{(k)}(0) &= \begin{cases} \hat{T}_n & (k = 1), \\ 0 & (k \geq 2). \end{cases} \end{aligned} \quad (\text{A6})$$

If we apply the PCUT up to second order in λ , the relevant terms in the series of Eq. (A4) are

$$\begin{aligned} \hat{U}^{(0)}(l) &= 1, & \hat{U}^{(1)}(l) &= -\frac{1}{4}(\hat{T}_{+2} - \hat{T}_{-2})(1 - e^{-4l}), \\ \hat{U}^{(2)}(l) &= \frac{1}{32}(\hat{T}_{+2} - \hat{T}_{-2})^2(1 - e^{-4l})^2 \\ &\quad - \frac{1}{16}[\hat{T}_{+2} + \hat{T}_{-2}, \hat{T}_0][1 - (1 + 4l)e^{-4l}], \\ \hat{T}_0^{(1)}(l) &= \hat{T}_0, & \hat{T}_0^{(2)}(l) &= \frac{1}{4}[\hat{T}_{+2}, \hat{T}_{-2}](1 - e^{-8l}), \end{aligned} \quad (\text{A7})$$

$$\hat{T}_{\pm 2}^{(1)}(l) = \hat{T}_{\pm 2} e^{-4l}, \quad \hat{T}_{\pm 2}^{(2)}(l) = \pm[\hat{T}_{\pm 2}, \hat{T}_0]l e^{-4l},$$

the basis transformation $\hat{U} \equiv \hat{U}(\infty)$ becomes

$$\begin{aligned} \hat{U} &= 1 + \frac{1}{4}(\hat{T}_{-2} - \hat{T}_{+2}) + \frac{1}{16}([\hat{T}_0, \hat{T}_{-2}] - [\hat{T}_{+2}, \hat{T}_0]) \\ &+ \frac{1}{32}(\hat{T}_{+2}\hat{T}_{+2} + \hat{T}_{-2}\hat{T}_{-2} - \hat{T}_{+2}\hat{T}_{-2} - \hat{T}_{-2}\hat{T}_{+2}), \end{aligned} \quad (\text{A8})$$

and the asymptotic Hamiltonian takes the form

$$\hat{H}' = 2\hat{Q} + \hat{T}_0 + \frac{1}{4}[\hat{T}_{+2}, \hat{T}_{-2}], \quad (\text{A9})$$

which indeed conserves the number of quasiexcitations. The same procedure can be continued to arbitrary order in λ , but the calculations quickly become cumbersome.

Since the ground state $|\Omega(\lambda)\rangle$ of \hat{H}' is the only state with no quasiexcitations, it has its own block in \hat{H}' . To express this state in terms of the physically transparent \hat{H}_0 excitations, we use the basis transformation: $|\Omega(\lambda)\rangle = \hat{U}|0\rangle$. When calculating the first two perturbative corrections to the ground state at $\lambda = 0$, the perturbed state $|\Omega(\lambda)\rangle$ needs to be properly normalized up to λ^4 . Applying the PCUT up to fourth order with the aid of a computer, the perturbed ground state becomes

$$\begin{aligned} |\Omega(\lambda)\rangle &= \hat{U}|0\rangle = \left[1 - \frac{N^2\lambda^2}{16} + \frac{(N^4 - 95N^2)\lambda^4}{512}\right]|0\rangle \\ &+ \left[\frac{\lambda}{4} - \frac{(N^2 - 15)\lambda^3}{64}\right] \sum_{2N^2} |\times \times\rangle \\ &+ \frac{\lambda^2}{4} \sum_{2N^2} \left| \begin{array}{c} \times \cdot \\ \cdot \times \end{array} \right\rangle + \frac{\lambda^2}{8} \sum_{2N^2} |\times \cdot \times\rangle \\ &+ \frac{\lambda^2}{8} \sum_{N^2} \left| \begin{array}{cc} \times & \times \\ \times & \times \end{array} \right\rangle + \frac{\lambda^2}{16} \sum_{2N^4 - 9N^2} \left| \left[\begin{array}{c} \times \\ \times \end{array} \right] \left[\begin{array}{c} \times \\ \times \end{array} \right] \right\rangle, \end{aligned} \quad (\text{A10})$$

where the equivalent states related to each other by translational and rotational symmetries are labeled by the relative positions of the star excitations (\times) in them, and the number of states in each equivalence class is given by the number below the corresponding sum. The notation $[\cdot \cdot \cdot][\cdot \cdot \cdot]$ means that there are two clusters of excitations that are independent of each other: they are not in a relative position characterizing any other equivalence class.

The ground state in Eq. (A10) is indeed properly normalized up to fourth order in λ because

$$\begin{aligned} \langle \Omega(\lambda) | \Omega(\lambda) \rangle &= \left[1 - \frac{N^2\lambda^2}{16} + \frac{(N^4 - 95N^2)\lambda^4}{512}\right]^2 \\ &+ 2N^2 \left[\frac{\lambda}{4} - \frac{(N^2 - 15)\lambda^3}{64}\right]^2 \\ &+ 2N^2 \left(\frac{\lambda^2}{4}\right)^2 + (2N^2 + N^2) \left(\frac{\lambda^2}{8}\right)^2 \\ &+ (2N^4 - 9N^2) \left(\frac{\lambda^2}{16}\right)^2 \\ &= 1 + O(\lambda^6). \end{aligned} \quad (\text{A11})$$

To calculate the Rényi entropy, we need to evaluate the expectation values of the products appearing in Eq. (11) for

the ground state. The expectation values having a contribution up to λ^4 to the Rényi entropy are

$$\begin{aligned} \langle \times \times \rangle &= 2 \left[\frac{\lambda}{4} - \frac{(N^2 - 15)\lambda^3}{64} \right] \left\{ \left[1 - \frac{N^2\lambda^2}{16} \right] \right. \\ &+ 4 \left(\frac{\lambda^2}{4} \right) + 4 \left(\frac{\lambda^2}{8} \right) + (2N^2 - 9) \left(\frac{\lambda^2}{16} \right) \left. \right\} \\ &= \frac{\lambda}{2} + \frac{15\lambda^3}{16} + O(\lambda^5), \end{aligned} \quad (\text{A12})$$

$$\left\langle \begin{array}{c} \times \cdot \\ \cdot \times \end{array} \right\rangle = 2 \left(\frac{\lambda^2}{4} \right) + 4 \left[\frac{\lambda}{4} \right]^2 = \frac{3\lambda^2}{4} + O(\lambda^4),$$

$$\langle \times \cdot \times \rangle = 2 \left(\frac{\lambda^2}{8} \right) + 2 \left[\frac{\lambda}{4} \right]^2 = \frac{3\lambda^2}{8} + O(\lambda^4),$$

$$\left\langle \begin{array}{cc} \times & \times \\ \times & \times \end{array} \right\rangle = 2 \left(\frac{\lambda^2}{8} \right) + 4 \left[\frac{\lambda}{4} \right]^2 = \frac{\lambda^2}{2} + O(\lambda^4),$$

$$\left\langle \left[\begin{array}{c} \times \\ \times \end{array} \right] \left[\begin{array}{c} \times \\ \times \end{array} \right] \right\rangle = 2 \left(\frac{\lambda^2}{16} \right) + 2 \left[\frac{\lambda}{4} \right]^2 = \frac{\lambda^2}{4} + O(\lambda^4),$$

where the notation is analogous to that in Eq. (A10). For example, $\langle \times \times \rangle \equiv \langle \Omega(\lambda) | \hat{A}_s^x \hat{A}_{s'}^x | \Omega(\lambda) \rangle$, where s and s' are any two nearest-neighbor stars. If the boundary ∂A consists of n closed loops with a combined length L and a total number of K corners that are sufficiently far away from each other, the Rényi entropy is given by

$$\begin{aligned} S_2^{\partial A} &= (L - n) - \log_2 \left(1 + L \langle \times \times \rangle^2 + K \left\langle \begin{array}{c} \times \cdot \\ \cdot \times \end{array} \right\rangle^2 \right. \\ &+ (L - K) \langle \times \cdot \times \rangle^2 + \frac{L(L - 3)}{2} \left\langle \left[\begin{array}{c} \times \\ \times \end{array} \right] \left[\begin{array}{c} \times \\ \times \end{array} \right] \right\rangle^2 \left. \right) \\ &= (L - n) - \frac{1}{\ln 2} \left[\frac{L}{4} \lambda^2 + \frac{63L + 27K}{64} \lambda^4 + O(\lambda^6) \right]. \end{aligned} \quad (\text{A13})$$

The perturbative corrections are linearly proportional to either L or K , and they are independent of n .

Now we consider a Wilson loop for a square region R with boundary length $L = 4D$ and corner number $K = 4$. According to Eq. (4), the Wilson loop is $W_R = 1$ for the unperturbed ground state $|0\rangle$ because $A_s^z = +1$ for all s . The structure of the perturbed ground state $|\Omega(\lambda)\rangle$ in Eq. (A10) shows that W_R can only be -1 instead of $+1$ if an odd number of excitations are inside R . Taking into account all possibilities up to fourth order in λ , the Wilson loop becomes

$$\begin{aligned} W_R &= 1 - 2 \left\{ L \left[\frac{\lambda}{4} - \frac{(N^2 - 15)\lambda^3}{64} \right]^2 + (2L - K) \left(\frac{\lambda^2}{4} \right)^2 \right. \\ &+ 2L \left(\frac{\lambda^2}{8} \right)^2 + K \left(\frac{\lambda^2}{8} \right)^2 \\ &+ L(2N^2 - L - 6) \left(\frac{\lambda^2}{16} \right)^2 \left. \right\} \end{aligned}$$

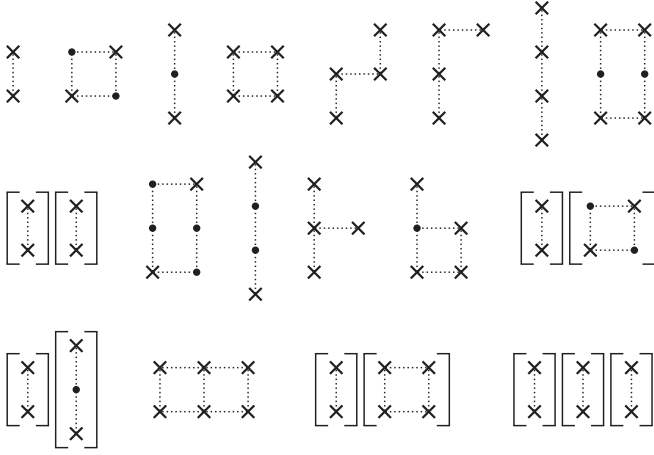


FIG. 7. Equivalence classes for the relative positions of the star excitations (\times) at the level of the third corrections (at order λ^6).

$$\begin{aligned}
 &= 1 - \frac{L}{8} \lambda^2 - \left(\frac{L}{2} - \frac{L^2}{128} - \frac{3K}{32} \right) \lambda^4 + O(\lambda^6) \\
 &= \exp \left\{ -L \left[\frac{\lambda^2}{8} + \frac{\lambda^4}{2} + O(\lambda^6) \right] + K \left[\frac{3\lambda^4}{32} + O(\lambda^6) \right] \right\}.
 \end{aligned} \tag{A14}$$

The perturbative corrections inside the exponential are linearly proportional to either L or K .

The expressions in Eqs. (A13) and (A14) give the first two corrections to $S_2^{\partial A}$ and W_R . With the aid of a computer, the third corrections proportional to λ^6 can be found in a similar manner. In this case, the state $|\Omega(\lambda)\rangle$ must be properly normalized up to sixth order in λ , and one needs to consider all the relative excitation positions shown in Fig. 7. Without including the detailed calculations, the final results for the Rényi entropy and the Wilson loop are

$$\begin{aligned}
 S_2^{\partial A} &= (L - n) - \frac{L}{\ln 2} \left[\frac{\lambda^2}{4} + \frac{63\lambda^4}{64} + \frac{503\lambda^6}{96} + O(\lambda^8) \right] \\
 &\quad - \frac{K}{\ln 2} \left[\frac{27\lambda^4}{64} + \frac{737\lambda^6}{256} + O(\lambda^8) \right],
 \end{aligned} \tag{A15}$$

$$\begin{aligned}
 W_R &= \exp \left\{ -L \left[\frac{\lambda^2}{8} + \frac{\lambda^4}{2} + \frac{7697\lambda^6}{3072} + O(\lambda^8) \right] \right. \\
 &\quad \left. + K \left[\frac{3\lambda^4}{32} + \frac{89\lambda^6}{128} + O(\lambda^8) \right] \right\}.
 \end{aligned} \tag{A16}$$

The features noticed after the first two corrections remain intact after the third corrections as well.

2. PCUT calculation at large magnetic field

When considering Eq. (24) in the $\mu \equiv \lambda^{-1} \ll 1$ limit, the PCUT procedure is entirely analogous to the one described in Appendix A1. The elementary excitations of the unperturbed Hamiltonian with $\mu = 0$ are flips of physical spins σ_i^z with an energy cost of 2 for each. The perturbed Hamiltonian with $\mu > 0$ can be written as

$$\hat{H} = 2\hat{Q} + \hat{T}_{+4} + \hat{T}_{+2} + \hat{T}_0 + \hat{T}_{-2} + \hat{T}_{-4}, \tag{A17}$$

where the respective terms take the explicit forms

$$\begin{aligned}
 \hat{Q} &= \frac{1}{2} \sum_i (1 - \hat{\sigma}_i^z), \\
 \hat{T}_n &= -\mu \sum_s \sum_{\pm} \prod_{i \in s} \hat{\sigma}_i^{\pm}.
 \end{aligned} \tag{A18}$$

The sum in \pm contains all inequivalent products of the four $\hat{\sigma}_i^{\pm}$ operators in which the number of the $\hat{\sigma}_i^+$ factors is $2 - n/2$ and that of the $\hat{\sigma}_i^-$ factors is $2 + n/2$. Applying the PCUT with $\hat{\eta}(l) \equiv \hat{T}_{+4}(l) + \hat{T}_{+2}(l) - \hat{T}_{-2}(l) - \hat{T}_{-4}(l)$ up to fourth order, we find that the perturbed ground state which is properly normalized up to μ^4 is given by

$$\begin{aligned}
 |\Omega(\lambda)\rangle &= \left[1 - \frac{N^2 \mu^2}{128} + \left(N^4 - \frac{62N^2}{9} \right) \frac{\mu^4}{32768} \right] |\uparrow\rangle \\
 &\quad + \left[\frac{\mu}{8} - \left(N^2 - \frac{2}{3} \right) \frac{\mu^3}{1024} \right] \sum_{N^2} \left| \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \right\rangle \\
 &\quad + \frac{\mu^2}{48} \sum_{2N^2} \left| \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \end{array} \right\rangle \\
 &\quad + \frac{\mu^2}{64} \sum_{\frac{1}{2}(N^4 - 5N^2)} \left| \left[\begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \right] \left[\begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \right] \right\rangle,
 \end{aligned} \tag{A19}$$

where the equivalent states are labeled by the relative positions of the spin excitations (\circ) in them [cf. Eq. (A10)].

We now consider a subsystem with a boundary ∂A containing L spins and L stars acting on these spins. The total number of corners is K as above. The diagonal elements of the density matrix $\rho_{\partial A}$ can be obtained directly in the basis of the physical spins σ_i^z . The element corresponding to $\sigma_i^z = +1$ for all L spins (no star excitations on the boundary) is

$$\begin{aligned}
 (\rho_{\partial A})_{00} &= \left[1 - \frac{N^2 \mu^2}{128} + \left(N^4 - \frac{62N^2}{9} \right) \frac{\mu^4}{32768} \right]^2 \\
 &\quad + (N^2 - L) \left[\frac{\mu}{8} - \left(N^2 - \frac{2}{3} \right) \frac{\mu^3}{1024} \right]^2 \\
 &\quad + (2N^2 - 3L) \left(\frac{\mu^2}{48} \right)^2 \\
 &\quad + \frac{1}{2} [N^4 - N^2(2L + 5) + (L^2 + 7L)] \left(\frac{\mu^2}{64} \right)^2 \\
 &= 1 - \frac{L}{64} \mu^2 - \left(\frac{5L}{8192} - \frac{L^2}{8192} \right) \mu^4 + O(\mu^6),
 \end{aligned} \tag{A20}$$

while the one corresponding to $\sigma_i^z = -1$ for any two neighboring spins and $\sigma_i^z = +1$ for the remaining $L - 2$ spins (one star excitation on the boundary) is

$$(\rho_{\partial A})_{11} = \left[\frac{\mu}{8} \right]^2 = \frac{\mu^2}{64} + O(\mu^4). \tag{A21}$$

Note that there are L ways of choosing two neighboring spins from ∂A . Since the contribution of the remaining diagonal

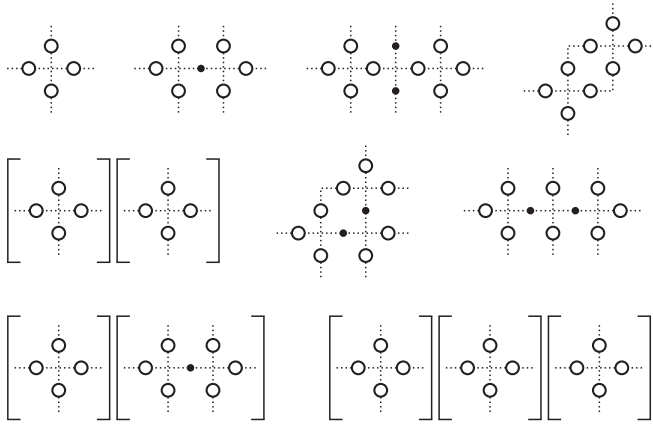


FIG. 8. Equivalence classes for the relative positions of the spin excitations (o) at the level of the third corrections (at order μ^6).

elements is $O(\mu^6)$ to the Rényi entropy, we find that

$$\begin{aligned} \text{Tr}[\hat{\rho}_{\partial A}^2] &= (\rho_{\partial A})_{00}^2 + L(\rho_{\partial A})_{11}^2 \\ &= 1 - \frac{L}{32} \mu^2 - \left(\frac{L}{1024} - \frac{L^2}{2048} \right) \mu^4 + O(\mu^6), \end{aligned} \quad (\text{A22})$$

and the Rényi entropy takes the form

$$S_2^{\partial A} = \frac{1}{\ln 2} \left[\frac{L}{32} \mu^2 + \frac{L}{1024} \mu^4 + O(\mu^6) \right]. \quad (\text{A23})$$

The perturbative corrections are again linearly proportional to the boundary length L . Furthermore, there are no terms proportional to K in the first two corrections determined here.

With the aid of a computer, the third correction to $S_2^{\partial A}$ can be calculated in a similar manner. In this case, the state $|\Omega(\lambda)\rangle$ must be properly normalized up to sixth order in μ , and one needs to consider all the relative excitation positions shown in Fig. 8. Without including the detailed calculations, the final result for the Rényi entropy is

$$\begin{aligned} S_2^{\partial A} &= \frac{L}{\ln 2} \left[\frac{\lambda^{-2}}{32} + \frac{\lambda^{-4}}{1024} + \frac{115\lambda^{-6}}{2\,359\,296} + O(\lambda^{-8}) \right] \\ &\quad - \frac{K}{\ln 2} \left[\frac{35\lambda^{-6}}{4\,718\,592} + O(\lambda^{-8}) \right]. \end{aligned} \quad (\text{A24})$$

After the third correction, there is a corner contribution proportional to K , but still no topological contribution proportional to n .

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