PHYSICAL REVIEW A **87**, 051801(R) (2013)

Efficient spin squeezing with optimized pulse sequences

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(Received 12 April 2013; published 14 May 2013)

Spin squeezed states are a class of entangled states of spins that have practical applications to precision measurements. In recent years spin squeezing with one-axis twisting (OAT) has been demonstrated experimentally with spinor Bose-Einstein condensates (BECs) with more than 10^3 atoms. Although the noise is below the standard quantum limit, the OAT scheme cannot reduce the noise down to the ultimate Heisenberg limit. Here we propose an experimentally feasible scheme based on optimized quantum control to greatly enhance the performance of OAT to approach the Heisenberg limit, requiring only an OAT Hamiltonian and the use of several coherent driving pulses. The scheme is robust against technical noise and can be readily implemented for spinor BECs or trapped ions with current technology.

DOI: 10.1103/PhysRevA.87.051801 PACS number(s): 42.50.Dv, 03.75.Gg, 03.67.Bg

Spin squeezed states [1] have attracted a lot of interest due to both their role in the fundamental study of many-particle entanglement and their practical application to precision measurements with Ramsey interferometers [2–6]. In recent years, much progress has been made on the experimental squeezing of a large number ($10^3 \sim 10^6$) of ultracold atoms [7–11]. Many of these experiments follow the so-called one-axis twisting (OAT) scheme, which is known to reduce the noise-to-signal ratio from the classical case by an amount that scales as $N^{-2/3}$ with the particle number N [1]. This reduction is not optimal yet and still above the so-called Heisenberg limit which scales as N^{-1}. There have been several theoretical proposals to enhance the OAT scheme [12,13]. For example, one of the approaches [13] involves inducing a better squeezing Hamiltonian, the so-called two-axis twisting (TAT) Hamiltonian, with Raman assisted coupling for trapped spinor Bose-Einstein condensates (BECs). This is hardware level engineering, requiring modification of a particular experimental setup and does not apply to other physical systems. Another approach [12] employs a digital quantum simulation technique to convert an OAT Hamiltonian to an effective TAT Hamiltonian by stroboscopically applying a large number of pulses. This software level solution is universal but sensitive to the accumulation of control errors. None of these proposals have been experimentally tested yet due to various difficulties.

Inspired by the idea of optimized quantum control, we propose an experimentally feasible scheme to greatly improve the performance of OAT, requiring only two or three additional coherent driving pulses to carry out collective spin rotations, which is a routine technique with the current technology. The scheme is shown to be robust to noise and imperfection in control pulses. Using this scheme, it is possible to generate more spin squeezing and detect a significantly larger entanglement depth for the many-particle atomic ensemble [5]. This scheme enhances the OAT squeezing on the software level and therefore can be applied to any physical system that is endowed with these operations. The idea of optimized squeezing may also be easily transferred to cases where the interaction term deviates from the OAT Hamiltonian.

We consider the general scenario of one-axis twisting independent of the underlying physical system with the Hamiltonian $H = \chi S_z^2$ ($S_z = \sum_i^N s_z^i$) (setting $\hbar = 1$). The system starts from a collective spin coherent state polarized along the x axis. As time goes on the initially homogenous spin fluctuation gets distorted and redistributed among different directions, and the direction along which spin fluctuation gets suppressed gradually changes over time. The squeezing is measured by the parameter ξ^2, defined as $\xi^2 = N\langle S_{\vec{n}}^2\rangle / |\langle S_x\rangle|^2$, where $\vec{n}$ is the direction along which spin fluctuation is minimized. The decreasing rate of ξ^2 slows down with time, and after the optimal squeezing point, ξ^2 increases again. Aside from the initial state, which is rotationally symmetric about the x axis, all the subsequent states breaks this symmetry and pick out a special direction, i.e., the direction along which fluctuation is minimized. It is well known that the TAT Hamiltonian $H_2 = \chi_2(S_x^2 - S_y^2)$ can produce better squeezing [1], which, after doing the Trotter decomposition with an infinitesimal time interval, could be seen as switching the squeezing axis back and forth very fast between two orthogonal directions [12]. To avoid the noise accumulation from a large number of switching pulses inherent in the Trotter expansion scheme, we take an alternative approach based on optimization of a few control pulses to maximize the squeezing of the final state. We consider an n-step squeezing protocol (where n is typically 2 or 3 for a practical scheme) defined as follows: at step j ($j = 1,2,\ldots,n$), we first apply an instantaneous collective spin rotation around the x axis, $U(\alpha_i) = \exp(-i\, S_x\alpha_i)$, and then let the state evolve under the OAT Hamiltonian $H = \chi S_z^2$ for a duration T_i. Effectively, we squeeze the state along a different axis lying in the y-z plane in each step, so the effective evolution operator can be written as

$$U(\theta_i, T_i) = \prod_{i=n}^{1} \exp\left(-i\, \chi S_{\theta_i}^2 T_i\right), \tag{1}$$

where $S_{\theta_j} \equiv \cos\theta_j S_z + \sin\theta_j S_y$ and the factors are arranged from right to left with increase of j. This evolution operator coincides with that of a quantum kicked-top model with n kicks. Since the initial state is assumed to be polarized along the x direction, which is symmetric around the x axis, θ_1 is irrelevant and can be chosen to be 0 (so no control pulse is needed for step 1). Therefore, for an n-step squeezing protocol,

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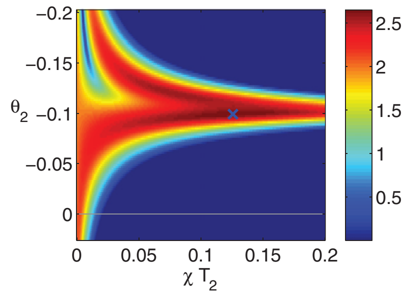

FIG. 1. (Color online) Squeezing $-\log(\xi^2)$ as a function of the control parameters θ_2 and T_2 for a typical value of T_1, calculated with $N = 2000$ spin-1/2 particles. See Eq. (1) and text for definition of θ_i and T_i. The cross symbol marks the point of optimal squeezing. The horizontal line $\theta_2 = 0$ corresponds to the case of the OAT scheme.

there are $(2n-1)$ tunable parameters: T_i and θ_i (excluding θ_1). The final squeezing parameter is thus a multivariable function $\xi^2(T_i,\theta_i)$. Note that as n becomes very large, our protocol includes the proposed sequence in [12] as a special case and so in principle our protocol can approach the Heisenberg limit as n grows. Our purpose is to find the best available squeezing $\xi^2(T_i,\theta_i)$ with a minimum number n of the time steps.

In the case of $n = 2$ or 3, the landscape of $\xi^2(T_i,\theta_i)$ in the parameter space is quite simple and well behaved. Take the $n = 2$ case as an example. For a typical value of T_1 smaller than the optimal OAT squeezing time, $-\log(\xi^2)$ as a function of θ_2 and T_2 is shown in Fig 1. The optimal squeezing point marked by the cross lies way off the OAT trajectory, the horizontal line with $\theta_2 = 0$. For the $n = 3$ case, with θ_2 and T_2 fixed near the optimal values of the $n = 2$ case, $-\log(\xi^2)$ as a function of θ_3 and T_3 shows a similar landscape. These solutions already exceed that of the OAT scheme by a large margin. The results indicate that the optimization technique with n as small as 2 or 3 suffices to significantly improve over the OAT scheme.

Next, we investigate performance of the optimized squeezing scheme, focusing on the scaling of the squeezing $\xi^2(T_i,\theta_i)$ as a function of the total particle number N. For a given set of parameters, we can numerically calculate the evolution operator in Eq. (1) by exactly diagonalizing the effective Hamiltonians $S_{\theta_i}^2$ and then obtain the squeezing parameter ξ^2. To account for the fact that in reality the coherent spin rotations cannot be generated instantaneously, in the numerical simulation we actually keep the OAT Hamiltonian on all the time, even during the spin rotations. However, we do assume the effective magnetic field B effecting the spin rotation to be much stronger than the squeezing Hamiltonian, $B \gg N\chi$, as is the case in experiments. We randomly sample from the parameter space for a large number of times, use these random samples as initial guesses to start unconstrained local optimization of the squeezing parameter, and pick the best one as our solution. Repeating this procedure for every system size N is extremely resource intensive especially when N gets as large as 10^5. Taking advantage of the fact that adding several more to 10^3 particles should not change the solution much, we can feed the previously found nonlocal optimal solution as an initial guess to the local optimizer of a larger system and obtain a near optimal solution quickly. In this way we managed to obtain (near) optimal solutions for systems all the way up to $N = 10^5$ particles, with only a cost of classical computing time on the order of tens of hours on a typical multicore computer. As shown in Fig 2, with $n = 2$, the squeezing parameter ξ^2 gets reduced by a significant amount already compared with the OAT scheme, and with $n = 3$, ξ^2 decreases further. The scaling of ξ^2 with the number of particles shows a clear power law $\xi^2 \sim 1/N^\beta$. A simple OAT scheme gives $\beta = 2/3$ and the TAT scheme gives $\beta = 1$ [1]. The Heisenberg limit of noise gives a bound $\beta \leqslant 1$ for the scaling, and this bound is saturated by the TAT scheme. Remarkably we observe that the optimized $n = 2,3$ protocols can give $\beta = 0.92$ and 0.98, respectively, very close to the ultimate Heisenberg limit. Moreover, the $n = 3$ optimized scheme has a smaller multiplicative constant compared with the TAT scheme, so in the realistic range of particle number $N \lesssim 10^6$, it actually outperforms the TAT scheme. This shows that a moderate alternation of the OAT scheme through optimization can significantly increase the spin squeezing.

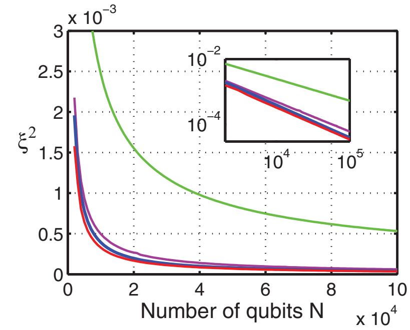

FIG. 2. (Color online) Scaling of the squeezing parameter ξ^2 with the number of qubits. Curves from top to bottom are for OAT, two-step optimized squeezing, TAT, and three-step optimized squeezing. Inset shows the same curves in log-log scale.

We have demonstrated a significant improvement over the conventional OAT by applying very few optimized control pulses. A cost of the proposed scheme is that it takes longer evolution time to achieve the optimal squeezing. A typical evolution of ξ^2 with time t is shown in Fig. 3. We notice that in general the $(i+1)$th squeezing step takes longer time than the ith step. Since the time cost in the first step is on the order of the optimal OAT duration, the overall duration of the proposed protocol is usually longer than that of the OAT scheme. An excessively long duration would be an obstacle in systems with short coherence time. The two relevant time

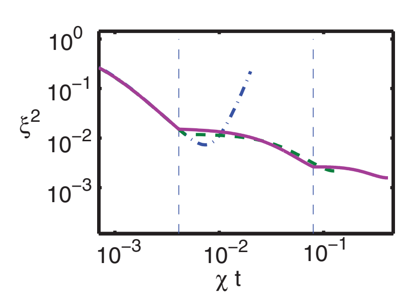

FIG. 3. (Color online) Evolution of the squeezing parameter ξ^2 with time, calculated with $N = 2000$ spin-1/2 particles. The dash-dot line is for OAT, the dashed line for the two-step optimized squeezing scheme, and the solid line for the three-step optimized squeezing.

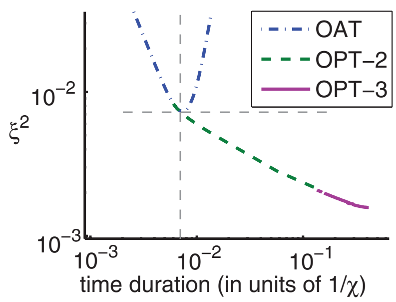

FIG. 4. (Color online) Constrained optimization of ξ^2 with the total time duration as a cost function. We take $1/\chi$ as the time unit. Achievable squeezing ξ^2 as a function of the total duration is shown, together with OAT, calculated with $N = 2000$ spin-1/2 particles. OPT-2 (3) stands for optimized squeezing sequence with $n = 2$ (3) segments. Horizontal and vertical dashed lines are guides to the eye.

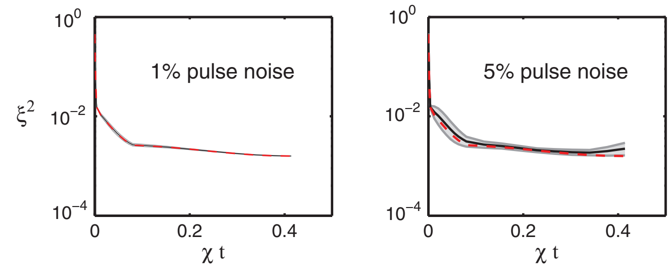

FIG. 5. (Color online) Optimized squeezing in the presence of control noise. We use the three-step optimization scheme as an example and assume all five control parameters in this scheme have the same magnitude of relative errors as specified in this figure. The dashed line is for the ideal case with no error in the control parameters, the solid line denotes the average of many random trajectories (50 random trials), and the shaded area marks the range of those trajectories. In the left panel, the shaded region is too small to be distinguished from the ideal case.

scales here are the coherence time τ and the inverse of interaction strength $1/\chi$. The time cost of the proposed scheme is around $0.01/\chi \sim 0.1/\chi$. If $\tau \gtrsim 0.1/\chi$ this scheme can be implemented without compromise. On the other hand, if that is not the case, decoherence effect would play a role and our unconstrained optimization no longer yields the best result. However, we can work around this problem by performing an optimization with the total duration added as a cost function and get a compromised optimal pulse sequence. By tuning the weight of the cost function we could obtain a continuous series of compromised optimal solutions as shown in Fig. 4. These solutions of two- and three-step schemes form two line segments, continuously connecting the optimal OAT squeezing protocol to that of the unconstrained optima, offering a tradeoff between the protocol duration and the squeezing magnitude. For each real experimental setup, one could correspondingly pick up the best point in accordance with the coherence time of the system. How much one can gain over the OAT scheme depends on how long the coherence time can reach.

Next we test noise resistance of the proposed scheme. There are only 3 (5) control parameters in the $n = 2$ (3) scheme, making the accumulation of control noise negligible. We have done numerical simulation of our scheme adding random pulse area and timing noise and confirmed the robustness of the squeezing parameter ξ^2 as shown in Fig. 5. This contrasts to the proposals [12,14] requiring a large number of coherent rotation pulses where control errors accumulate and significantly degrade the performance. Thus our proposed scheme offers a useful alternative to the previous works. Another practical issue related to control noise is the uncertainty in the number of particles in a real experiment. Our pulse scheme depends on the number of particles N while in experiments such as ultracold gas we do not typically know the number N exactly. Fortunately we notice that the control parameters vary slowly with N, e.g., in going from $N = 1900$ to $N = 2100$, the control parameters only vary by 1%–5%. So a ±5% uncertainty in N at $N = 2000$ is equivalent to an extra noise below 5% in the control parameters, to which ξ^2 is not so sensitive, as we have shown in Fig. 5.

Finally we discuss possible physical realizations of the scheme proposed here. The scheme only requires two ingredients: the nonlinear collective spin interaction S_z^2 and the ability to rotate the collective spin around an orthogonal axis, say x. Several experimental systems meet these requirements, e.g., trapped ions and spinor BECs. In trapped ion systems, depending on the ion species, one can use bichromatic lasers or two pairs of Raman laser beams (the Mølmer-Sørensen scheme) to induce the S_z^2 or S_x^2 type of interaction. The strength of this interaction χ can reach kHz scale, giving $1/\chi \sim$ ms. The coherence time usually exceeds $1/\chi$ and our scheme can apply without compromise. Collective spin rotation can be simply done by shining a laser on all the ions driving the corresponding single-qubit $\sigma_{x/y}$ or rotation. The rotation pulses have durations much shorter than $1/\chi$. While linear Paul traps [15] can now coherently control only about a dozen ions, too few for the purpose of spin squeezing, planar Penning traps can manipulate more than 200 ions [16]. For the purpose of precision measurement, 200 ions may seem less impressive than 10^5 particles, but we show that using our scheme we can create genuine multiparticle entangled states with a significantly larger entanglement depth. The entanglement depth, defined in [5], is a way to measure how many particles within the whole sample have been prepared in a genuine entangled state. Our result is shown in Fig. 6. In

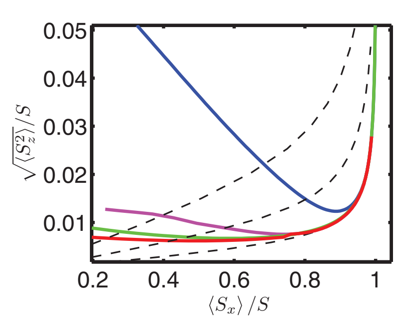

FIG. 6. (Color online) Entanglement depth achievable with different approaches for 200 spin-1/2 particles. The solid lines from top to bottom correspond respectively to the OAT scheme, the two-step optimized squeezing, the TAT, and the three-step optimized squeezing. The dashed lines from top to bottom correspond to the optimal squeezing for 50, 100, and 200 particles, respectively. Lying below the curve of optimal squeezing for n particles is a certificate of genuine n-particle entanglement.

this figure, a point lying below the optimal squeezing curve of n particles corresponds to a state that contains genuine n-particle entanglement. Our scheme produces states that lie below the OAT states in a large range of $\langle S_x \rangle$ values, which means that experimentally one can achieve a significantly larger entanglement depth by this optimization technique.

Another class of physical systems is a spinor Bose-Einstein condensate of atoms with two chosen internal states mimicking spin-$1/2$ particles [8,9]. The desired S_z^2 interaction is induced by spin-dependent s-wave scattering as proposed in [4]. Coherent laser pulses illuminating the whole condensate can implement spin rotations similar to the trapped ion case. However, the strength of the S_z^2 interaction is much smaller than in the trapped ion case, $\chi = 0.3 \sim 0.5$ Hz as reported in [8,9]. The coherence time for the spinor BEC is also shorter. Hence we typically need to apply the compromised scheme, using the actual coherence time and interaction strength of the system as input parameters.

In summary, we have proposed a method based on optimization to significantly enhance spin squeezing using the one-axis twisting Hamiltonian. To achieve significant improvement in spin squeezing, we need to apply only one or two global rotation pulses at an appropriate evolution time and with optimized rotation angles. Using two pulses, the final squeezing is very close to the Heisenberg limit already. Compared to the previous proposal [12], apart from requiring a simpler pulse sequence, the major advantage of this method is the robustness to control noise due to the very small number of coherent pulses used. A scheme involving a large number of pulses usually suffers the accumulation and amplification of control errors in each pulse and tolerates only a very small technical noise. The major drawback that limits the applicability of our proposal is the longer evolution time compared to that in [12], although still being faster than adiabatic preparation. We believe this proposal can be readily applied in certain experimental systems where coherence time is not the bottleneck, without significant modification of the setup. Combining our proposal with a continuous-wave form optimization and extending our results to larger spin particles may be interesting future directions.

Note added. Recently, we became aware of a previous work [17] where a similar Hamiltonian was considered and optimal control techniques were used to obtain a continuous-wave form of effective magnetic field for squeezing a collection of $F = 3$ spins.

This work was supported by the NBRPC (973 Program) 2011CBA00300 (2011CBA00302), the IARPA MUSIQC program, the ARO and the AFOSR MURI programs, and the DARPA OLE program.

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