A Probabilistic Approach to Time Delay Estimation

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***Abstract*—Time delay estimation is a very general problem with wide range of applications. When noisy repetitive signals are observed, the noise cancellation is achieved by averaging perfectly aligned signals. A time delay estimator is developed for determining time delay between signals received on different trials in the presence of uncorrelated noise. The estimator is based on a probabilistic generative model for delayed signals, and tries to find the delay and the source signal simultaneously so that maximum likelihood is achieved. An iterative method based on the Expectation-Maximization algorithm is used for finding maximum likelihood estimate of parameters. The estimator has been tested on three types of synthetic signals. The result shows that it can tolerate 5 to 10dB more noise while achieving the same performance as cross-correlation estimator.**

I. Introduction

Recurrent signals play an important part in many signal processing applications. For example, the estimation of a transient signal usually needs the average of a lot of identically repeated experiments to enhance the signal to noise ratio (SNR) in the physical chemistry field. Biomedical signals such as electrocardiograms or evoked potentials are also good examples of recurrent signals.

This recurrence is used in the signal averaging process in order to improve the SNR or to cancel random artefacts. This averaging process is efficient when the recurrent signals are perfectly aligned. The effect of the misalignment can be approximated within the signal convolution with the low-pass filter whose impulse response is the probability density function of this misalignment. This filter will reduce the SNR improvement. The loss of the high frequency information can also reduce the relevance of the mean signal.

Time Delay Estimation (TDE) has been intensively studied in the field of signal processing and system control [1]–[6]. Knapp and Carter [3] used a pair of receiver prefilters and a cross-correlator to find the relative time delay between two input sequences. Carter [4] extended this approach to multiple input sequences. Ljung [7] formulated this problem as optimizing the response of a parameterized system together with a time delay. Ran [2] discussed the time delay estimation methods using fractional fourier transform. Most of these works require some kind of prior knowledge of the source signals, which is difficult to obtain in some circumstances. This paper focuses on the situation where no such prior knowledge is available, and develops a method to find the relative time delay between data sequences.

In this paper, the problem is formulated as finding parameters to maximize likelihood in a probabilistic model for time delay estimation. An iterative algorithm is developed for joint estimation of time delay and source signal based on the Expectation-Maximization (EM) algorithm. The idea of this algorithm is to calculate the conditional probability distribution of the time delay based on current estimate of the source signal, and then use it to better estimate the source signal. EM algorithm [8] is a general method for solving maximum likelihood estimation problems, and has been successfully applied to estimate the time delay in multipath scenario [9].

This paper is organized as follows. The formulation of the problem and the probabilistic model is presented in section II. Our algorithm is then introduced in section III. After that some experiment results are carried out to demonstrate capabilities and some other aspects of the algorithm in section IV.

II. Probabilistic Model of Time Delay

Suppose that there are M channels and N trials in an experiment. For each channel i there is a source signal S_i related to this channel. For each trial j there is an inherent time delay T_j on this trial. For each pair of channel i and trial j, the signal received by channel i on trial j is a delayed signal with some noise:

$$X_{ij}(t) = S_i(t - T_j) + n_{ij}(t)$$

With these $M \times N$ signal sequences X_{ij} as data, the problem is to estimate the inherent time delay T_j of each trial and also try to retrieve the source signals S_i for each channel as well. It is assumed that the noises are independent, additive gaussian noises with zero mean and unknown variance. It is also assumed that no prior knowledge about the source signals is available. Finally for simplicity, the sampling rate is assumed to be high enough that all time delay variables can be approximately treated as discrete variables.

A probabilistic model can be used to describe the signal generation process, as shown in Fig. 1.Here S_i denotes the source signal on channel i. Since there is no prior knowledge about S_i, it is uniformly distributed among all possible signals. $p_j(\cdot)$ is prior distribution of the time delay T_j on trial j, it could be either uniform or non-uniform, based on the prior knowledge about the distribution of time delay. σ_j^2 denotes the variance of gaussian noise on trial j, it is a fixed but unknown

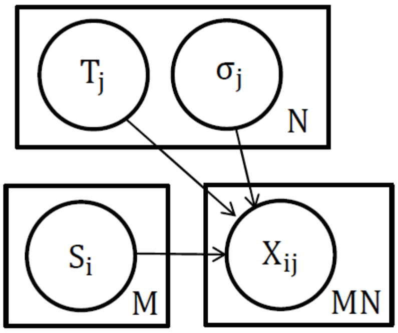

Fig. 1. The Graphical Model

parameter. The distribution of each parameter is defined as follow:

- $S_i \sim Uniform(\cdot)$
- $T_j \sim p_j(\cdot)$
- $X_{ij} \sim \mathcal{N}(S_i(t-T_j), \sigma_j^2 I)$

III. Maximum Likelihood Estimation via the EM Algorithm

The EM algorithm, introduced by Dempster et al. [8], is an iterative method for finding maximum likelihood or maximum a posteriori (MAP) estimates of parameters in probabilistic models, where the model depends on unobserved latent variables. EM algorithm generally alternates between two steps: expectation, which calculates the expected log-likelihood using the current estimate of the parameters; and maximization, which maximize the expected log-likelihood calculated in the E step.

We utilize the EM algorithm to find MAP estimates of parameters. Let $\mathcal{L}(\mathbf{S}, \mathbf{T})$ denotes the log-likelihood of the model. It can be calculated as follow:

$$\begin{aligned}\mathcal{L}(\mathbf{S}, \mathbf{T}) = \sum_j \log p_j(T_j) - \sum_{i,j,t} (\log \sqrt{2\pi}\sigma_j \\ + \frac{(X_{ij}(t) - S_i(t-T_j))^2}{2\sigma_j^2})\end{aligned} \tag{1}$$

In E step, the algorithm needs to calculate the posterior distribution of the time delay $\mathbf{T}$, then use this posterior distribution to calculate the expected log-likelihood. The conditional probability distribution of T_j on previous estimate of parameters $\mathbf{S}^{(i)}$ is first calculated, using the fact that the conditional probability of T_j is proportional to the complete likelihood:

$$P(T_j|\mathbf{S}^{(i)}) = \frac{P(T_j, \mathbf{S}^{(i)})}{\sum_k P(T_j = k, \mathbf{S}^{(i)})} \tag{2}$$

Where,

$$\begin{aligned}P(T_j, \mathbf{S}) = p_j(T_j) \exp(-\sum_{i,t} (\log \sqrt{2\pi}\sigma_j \\ + \frac{(X_{ij}(t) - S_i(t-T_j))^2}{2\sigma_j^2}))\end{aligned} \tag{3}$$

The expected log-likelihood is then calculated as follow(using the fact that T_j are independent conditioned on $\mathbf{S}$):

$$\begin{aligned}\mathbf{E_T}(\mathcal{L}(\mathbf{S}, \mathbf{T})) &= \sum_{\mathbf{T}} \mathcal{L}(\mathbf{S}, \mathbf{T}) P(\mathbf{T}|\mathbf{S}^{(i)}) \\ &= \sum_{\mathbf{T}} \mathcal{L}(\mathbf{S}, \mathbf{T}) \prod_j P(T_j|\mathbf{S}^{(i)})\end{aligned} \tag{4}$$

In M step, the expected log-likelihood is maximized towards $\mathbf{S}$. Substituting Eq. 2 and Eq. 3 into Eq. 4:

$$\mathbf{E_T}(\mathcal{L}(\mathbf{S}, \mathbf{T})) = \sum_j \sum_{T_j} P(T_j|\mathbf{S}^{(i)}) \mathcal{L}(T_j, \mathbf{S}) \tag{5}$$

Where $\mathcal{L}(T_j, \mathbf{S})$ is the log-likelihood of T_j:

$$\begin{aligned}\mathcal{L}(T_j, \mathbf{S}) = \log p_j(T_j) - \sum_{i,t} (\log \sqrt{2\pi}\sigma_j \\ + \frac{(X_{ij}(t) - S_i(t-T_j))^2}{2\sigma_j^2})\end{aligned}$$

The parameter $\mathbf{S}$ used in next iteration is found by maximizing the expected log-likelihood:

$$\mathbf{S}^{(i+1)} = \sup_{\mathbf{S}} \mathbf{E_T}(\mathcal{L}(\mathbf{S}, \mathbf{T}))$$

In order to find $\mathbf{S}^{(i+1)}$, we calculate the derivative of the expected log-likelihood on S_i, and set it to zero:

$$\frac{\partial \mathbf{E_T}(\mathcal{L})}{\partial S_i} = \sum_j \sum_{T_j} P(T_j|\mathbf{S}^{(i)}) \frac{X_{ij}(t) - S_i(t-T_j)}{\sigma_j^2} = 0$$

Then S_i can be calculated using the following formula:

$$S_i(t) = \frac{1}{\sum_j \frac{1}{\sigma_j^2}} \sum_j \frac{1}{\sigma_j^2} \sum_{T_j} P(T_j|\mathbf{S}^{(i)}) X_{ij}(t+T_j) \tag{6}$$

The outline of algorithm is given in pseudocode as follow:

```
i ← 0
Randomly initialize S^(0)
repeat
    for all trial j and all possible value k do
        Calculate P(T_j = k|S^(i)) using equation (2)
    end for
    Calculate S^(i+1) using equation (6)
    i ← i + 1
until ||S^(i) − S^(i−1)|| < ε
for all trial j do
    T_j ← sup_k P(T_j = k|S^(i))
end for
```

The noise variance σ_j^2 is usually pre-estimated manually and then fixed during the algorithm. It can be estimated before the experiment using the data segment with no signal. Although it can be viewed as explicit parameter and optimized by EM algorithm, introducing this parameter does not improve the performance(see section IV for more details), and may harm the efficiency of the algorithm, also potentially cause the problem of overfitting.

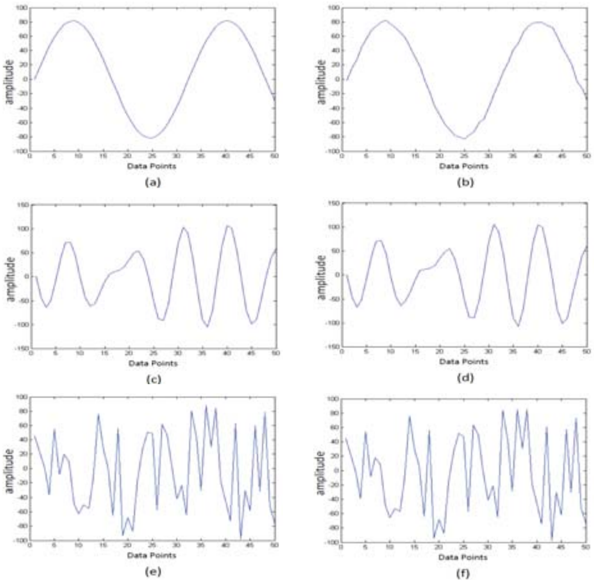

Fig. 2. Source signal(Left) and retrieved signal(Right). Top: single-frequency signal, Middle: signal with limited bandwidth, Bottom: random signal. In these experiments SNR = 10dB and number of trials is 100.

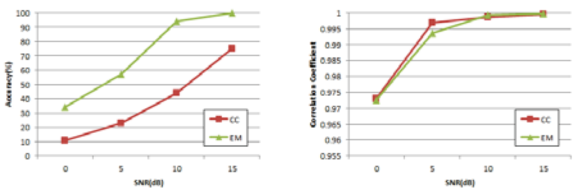

Fig. 3. The comparison between cross-correlation(CC) method and EM method on single frequency signals(computer simulation), the left figure shows the time delay estimation accuracy on different SNR scenarios, the right figure shows the correlation coefficient of the source signal and retrieved signal

IV. Experiment Result

In order to demonstrate our probabilistic approach's capabilities, our work was compared with conventional cross-correlation based algorithm on synthetic signals. Three different types of signals were synthesized with uniformly random time delay and white gaussian noise. Fig. 2 shows the signal used to generate data and the signal retrieved by our algorithm. Fig. 3 shows the result on single frequency signal. Fig. 4 shows the result on signal with limited bandwidth. Fig. 5 shows the result on completely random signal. In all scenarios, our approach consistently outperformed the cross-correlation algorithm.

We also studied the effect of parameters in the algorithm. As shown in Fig. 6, the accuracy of time delay estimation increases as the number of channels grows. Fig. 7 shows that the retrieved signal gets closer to the original source signal if we have more trials.

Since in our method, the parameter σ_j, which denotes the noise variance on each trial, is manually estimated and fixed in the algorithm. Inaccurate estimation of σ_j may affect the performance of our algorithm. We studied the effect of inaccurate estimation. Fig. 8 shows that underestimating the noise variance, as long as the estimated variance is not far apart from the true value, will not harm the performance. However, overestimating the noise variance may greatly reduce the performance.

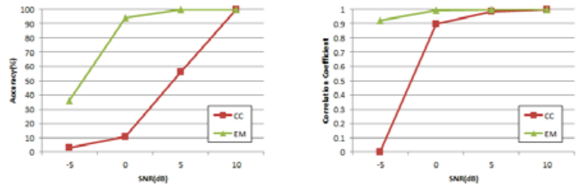

Fig. 4. The comparison between cross-correlation(CC) method and EM method on signals with limited bandwidth(computer simulation), the left figure shows the time delay estimation accuracy on different SNR scenarios, the right figure shows the correlation coefficient of the source signal and retrieved signal

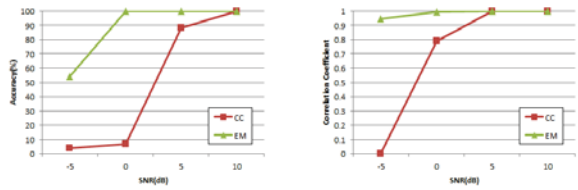

Fig. 5. The comparison between cross-correlation(CC) method and EM method on random signals(computer simulation), the left figure shows the time delay estimation accuracy on different SNR scenarios, the right figure shows the correlation coefficient of the source signal and retrieved signal

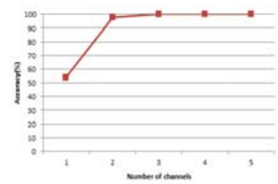

Fig. 6. The relationship between the number of channels and the accuracy of time delay estimation

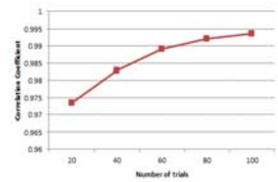

Fig. 7. The relationship between the number of trials and the correlation coefficient between retrieved signal and source signal

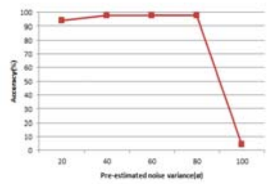

Fig. 8. The accuracy of EM Algorithm with different pre-estimated value of σ(the standard deviation of noise), the true value of σ is 80

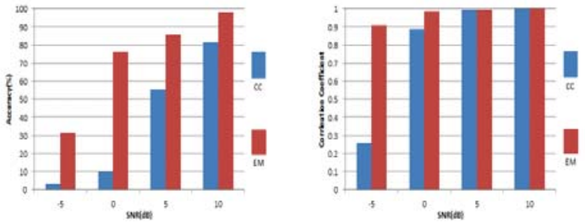

Fig. 9. The overall comparison between cross-correlation(CC) method and EM method(computer simulation), the left figure shows the average time delay estimation accuracy on different SNR scenarios, the right figure shows the average correlation coefficient of the source signal and retrieved signal

Fig. 9 shows the overall comparison between EM and cross-correlation method. It shows that our method can tolerate 5 to 10dB more noise while achieving the same performance as cross-correlation method.

V. Discussion

Our method usually took about 20 to 50 iterations to converge. The convergence rate depends on both input SNR and the initial value. A good initial value can greatly speed up the convergence rate, and could be manually set according to the prior knowledge on source signals, if any.

Fast Fourier Transform, a common method to speed up the cross-correlation process, may also be used in our algorithm. Equation (3) and (6), the most time-consuming steps in our methods, both involves the convolution of two signals, which can be efficiently solved by FFT.

Our method, as well as most TDE methods, assumed additive white noise. So it would be an interesting study to consider non-white noises, such as time-dependent noises.

In many applications, the time delay are caused by the distance between the signal source and sensors, which would also weaken the amplitude of the signals. Our work didn't take this factor into account, and may possibly reduce the accuracy of time delay estimation.

Future work will consider the extension of the model to incorporate the prior knowledge of source signals, such as parameterized family of signals, and deal with the situation where the amplitude of signals are no longer invariant among trials.

VI. Conclusion

Conventional cross-correlation algorithm could not tolerant large noises. We have developed a new time delay estimation method based on statistical framework, which aims to find the maximum likelihood estimation of parameters(time delay and source signals) in a probabilistic, using the expectation-maximization algorithm. The utility of our method has been demonstrated on three different kinds of synthetic signal-s, where it outperformed conventional cross-correlation approach, achieving the same performance while tolerating 5 to 10dB more noise.

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