Scalable Implementation of Boson Sampling with Trapped Ions

C. Shen,* Z. Zhang, and L.-M. Duan
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA and Center for Quantum Information, IIIS, Tsinghua University, Beijing 100084, China
(Received 17 October 2013; published 6 February 2014)

Boson sampling solves a classically intractable problem by sampling from a probability distribution given by matrix permanents. We propose a scalable implementation of boson sampling using local transverse phonon modes of trapped ions to encode the bosons. The proposed scheme allows deterministic preparation and high-efficiency readout of the bosons in the Fock states and universal mode mixing. With the state-of-the-art trapped ion technology, it is feasible to realize boson sampling with tens of bosons by this scheme, which would outperform the most powerful classical computers and constitute an effective disproof of the famous extended Church-Turing thesis.

DOI: 10.1103/PhysRevLett.112.050504 PACS numbers: 03.67.Ac, 37.10.Ty, 89.70.Eg

What is the ultimate computational power of physical devices? That is a deep question of great importance for both physics and computer science. The famous extended Church-Turing thesis (ECTT) postulates that a (classical) probabilistic Turing machine can efficiently simulate the computational power of any physical devices ("efficiently" here means with a polynomial overhead) [1]. The recent development in quantum computation brings doubt to the ECTT with the discovery of superfast quantum algorithms. The most well-known example is Shor's algorithm to factorize a large number in polynomial time with a quantum computer [2]. Classically, whether factoring is hard is not settled (a "hard" problem means its solution requires exponential time). No efficient classical algorithm has been found yet to solve factoring, but it would not be very surprising if one finds one, as this will not induce dramatic change to the computational complexity theory.

Recently, Ref. [1] introduced another problem, called boson sampling, which is hard for classical computers but can be solved efficiently with a quantum machine. Boson sampling is defined as a problem to predict the probabilities of the measurement outcomes in the Fock basis for M bosonic modes, which start in definite Fock states and undergo a series of mode mixing defined, in general, by a unitary matrix. By definition, this problem can be efficiently solved with a quantum machine, but classically its solution requires sampling of a probability distribution given by matrix permanents with an exponentially large number of possible outcomes. Computation of the matrix permanent is known to be #P hard (much harder than the more well-known class of the NP-hard problems) [3]. Reference [1] rigorously proved that boson sampling is classically intractable unless the so-called polynomial hierarchy in the computational complexity theory collapses, which is believed to be extremely unlikely. In this sense, compared with the factoring problem, although boson sampling has no immediate practical applications, it is a problem much harder for classical computers to solve. A demonstration of boson sampling with a quantum machine thus constitutes an effective disproof of the famous ECTT. Because of this far-reaching theoretical implication, experimental demonstration of the boson sampling has raised strong interest recently. Several publications this year have reported proof-of-principle demonstrations of the boson sampling with up to three photons [4–7]. The key challenge for the next-step experiments is to scale up the number of bosons. The demonstration using photons based on the spontaneous parametric down-conversion source has difficulty in terms of scalability [4–7]. The success probability decreases very rapidly with the number of photons due to the probabilistic nature of the single-photon source and the significant photon loss caused by the detector and the coupling inefficiencies. This, in practice, limits the number of bosons below 10, which is still within the simulation range of classical computers.

In this Letter, we propose a scalable scheme to realize boson sampling using the transverse phonon modes of trapped ions. Compared with the implementation using photons, this scheme has the following desirable features: First, the Fock states of the phonons can be prepared in a deterministic fashion and there is no limitation to the number of bosons that one can realize with this system. We encode the bosons using the local transverse phonon modes [8], and the state initialization can be done through simple Doppler cooling and one step of the sideband cooling that applies to any number of ions. Second, we find a technique to do projective detection of the phonon numbers for all the ions through sequential spin quantum jump measurements. This gives an implementation of number-resolving phonon detectors with near perfect efficiency, much higher than the efficiency of typical single-photon detectors. Finally, we prove that universal coherent mixing of different phonon modes can be achieved through a combination of the inherent

Coulomb interaction and simple laser-induced phase shifts of the ions. Through this scheme, it is feasible to realize boson sampling for tens of phonons with the state-of-the-art trapped ion technology. This scale has gone beyond the simulation capability of any classical computers and corresponds to the most interesting experimental region for a test of the ECTT [1,9].

The problem of boson sampling is defined as follows: we have M input bosonic modes a_i $(i = 1, 2, \ldots, M)$, which undergo coherent mode mixing described in general by a unitary matrix Λ, with the output modes given by $b_i = \sum_j^M \Lambda_{ij} a_j$. The input modes are prepared in a Fock (number) state $|\mathbf{T}\rangle = |t_1, t_2, \ldots, t_M\rangle$, where t_i is an integer denoting the occupation number of the mode a_i. We measure the output modes b_i in the Fock basis, and the probability to get the outcome $|\mathbf{S}\rangle = |s_1, s_2, \ldots, s_M\rangle$ is given by [1,10]

$$P(\mathbf{S}|\mathbf{T}) = \frac{|\mathrm{Per}(\Lambda^{(\mathbf{S},\mathbf{T})})|^2}{\prod_{j=1}^M s_j! \prod_i^M t_i!}, \tag{1}$$

where $\mathrm{Per}(\cdot)$ denotes the matrix permanent and $\Lambda^{(\mathbf{S},\mathbf{T})}$ is a submatrix of Λ formed by taking s_j copies of the jth column and t_i copies of the ith row of the matrix Λ. Since the total number of bosons is conserved $N = \sum_i^M a_i^\dagger a_i = \sum_j^M b_j^\dagger b_j$, the submatrix $\Lambda^{(\mathbf{S},\mathbf{T})}$ has dimension $N \times N$. Because of the hardness to calculate the matrix permanent, it becomes impossible to sample the probability distribution $P(\mathbf{S}|\mathbf{T})$ with any classical computer when the number of bosons N increases beyond 20–30. An experimental demonstration of a quantum machine that can successfully perform this job therefore provides strong evidence against the ECTT.

To realize boson sampling with trapped ions, we consider a chain of ions in a linear Paul trap with the transverse trapping frequency ω_x significantly larger than the axial one ω_z. The bosons are represented by the local transverse phonon modes a_i associated with each ion i $(i = 1, 2, \ldots, M)$, all with the oscillation frequency ω_x. The Coulomb interaction between the ions introduces a small perturbation to the oscillation frequency of the local phonon modes, with the interaction Hamiltonian described by [11]

$$H_c = \sum_{1 \le i < j \le M} \hbar t_{i,j} (a_i^\dagger a_j + a_i a_j^\dagger), \tag{2}$$

where the hopping rates $t_{i,j} = t_0/|z_{i0} - z_{j0}|^3$ and $t_0 = e^2/(8\pi\epsilon_0 m \omega_x)$. Here, z_{i0} denotes the axial equilibrium position of the ith ion with mass m and charge e. The Hamiltonian (2) is valid under the condition $t_{i,j} \ll \omega_x$, which is always satisfied for the parameters considered in this Letter. To make the scheme more scalable and eliminate the challenging requirement of resolving phonon sidebands for a large ion chain, we use the local transverse phonon modes to represent the target bosons instead of the conventional normal modes.

To initialize the local phonon modes a_i to the desired Fock states, first we cool them to the ground state by laser cooling. The routine Doppler cooling achieves a temperature $T_D \sim \hbar\Gamma/(2k_B)$ (Γ is the natural bandwidth of the excited state and k_B is the Boltzmann constant), with the corresponding thermal phonon number $\bar{n}_x = k_B T_D/\hbar\omega_z \sim \Gamma/(2\omega_x)$, which is about 1–2 under typical values of $\omega_x \approx 2\pi \times (5 - 10)$ MHz and $\Gamma \approx 2\pi \times 20$ MHz. The sideband cooling can further push the transverse modes to the ground state with $\bar{n}_x \approx 0$ [12]. For the axial modes, we only require their thermal motion to be much less than the ion spacing, which is satisfied already under routine Doppler cooling. As all the local transverse modes have the same frequency (with $t_{i,j} \ll \omega_x$), we only need to apply one step of the sideband cooling independent of the number of ions, with the laser detuning set at $-\omega_x$. The off-resonant process in the sideband cooling limits $\bar{n}_x \sim \gamma/\omega_x$, where γ is the rate of the sideband cooling which needs to be comparable with the phonon hopping rate $t_{i,i+1}$. For a harmonic trap, we take $l_0 = [e^2/(4\pi\epsilon_0 m \omega_z^2)]^{1/3}$ as the length unit so that the ion spacings in this unit take universal dimensionless values (of the order of 1) independent of the ion species and the trap frequency [13]. The hopping rate $t_{i,i+1} \sim t_0/l_0^3 = \omega_z^2/(2\omega_x)$ and the thermal phonon number after the sideband cooling $\bar{n}_x \sim t_{i,i+1}/\omega_x \sim \omega_z^2/(2\omega_x^2) < 10^{-2}$ with a typical $\omega_z \approx 2\pi \times (0.3–1)$ MHz. After cooling of all the transverse modes to the ground state, we can then set them to any desired Fock states through a sequence of laser pulses blue detuned at ω_x [14]. Note that the ion spacing is about or larger than 10 μm under our choice of the parameters, and under such a spacing, it is reasonable to assume individual addressing of different ions with focused laser beams. The focused beam can prepare different local modes a_i to different Fock states $|n_i\rangle$. For implementation of the boson sampling, without loss of generality, we actually can choose $n_i = 1$, which requires only one pulse for preparation. To make the phonon hopping negligible during the preparation step, the sideband Rabi frequency Ω needs to be large compared with the hopping rate $t_{i,i+1} \sim \omega_z^2/(2\omega_x) \sim 2\pi \times (10–100)$ kHz, which is easy to satisfy under typical laser power.

After the state initialization, we need to coherently mix different phonon modes. The inherent Coulomb interaction described by the Hamiltonian (1) serves this purpose; however, it is constantly on without a tuning knob and we need to introduce additional control parameters to realize different unitary transformations between the M modes. To achieve this goal, we introduce a simple operation which induces a controllable phase shift for any local phonon mode at any desired time. A laser pulse with duration t_p and detuning δ to the sideband induces an additional Hamiltonian $H_i = \hbar(\Omega_i^2/\delta) a_i^\dagger a_i$ (Ω_i is the sideband Rabi frequency applied to the target ion i), which gives a phase shift $U_{\phi_i} = e^{i\phi_i a_i^\dagger a_i}$ to the mode a_i with $\phi_i = \Omega_i^2 t_p/\delta$. We choose $\Omega_i^2/\delta \gg t_{i,i+1}$ so that the pulse can be considered to be instantaneous over the time scale of phonon tunneling.

The operation U_{ϕ_i} and the Coulomb interaction H_c together are universal in the sense that a combination of them can make any unitary transformation on the M phonon modes represented by the $M \times M$ matrix Λ. Now, we prove this statement. It is known that any unitary transformation Λ on M bosonic modes can be decomposed as a sequence of neighboring beam-splitter types of operations and individual phase shifts [15]. The beam-splitter operation for the modes $(j, j+1)$ is represented by the Hamiltonian $H_{\rm bs}^{(j)} = \hbar t_{j,j+1}(a_j a_{j+1}^\dagger + a_{j+1} a_j^\dagger)$. To realize $H_{\rm bs}^{(j)}$, we just need to cut off all the other interaction terms in the Coulomb Hamiltonian given by Eq. (1) except for a specific pair $(j, j+1)$. This can be achieved through the idea of dynamical decoupling using the fast phase shifts U_{ϕ_i} with $\phi_i = \pi$ [16]. Note that a Hamiltonian term $H_{ij} = \hbar t_{i,j}(a_i a_j^\dagger + a_j a_i^\dagger)$ can be effectively turned off for an evolution time t if we apply an instantaneous π-phase shift $U_{\phi_i=\pi}$ at time $t/2$ to the mode a_i to flip the sign of H_{ij} to $-H_{ij}$ for the second half-period of the evolution. The interaction Hamiltonian H_c has long-range tunneling, but it decays fast with distance d through $1/d^3$ scaling. If we take the first order approximation to keep only the nearest neighbor tunneling, the Hamiltonian has the form $H_{\rm NN} = \sum_{i=1}^{M-1} \hbar t_{i,i+1}(a_i^\dagger a_{i+1} + a_i a_{i+1}^\dagger)$. The Hamiltonian $H_{\rm NN}$ can be used to realize the required coupling $H_{\rm bs}^{(j)}$ for an arbitrary j if we apply π-phase shifts at time $t/2$ to every other mode in the ion chain except for the pair $(j, j+1)$, as illustrated in Fig. 1(a). This kind of decoupling can be extended, and we can simulate the Hamiltonian $H_{\rm NN}$ (and thus $H_{\rm bs}^{(j)}$) with the original long-range Hamiltonian H_c to an arbitrary order of approximation. Suppose we cut the interaction range in H_c to the kth order (i.e., we neglect the terms in H_c that scale as $1/d_{ij}^3$ with $|i-j| > k$); we can shrink the interaction range from k to $k-1$ by applying one step of dynamical decoupling with the pattern of π-phase shifts illustrated in Fig. 1(b). This step can be continued until one reaches $H_{\rm NN}$ through concatenation of

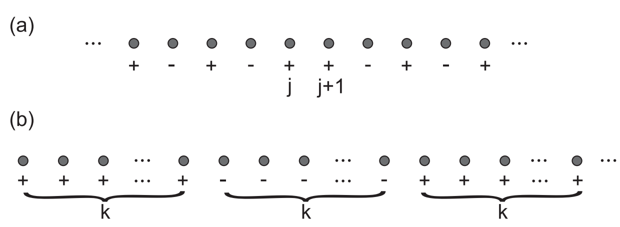

FIG. 1. Control of the tunneling Hamiltonian through the dynamical decoupling. The negative signs in (a) and (b) denote the set of ions to be applied to a π-phase shift at half of the evolution time, while the positive signs denote the ions left intact. (a) The π-phase pattern to turn off other tunneling terms in $H_{\rm NN}$ except for a neighboring pair $j, j+1$. (b) The π-phase pattern to shrink the tunneling range of the Hamiltonian from k to $k-1$.

the dynamical decoupling [16]. This proves that the Coulomb interaction Hamiltonian H_c, together with the phase shifts U_{ϕ_i} on single ions, can realize any beam-splitter operations and thus be universal for the construction of arbitrary unitary operations on the M phononic modes.

We should note that the above proof of universality based on the idea of dynamical decoupling is intuitively straightforward but may be cumbersome to realize in practice. For a real experiment, we suggest using optimization methods to design the control sequence for any given unitary. Alternatively, one can randomly generate a sequence of phase shifters and insert them to the evolution to sample unitaries from the group SU(N) randomly [17]. Because of the universality of the device, we are guaranteed to reach almost any corner of the space SU(N). In both approaches, the truncation of the Coulomb interaction range is not necessary.

The final step of the boson sampling is detection of all the phononic modes in the Fock basis. The conventional method of measuring the phonon number distribution of a single mode by recording the spin oscillation from red or blue sideband pulses is not applicable here, as it cannot measure correlation of different phonon modes in the Fock basis [14]. What we need is a projective measurement of each mode in the Fock basis which gives information of arbitrary high order correlations between different modes. For trapped ions, a projective measurement of its spin (internal) state can be done with a very high efficiency through the quantum jump technique using a cycling transition. However, the spin detection gives only binary measurement outcomes ("dark" or "bright"). We need to figure out a way to perform projective measurements of the Fock states (with multiple possible outcomes) for each phonon mode through the binary spin detection. This is achieved through a consecutive detection scheme with the following steps: (1) First, to illustrate the idea, we consider a single ion with its phonon mode in an arbitrary state $\sum_n c_n |n\rangle$ and its spin prepared in the dark state $|D\rangle$ [see Fig. 2(a)]. (2) Through the well-known adiabatic transition technique [18], we make a complete population transfer from $|n+1\rangle|D\rangle$ to $|n\rangle|B\rangle$ for all the Fock components $|n\rangle$ by chirping the frequency of a laser pulse across the red detuning at $-\omega_x$ [see Fig. 2(b) for the population distribution after this step]. (3) We make a carrier transition $|n\rangle|D\rangle \rightleftarrows |n\rangle|B\rangle$ with a π pulse to flip the dark and the bright states [see Fig. 2(c)]. (4) After this step, we immediately measure the spin state of the ion through the quantum jump detection. With probability $|c_0|^2$, the outcome is bright. In this case, the measurement is finished and we know the phonon is in the $|n=0\rangle$ state. Otherwise, the spin is in the dark state and the phonon is in the $|n \geq 1\rangle$ components [see Fig. 2(d) for the population distribution in this case]. When the spin is in the dark state, the ion does not scatter any photons during the quantum jump measurement. So, its phonon state will not be influenced by this measurement.

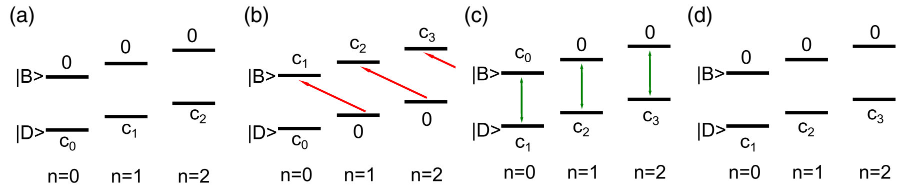

FIG. 2 (color online). A consecutive measurement scheme to perform projective detection of the phonon mode in the Fock basis. (a) The initial state configuration right before the measurement. (b)–(d) The state configuration after the blue sideband transition, the carrier transition, and the quantum jump detection. These three steps are repeated until one finally registers the bright state (see the text for details).

This feature is important for this consecutive measurement scheme. (5) Now, with the phonons in the $|n \geq 1\rangle$ components, we just repeat steps (2)–(4) until finally we get the outcome bright for the spin detection. We conclude that the phonon is in the Fock state $|n = l\rangle$ if the outcome bright occurs (with probability $|c_l|^2$) after l repetitions of the above steps. (6) The above consecutive measurement scheme can be extended straightforwardly to measure M local phonon modes in the Fock basis independently with M ions. The only requirement is that the phonon tunneling between different modes is negligible during the measurement process. The slowest step of the measurement is the quantum jump detection of the ion spin state. Recently, there has been an experimental report of high-efficiency ($> 99\%$) spin state detection within 10 μs detection time [19]. The typical phonon hopping rate between the neighboring ions in our scheme is in the range of $t_{i,i+1} \sim 2\pi \times (10–100)$ kHz, and this hopping rate can be significantly reduced during the detection through either an expansion of the ion chain along the z direction right before the direction by lowering the axial trap frequency or application of a few dynamical decoupling pulses to turn off the neighboring tunneling during the detection. As the hopping scales as $1/d^3$, a moderate increase of the effective distance d will significantly reduce the tunneling and push it below the kHz level. We should note that for the boson sampling algorithm, the output phonon number per mode is typically small (the conventional photon detectors actually can only distinguish 0 and 1 photons), and the number of repetitions in our consecutive measurement scheme is either zero or very few in most cases.

In summary, we have proposed a scalable scheme to realize the boson sampling algorithm by use of the local transverse phonon modes of trapped ions. The scheme allows deterministic preparation and high-efficiency readout of the phonon Fock states and universal manipulation of the phonon modes through a combination of inherent Coulomb interaction and individual phase shifts. Several dozens of ions have been successfully trapped experimentally to form a linear chain, and, in principle, there is no limitation to the number of ions that can be manipulated in a linear Paul trap by use of anharmonic axial potentials [20]. This scheme thus opens the perspective to realize boson sampling for dozens of phonons with the state-of-the-art trapped ion technology, which would beat the capability of any classical computers and give the first serious experimental test of the extended Church-Turing thesis.

We thank Kihwan Kim for discussions. This work was supported by the NBRPC (973 Program) 2011CBA00300 (2011CBA00302), the IARPA MUSIQC Program, the ARO and the AFOSR MURI Programs, and the DARPA OLE Program.

*chaoshen@umich.edu

[1] S. Aaronson and A. Arkhipov, in *Proceedings of the ACM Symposium on Theory of Computing (ACM, New York, 2011), p. 333.*

[2] P. W. Shor, SIAM J. Comput. **26**, 1484 (1997); P. W. Shor, *Algorithms for Quantum Computation: Discrete Logarithms and Factoring, Proceedings of 35th Annual Symposium on Foundations of Computer Science* edited by S. Goldwasser (IEEE Computer Society Press, Washington, DC, USA, 1994).

[3] L. G. Valiant, Theor. Comput. Sci. **8**, 189 (1979).

[4] J. B. Spring *et al.*, Science **339**, 798 (2013).

[5] M. A. Broome, A. Fedrizzi, S. Rahimi-Keshari, J. Dove, S. Aaronson, T. C. Ralph, and A. G. White, Science **339**, 794 (2013).

[6] M. Tillmann, B. Dakić, R. Heilmann, S. Nolte, A. Szameit, and P. Walther, Nat. Photonics **7**, 540 (2013).

[7] A. Crespi, R. Osellame, R. Ramponi, D. J. Brod, E. F. Galvão, N. Spagnolo, C. Vitelli, E. Maiorino, P. Mataloni, and F. Sciarrino, Nat. Photonics **7**, 545 (2013).

[8] S.-L. Zhu, C. Monroe, and L.-M. Duan, Phys. Rev. Lett. **97**, 050505 (2006).

[9] S. Aaronson and A. Arkhipov, arXiv:1309.7460; C. Gogolin, M. Kliesch, L. Aolita, and J. Eisert, arXiv:1306.3995.

[10] S. Scheel, arXiv:quant-ph/0406127.

[11] D. Porras and J. I. Cirac, Phys. Rev. Lett. **93**, 263602 (2004); X.-L. Deng, D. Porras and J. I. Cirac, Phys. Rev. A **77**, 033403 (2008).

[12] F. Diedrich, J. C. Bergquist, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **62**, 403 (1989); C. Monroe, D. Meekhof, B. E. King, S. Jefferts, W. M. Itano, D. J. Wineland, and P. Gould, Phys. Rev. Lett. **75**, 4011 (1995); B. E. King, C. Wood, C. Myatt, Q. Turchette, D. Leibfried, W. M. Itano, C. Monroe, and D. J. Wineland, Phys. Rev. Lett. **81**, 1525 (1998); G. Poulsen, Y. Miroshnychenko, and M. Drewsen, Phys. Rev. A **86**, 051402(R) (2012).

[13] D. F. V. James, Appl. Phys. B **66**, 181 (1998).

[14] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **76**, 1796 (1996).
[15] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).
[16] For a review, see H. K. Ng, D. A. Lidar, and J. Preskill, Phys. Rev. A **84**, 012305 (2011).
[17] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.112.050504 for a classical simulation of the random generation of unitary transformations, together with a discussion of robustness to control errors.
[18] T. Watanabe, S. Nomura, K. Toyoda, and S. Urabe, Phys. Rev. A **84**, 033412 (2011).
[19] R. Noek *et al.*, arXiv:1304.3511.
[20] G.-D. Lin, S.-L. Zhu, R. Islam, K. Kim, M.-S. Chang, S. Korenblit, C. Monroe, and L.-M. Duan, Europhys. Lett. **86**, 60 004 (2009).