

An Improved FPTAS for 0-1 Knapsack

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Abstract

The 0-1 knapsack problem is an important NP-hard problem that admits fully polynomial-time approximation schemes (FPTASs). Previously the fastest FPTAS by Chan (2018) with approximation factor $1 + \varepsilon$ runs in $\tilde{O}(n + (1/\varepsilon)^{12/5})$ time, where \tilde{O} hides polylogarithmic factors. In this paper we present an improved algorithm in $\tilde{O}(n + (1/\varepsilon)^{9/4})$ time, with only a $(1/\varepsilon)^{1/4}$ gap from the quadratic conditional lower bound based on $(\min, +)$ -convolution. Our improvement comes from a multi-level extension of Chan’s number-theoretic construction, and a greedy lemma that reduces unnecessary computation spent on cheap items.

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1 Introduction

1.1 Background

In the *0-1 knapsack* problem, we are given a set I of n items where each item $i \in I$ has weight w_i and profit p_i , and we want to select a subset $J \subseteq I$ such that $\sum_{j \in J} w_j \leq W$ and $\sum_{j \in J} p_j$ is maximized.

The 0-1 knapsack problem is a fundamental optimization problem in computer science and is one of Karp’s 21 NP-complete problems [8]. An important field of study on NP-hard problems is to find efficient approximation algorithms. A $(1 + \varepsilon)$ -approximation algorithm (for a maximization problem) outputs a value SOL such that $\text{SOL} \leq \text{OPT} \leq (1 + \varepsilon) \cdot \text{SOL}$, where OPT denotes the optimal answer. The 0-1 knapsack problem is one of the first problems that were shown to have fully polynomial-time approximation schemes (FPTASs), i.e., algorithms with approximation factor $1 + \varepsilon$ for any given $0 < \varepsilon < 1$ and running time polynomial in both n and $1/\varepsilon$.

There has been a long line of research on finding faster FPTASs for the 0-1 knapsack problem, as summarized in Table 1. The first algorithm with only subcubic dependence on $1/\varepsilon$ was due to Rhee [15]. Very recently, Chan [3] gave an elegant algorithm for the 0-1 knapsack problem in deterministic $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{5/2} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ via simple combination of the SMAWK algorithm [1] and a standard divide-and-conquer technique. The speedup of superpolylogarithmic factor $2^{\Omega(\sqrt{\log(1/\varepsilon)})}$ is due to recent progress on $(\min, +)$ -convolution [2, 16, 4]. Using an elementary number-theoretic lemma, Chan further improved the algorithm to $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{12/5} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time, and obtained two new algorithms running in $\tilde{O}(\frac{1}{\varepsilon} n^{3/2})$ and $O((\frac{1}{\varepsilon})^{4/3} n + (\frac{1}{\varepsilon})^2) / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}$ time respectively, which are faster for small n .

FPTASs on several special cases of 0-1 knapsack are also of interest. For the *unbounded knapsack* problem, where every item has infinitely many copies, Jansen and Kraft [7] obtained an $O(n + (\frac{1}{\varepsilon})^2 \log^3 \frac{1}{\varepsilon})$ -time algorithm; the unbounded version can be reduced to 0-1 knapsack with only a logarithmic blowup in the problem size [5]. For the *subset sum*

■ **Table 1** FPTASs for 0-1 knapsack

$O(n \log n + (\frac{1}{\varepsilon})^4 \log \frac{1}{\varepsilon})$	Ibarra and Kim [6]	1975
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^4)$	Lawler [13]	1979
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^3 \log^2 \frac{1}{\varepsilon})$	Kellerer and Pferschy [11]	2004
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{5/2} \log^3 \frac{1}{\varepsilon})$ (randomized)	Rhee [15]	2015
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{12/5} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$	Chan [3]	2018
$O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$	This work	
$O(\frac{1}{\varepsilon} n^3)$	Textbook algorithm	
$O(\frac{1}{\varepsilon} n^2)$	Lawler [13]	1979
$O((\frac{1}{\varepsilon})^2 n \log \frac{1}{\varepsilon})$	Kellerer and Pferschy [10]	1999
$\tilde{O}(\frac{1}{\varepsilon} n^{3/2})$ (randomized, Las Vegas)	Chan [3]	2018
$O(((\frac{1}{\varepsilon})^{4/3} n + (\frac{1}{\varepsilon})^2) / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$	Chan [3]	2018
$O(((\frac{1}{\varepsilon})^{3/2} n^{3/4} + (\frac{1}{\varepsilon})^2) / 2^{\Omega(\sqrt{\log(1/\varepsilon)})} + n \log \frac{1}{\varepsilon})$	This work	

problem, where every item has $p_i = w_i$, Kellerer et al. [9] obtained an algorithm with $O(\min\{n/\varepsilon, n + (\frac{1}{\varepsilon})^2 \log \frac{1}{\varepsilon}\})$ running time, which will be used in our algorithm as a subroutine. For the *partition* problem, which is a special case of the subset sum problem where $W = \frac{1}{2} \sum_{i \in I} w_i$, Mucha et al. [14] obtained an algorithm with a subquadratic $\tilde{O}(n + (\frac{1}{\varepsilon})^{5/3})$ running time.

On the lower bound side, recent reductions showed by Cygan et al. [5] and Künnemann et al. [12] imply that 0-1 knapsack and unbounded knapsack have no FPTAS in $O((n + \frac{1}{\varepsilon})^{2-\delta})$ time, unless (min, +)-convolution has truly subquadratic algorithm [14]. It remains open whether 0-1 knapsack has a matching upper bound.

1.2 Our results

In this paper we present improved FPTASs for the 0-1 knapsack problem. Our results are summarized in the following two theorems.

► **Theorem 1.** *There is a deterministic $(1 + \varepsilon)$ -approximation algorithm for 0-1 knapsack with running time $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$.*

► **Theorem 2.** *For $n = O(\frac{1}{\varepsilon})$, there is a deterministic $(1 + \varepsilon)$ -approximation algorithm for 0-1 knapsack with running time $O\left(n^{3/4} (\frac{1}{\varepsilon})^{3/2} + (\frac{1}{\varepsilon})^2\right) / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}$.*

Theorem 2 gives the current best time bound for $(\frac{1}{\varepsilon})^{2/3} \ll n \ll \frac{1}{\varepsilon}$, improving upon the previous $O((\frac{1}{\varepsilon})^{4/3} n + (\frac{1}{\varepsilon})^2) / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}$ algorithm by Chan [3]. For $n \ll (\frac{1}{\varepsilon})^{2/3}$, Chan's $\tilde{O}(\frac{1}{\varepsilon} n^{3/2})$ time randomized algorithm [3] remains the fastest.

For $n \gg \frac{1}{\varepsilon}$, Theorem 1 gives a better time bound, improving upon the previous $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{12/5} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ algorithm by Chan [3].

1.3 Outline of our algorithm

We give an informal overview of our improved algorithm for 0-1 knapsack.

Using a known reduction [3], it suffices to solve an easier instance of 0-1 knapsack where profits of all items satisfy $p_i \in [1, 2]$. Here “solving an instance” means approximating the function $f(x) := [\text{maximum total profit of items with at most } x \text{ total weight}]$ for all $x \geq 0$,

rather than for just a single point $x = W$. In this restricted case, simple greedy (sorting according to *unit profits* p_i/w_i) gives an additive error of at most $\max_j p_j = O(1)$, so it suffices to approximate the capped function $\min\{\varepsilon^{-1}, f(x)\}$ with approximation factor $1+O(\varepsilon)$. Chan gave an algorithm that $(1+\varepsilon)$ -approximates $\min\{B, f(x)\}$ in $\tilde{O}(n+\varepsilon^{-2}B^{1/2})$ time (implied by [3, Lemma 7]), which immediately implies an $\tilde{O}(n+\varepsilon^{-5/2})$ time FPTAS by setting $B = \varepsilon^{-1}$.

Greedy. Now we explain how to use a greedy argument (described in detail in Section 5) to improve this algorithm to $\tilde{O}(n+\varepsilon^{-7/3})$ time. We sort all items (with $p_i \in [1, 2]$) in non-increasing order of unit profits p_i/w_i , and divide them into three subsets H, M, L (items with high, medium, low unit profits), where H contains the top $\Theta(\varepsilon^{-1})$ items, and L contains all items i for which $p_i/w_i \leq (1-\varepsilon^{2/3}) \cdot \min_{h \in H}\{p_h/w_h\}$, so there is a gap between the unit profits of H -items and L -items. Intuitively, there are sufficiently many H -items available, so it's not optimal to include too many cheap L -items when the knapsack capacity is not very big. To be more precise, we prove that in any optimal solution we care about (i.e., having optimal total profit smaller than ε^{-1}), the total profit contributed by L -items cannot exceed $O(\varepsilon^{-2/3})$. Hence, for subset L we only need to approximate up to $B = \Theta(\varepsilon^{-2/3})$ in $\tilde{O}(n+\varepsilon^{-2}B^{1/2}) = \tilde{O}(n+\varepsilon^{-7/3})$ time. Subset H has only $O(\varepsilon^{-1})$ items and can be solved using Chan's $\tilde{O}(\varepsilon^{-4/3}n+\varepsilon^{-2})$ algorithm in $\tilde{O}(\varepsilon^{-7/3})$ time. To solve subset M , we round down the profit value p_i for every item $i \in M$, so that the unit profit p_i/w_i becomes a power of $(1+\varepsilon)$. Then there are $O(\varepsilon^{-1/3})$ distinct unit profit values in M . Items with the same unit profit can be solved together using the efficient FPTAS for *subset sum* by Kellerer et al. [9] in $\tilde{O}(n+\varepsilon^{-2})$ time. Finally we merge the results for H, M, L . The total time complexity is $\tilde{O}(n+\varepsilon^{-7/3})$.

Multi-level number-theoretic construction. The above approach invokes two of Chan's algorithms: an $\tilde{O}(n+\varepsilon^{-2}B^{1/2})$ algorithm (useful for small B) and an $\tilde{O}(\varepsilon^{-4/3}n+\varepsilon^{-2})$ algorithm (useful for small n). The key ingredient in these algorithms is a number-theoretic lemma: we can $(1+\varepsilon)$ -approximate all profit values $p_i \in [1, 2]$ by multiples of elements from a small set $\Delta \subset [\delta, 2\delta]$ of size $|\Delta| = \tilde{O}(\frac{\delta}{\varepsilon})$ (small $|\Delta|$ can reduce the additive error incurred from rounding).

Chan obtained an $\tilde{O}(n+\varepsilon^{-2}B^{2/5})$ time algorithm using some additional tricks. First, evenly partition Δ into r subsets $\Delta^{(1)}, \dots, \Delta^{(r)}$, and divide the items into $P = P^{(1)} \cup \dots \cup P^{(r)}$ accordingly, so that profit values from $P^{(j)}$ are approximated by $\Delta^{(j)}$ -multiples. To $(1+\varepsilon)$ -approximate the profit function f_j for each $P^{(j)}$, pick a threshold $B_0 \ll B$, and return the combination of a $(1+\varepsilon)$ -approximation of $\min\{f_j, B_0\}$ and an εB_0 -additive-approximation of $\min\{f_j, B\}$. Since the size of $\Delta^{(j)}$ is only $|\Delta|/r$, the latter function can be approximated faster when $r \gg 1$. Finally, merge f_j over all $1 \leq j \leq r$. By fine-tuning the parameters r, δ, B_1 , the time complexity is improved to $\tilde{O}(n+\varepsilon^{-2}B^{2/5})$.

Our new algorithm extends this technique to *multiple levels*. To $(1+\varepsilon)$ -approximate the profit function f_j for each $P^{(j)}$, we will pick $B_0 \ll B_1 \ll \dots \ll B_{d-1} \ll B_d \approx B$, and compute the εB_{i-1} -additive-approximation of $\min\{f_j, B_i\}$, for all $i \in [d]$. An issue of this multi-level approach is that, different levels have different optimal parameters δ_i and different $\Delta_i^{(1)}, \dots, \Delta_i^{(r)}$, but we have to stick to the same partition of items $P = P^{(1)} \cup \dots \cup P^{(r)}$ over all levels. We overcome this issue by enforcing that $\Delta_i^{(j)}$ at level i must be generated by multiples of elements from $\Delta_{i-1}^{(j)}$ at level $i-1$, so that $P^{(j)}$ can be approximated by $\Delta_i^{(j)}$ -multiples for all levels. To achieve this, we need a multi-level version of the number-theoretic lemma. We will discuss this part in detail in Section 4.

Using this multi-level construction, we obtain algorithms in $\tilde{O}(n + \varepsilon^{-2}B^{1/3})$ time and $\tilde{O}(\varepsilon^{-3/2}n^{3/4} + \varepsilon^{-2})$ time. Combining these improved algorithms with the greedy argument previously described (the threshold which splits M and L needs to be adjusted accordingly), we obtain an algorithm in $\tilde{O}(n + \varepsilon^{-9/4})$ time as claimed in Theorem 1.

2 Preliminaries

Throughout this paper, $\log x$ stands for $\log_2 x$, and $\tilde{O}(f)$ stands for $O(f \cdot \text{poly}(\log(f)))$.

We will describe our algorithm with approximation factor $1 + O(\varepsilon)$, which can be lowered to $1 + \varepsilon$ if we scale down ε by a constant factor at the beginning.

We are only interested in the case where $n = O(\varepsilon^{-4})$. For greater n , Lawler's $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^4)$ algorithm [13] is already near-optimal. Hence we assume $\log n = O(\log \varepsilon^{-1})$.

Assume $0 < w_i \leq W$ and $p_i > 0$ for every item i . Then a trivial lower bound of the maximum total profit is $\max_j p_j$. At the beginning, we discard all items i with $p_i \leq \frac{\varepsilon}{n} \max_j p_j$. Since the total profit of discarded items is at most $\varepsilon \max_j p_j$, the optimal total profit is only reduced by a factor of $1 + O(\varepsilon)$. So we can assume that $\frac{\max_j p_j}{\min_j p_j} \leq \frac{n}{\varepsilon}$.

We adopt Chan's terminology in presenting our algorithm. For a set I of items, define the profit function

$$f_I(x) = \max \left\{ \sum_{i \in J} p_i : \sum_{i \in J} w_i \leq x, \quad J \subseteq I \right\}$$

over non-negative real numbers $x \geq 0$. Note that f_I is a monotone (nondecreasing) step function. The *complexity* of a monotone step function refers to the number of its steps.

We say that a function \tilde{f} approximates a function f with factor $1 + \varepsilon$ if $\tilde{f}(x) \leq f(x) \leq (1 + \varepsilon)\tilde{f}(x)$ for all $x \geq 0$. We say that \tilde{f} approximates f with additive error δ if $\tilde{f}(x) \leq f(x) \leq \tilde{f}(x) + \delta$ for all $x \geq 0$. Our goal is to approximate f_I with factor $1 + O(\varepsilon)$ on the input item set I .

Let I_1, I_2 be two disjoint subsets of items, and $I = I_1 \cup I_2$. We have $f_I = f_{I_1} \oplus f_{I_2}$, where \oplus denotes the $(\max, +)$ -convolution, defined by $(f \oplus g)(x) = \max_{0 \leq x' \leq x} (f(x') + g(x - x'))$. If two non-negative monotone step functions f, g are approximated with factor $1 + \varepsilon$ by functions \tilde{f}, \tilde{g} respectively, then $f \oplus g$ is also approximated by $\tilde{f} \oplus \tilde{g}$ with factor $1 + \varepsilon$.

For a monotone step function f with range¹ contained in $\{0\} \cup [A, B]$, we can obtain a function \tilde{f} with complexity only $O(\varepsilon^{-1} \log(B/A))$ which approximates f with factor $1 + \varepsilon$, by simply rounding f down to powers of $(1 + \varepsilon)$. For our purposes, B/A will be bounded by polynomial of n and $1/\varepsilon$, hence we may always assume that the approximation results are monotone step functions with complexity $\tilde{O}(\varepsilon^{-1})$.

For an item set I with the same profit $p_i = p$ for every item $i \in I$, the step function f_I can be exactly computed in $O(n \log n)$ time by simple greedy: the function values are $0, p, 2p, \dots, np$ and the x -breakpoints are $w_1, w_1 + w_2, \dots, w_1 + \dots + w_n$, after sorting all w_i 's in nondecreasing order. We say that a monotone step function is p -uniform if its function values are of the form $0, p, 2p, \dots, lp$ for some l . We say that a p -uniform function is *pseudo-concave* if the differences of consecutive x -breakpoints are nondecreasing from left to right. In the previous case where all p_i 's are equal to p , f_I is indeed p -uniform and pseudo-concave.

¹ Here *range* refers to the set of possible output values of the function.

3 Chan's techniques

In this section we review several useful lemmas by Chan [3].

3.1 Merging profit functions

► **Lemma 3** ([3, Lemma 2(i)]). *Let f_1, \dots, f_m be monotone step functions with total complexity $O(n)$ and ranges contained in $\{0\} \cup [A, B]$. Then we can compute a monotone step function that approximates $f_1 \oplus \dots \oplus f_m$ with factor $1 + O(\varepsilon)$ and complexity $\tilde{O}(\frac{1}{\varepsilon} \log B/A)$ in $O(n) + \tilde{O}((\frac{1}{\varepsilon})^2 m / 2^{\Omega(\sqrt{\log(1/\varepsilon)})} \log B/A)$ time.*

► **Remark 4.** Lemma 3 is proved using a divide-and-conquer method, which was also used previously in [10]. The speedup of superpolylogarithmic factor $2^{\Omega(\sqrt{\log(1/\varepsilon)})}$ is due to recent progress on $(\min, +)$ -convolution [2, 16, 4].

Lemma 3 enables us to focus on a simpler case where all $p_i \in [1, 2]$. For the general case, we divide the items into $O(\log \frac{\max_j p_j}{\min_j p_j}) = O(\log \varepsilon^{-1})$ groups, each containing items with $p_i \in [2^j, 2^{j+1}]$ for some j (which can be rescaled to $[1, 2]$), and finally merge the profit functions of all groups by using Lemma 3 in $\tilde{O}(n + \varepsilon^{-2})$ time.

Assuming ε^{-1} is an integer and every $p_i \in [1, 2]$, we can round every p_i down to a multiple of ε , introducing only a $1 + \varepsilon$ error factor. Then there are only $m = O(\varepsilon^{-1})$ distinct values of p_i . For every value of p_i , the corresponding profit function f_i is p_i -uniform and pseudo-concave, and can be obtained by simple greedy (as discussed in Section 2).

3.2 Approximating big profit values using greedy

When all p_i 's are small, simple greedy gives good approximation guarantee when the answer is big enough.

► **Lemma 5.** *Suppose $p_i \in [1, 2]$ for all $i \in I$. For $B = \Omega(\varepsilon^{-1})$, the function f_I can be approximated with additive error $O(\varepsilon B)$ in $O(n \log n)$ time.*

Proof. Sort the items in nonincreasing order of unit profit p_i/w_i . Let \tilde{f} be the monotone step function resulting from greedy, with function values $0, p_1, p_1 + p_2, \dots, p_1 + \dots + p_n$ and x -breakpoints $0, w_1, w_1 + w_2, \dots, w_1 + \dots + w_n$. It approximates f_I with an additive error of $\max_i p_i \leq 2 \leq O(\varepsilon B)$ for $B = \Omega(\varepsilon^{-1})$. ◀

When every $p_i \in [1, 2]$, we set $B = \varepsilon^{-1}$ and let f_H denote the result from greedy (Lemma 5). Then we only need to obtain a function f_L which $1 + O(\varepsilon)$ approximates $\min\{f_I, B\}$, and finally return $\max\{f_L, f_H\}$ as a $1 + O(\varepsilon)$ approximation of f_I (because when $f_I(x)$ exceeds B , an additive error $O(\varepsilon B)$ implies $1 + O(\varepsilon)$ approximation factor).

3.3 Approximation using Δ -multiples of small set Δ

For a set Δ of numbers, we say that p is a Δ -multiple if it is a multiple of δ for some $\delta \in \Delta$.

► **Lemma 6** ([3, Lemma 5]). *Let f_1, \dots, f_m be monotone step functions with ranges contained in $[0, B]$. Let $\Delta \subset [\delta, 8\delta]$. If every f_i is p_i -uniform and pseudo-concave for some $p_i \in [1, 2]$ which is a Δ -multiple, then we can compute a monotone step function that approximates $\min\{f_1 \oplus \dots \oplus f_m, B\}$ with additive error $O(|\Delta|\delta)$ in $\tilde{O}(Bm/\delta)$ time.*

► **Remark 7.** An intuition of Lemma 6 is as follows. When p_i 's are exact multiples of δ , standard dynamic programming algorithm maintains a result array of length B/δ , and adding a new f_i to the result can be done in linear time (by exploiting the pseudo-concavity of f_i using the SMAWK algorithm²). Now if the next p_i to be considered is a multiple of $\delta' \neq \delta$, we first have to round down the current results to multiples of δ' , introducing an additive error of δ' . We round our results for $|\Delta| - 1$ times, so smaller $|\Delta|$ implies smaller overall additive error.

► **Corollary 8.** *Let f_1, \dots, f_m be monotone step functions with ranges contained in $[0, B]$. If every f_i is p_i -uniform and pseudo-concave for some $p_i \in [1, 2]$, then we can compute a monotone step function that approximates $\min\{f_1 \oplus \dots \oplus f_m, B\}$ with factor $1 + O(\varepsilon)$ in $\tilde{O}(\varepsilon^{-1}Bm)$ time.*

Proof. Assuming ε^{-1} is an integer, adjust every p_i down to the nearest multiple of ε , and adjust f_i accordingly. This introduces a $1 + \varepsilon$ overall error factor. Then use Lemma 6 with $\delta = \varepsilon, \Delta = \{\varepsilon\}$ to compute the desired function in $\tilde{O}(Bm\varepsilon^{-1})$ time. ◀

4 Extending Chan's number-theoretic construction

As mentioned in Section 1.3, the main results of this section are two approximation algorithms in $\tilde{O}(n + \varepsilon^{-2}B^{1/3})$ and $\tilde{O}(\varepsilon^{-3/2}n^{3/4} + \varepsilon^{-2})$ time respectively (the latter time bound assumes $n = O(1/\varepsilon)$). These results rely on Lemma 6.

4.1 Number-theoretic construction

To avoid checking boundary conditions, from now on we assume $\varepsilon > 0$ is sufficiently small.

Our algorithm extends Chan's technique by using a multi-level structure defined as follows.

► **Definition 9.** *For fixed parameters $\delta_1, \delta_2, \dots, \delta_d$ satisfying condition*

$$\varepsilon \leq \delta_1, \delta_i \leq \delta_{i+1}/2, \delta_d \leq 1/8 \quad (1)$$

and a finite real number set $\Delta_1 \subset [\delta_1, 8\delta_1]$, a set tower $(\Delta_1, \Delta_2, \dots, \Delta_d)$ generated by Δ_1 is defined by recurrence³

$$\Delta_{i+1} := [\delta_{i+1}, 8\delta_{i+1}] \cap \bigcup_{k \in \mathbb{Z}} k\Delta_i, \quad i = 1, 2, \dots, d-1. \quad (2)$$

We refer to Δ_1 as the base set and Δ_d as the top set of this set tower. We also say that the base set Δ_1 generates the top set Δ_d .

If Δ_d^ is the top set generated by a singleton base set $\Delta_1^* = \{x\}$, then for every $y \in \Delta_d^*$ we say x generates y .*

We have the following simple facts about set towers.

► **Proposition 10.** *If x generates y then $x \in \Delta_1$ implies $y \in \Delta_d$. Conversely, for every $y \in \Delta_d$, there exists $x \in \Delta_1$ which generates y , and for every $1 \leq i \leq d$ there exists $z \in \Delta_i$ such that both y/z and z/x are integers.*

² The SMAWK algorithm [1] finds all row-minima in an $n \times n$ matrix A satisfying the *Monge property* $A[i, j] + A[i+1, j+1] \leq A[i, j+1] + A[i+1, j]$ using only $O(n)$ queries.

³ For a number k and a set A of numbers, $kA := \{ka : a \in A\}$.

► **Proposition 11.** For any $1 \leq i \leq d$, $|\Delta_i| \leq 8^{i-1}(\delta_i/\delta_1)|\Delta_1|$, and we can compute Δ_i in $\tilde{O}(8^{i-1}(\delta_i/\delta_1)|\Delta_1|)$ time given Δ_1 as input.

Proof. For $2 \leq i \leq d$, we have

$$|\Delta_i| = \left| [\delta_i, 8\delta_i] \cap \bigcup_{k \in \mathbb{Z}} k\Delta_{i-1} \right| \leq \sum_{x \in \Delta_{i-1}} 8\delta_i/x \leq |\Delta_{i-1}| 8\delta_i/\delta_{i-1}.$$

The proof of size upper bounds follows by induction. Elements of Δ_i can be generated straightforwardly within the time bound. ◀

► **Lemma 12.** Let T_1, T_2, \dots, T_d be positive real numbers satisfying $T_1 \geq 2$ and $T_{i+1} \geq 2T_i$. There exist at least $T_d/(\log T_d)^{O(d)}$ integers t satisfying the following condition: t can be written as a product of integers $t = n_1 n_2 \cdots n_d$, such that $n_1 n_2 \cdots n_i \in (T_i/2, T_i]$ for every $1 \leq i \leq d$.

The proof of Lemma 12 is deferred to Appendix A. Lemma 12 helps us prove the following fact, which is a multi-level extension of [3, Lemma 6].

► **Lemma 13.** For any parameters $\delta_1, \dots, \delta_d$ satisfying condition (1), there exists a base set Δ_1 of size $\frac{\delta_1}{\varepsilon} \cdot (\log \varepsilon^{-1})^{O(d)}$, such that every $p \in [1, 2]$ can be approximated by a Δ_d -multiple with additive error $O(\varepsilon)$, where Δ_d is the top set generated by Δ_1 .

This base set Δ_1 can be constructed in $\tilde{O}(\varepsilon^{-1}\delta_1^{-1})$ time deterministically.

Proof. Let $P = \{1, 1 + \varepsilon, 1 + 2\varepsilon, \dots, 1 + \lfloor \frac{1}{\varepsilon} \rfloor \varepsilon\}$. It suffices to approximate every value $p \in P$ with additive error ε using Δ_d -multiples. For any $p \in P$ and $y \in \Delta_d \subset [\delta_d, 8\delta_d]$, p is approximated with additive error ε by a multiple of y if and only if $y \in \bigcup_{j \in \mathbb{Z}} \left[\frac{p-\varepsilon}{j}, \frac{p}{j} \right]$.

Our constructed base set Δ_1 will satisfy $\Delta_1 \subset [\delta_1, 4\delta_1]$. Suppose integers k_1, k_2, \dots, k_{d-1} satisfy

$$k_1 k_2 \cdots k_{i-1} \in [\delta_i/\delta_1, 2\delta_i/\delta_1], \quad \text{for every } 2 \leq i \leq d. \quad (3)$$

Then by Definition 9, for any $x \in \Delta_1 \subset [\delta_1, 4\delta_1]$, we have $x k_1 k_2 \cdots k_{i-1} \in \Delta_i$ for every $2 \leq i \leq d$.

For any integer j satisfying

$$k_1 k_2 \cdots k_{d-1} j \in [p/(4\delta_1), p/(2\delta_1)], \quad (4)$$

the interval $\left[\frac{p-\varepsilon}{k_1 k_2 \cdots k_{d-1} j}, \frac{p}{k_1 k_2 \cdots k_{d-1} j} \right]$ is contained in $[\delta_1, 4\delta_1]$.

We say an integer K is *good* for p , if K can be expressed as a product of integers $k_1 k_2 \cdots k_{d-1} j$ satisfying conditions (3) and (4). For such K , any $x \in \left[\frac{p-\varepsilon}{K}, \frac{p}{K} \right] \cap \Delta_1$ generates an element $y = x k_1 k_2 \cdots k_{d-1} j \in \Delta_d \cap \left[\frac{p-\varepsilon}{j}, \frac{p}{j} \right]$ such that p can be approximated by a multiple of y with additive error ε .

By Lemma 12, the number of good integers K for p is at least

$$\frac{p/(4\delta_1)}{(\log(p/(4\delta_1)))^{O(d)}} = \Omega\left(\frac{\delta_1^{-1}}{(\log \varepsilon^{-1})^{O(d)}}\right),$$

and at most $p/(2\delta_1) = O(\delta_1^{-1})$, by (4). Using conditions (3) and (4) we can compute all these K 's by simple dynamic programming. We denote the union of their associated intervals by

$$I_p := \bigcup_{K \text{ good for } p} \left[\frac{p-\varepsilon}{K}, \frac{p}{K} \right] \subset [\delta_1, 4\delta_1]. \quad (5)$$

Note that these intervals are disjoint since $p/(K+1) \leq (p-\varepsilon)/K$, so the total length of I_p can be lower-bounded as

$$\lambda(I_p) \geq \frac{\delta_1^{-1}}{(\log \varepsilon^{-1})^{O(d)}} \cdot \frac{p - (p - \varepsilon)}{\max K} \geq \frac{\varepsilon}{(\log \varepsilon^{-1})^{O(d)}}. \quad (6)$$

We have seen that p is approximated by a Δ_d -multiple with additive error ε as long as $\Delta_1 \cap I_p \neq \emptyset$. We compute I_p for every $p \in P$, and use the standard greedy algorithm (for Hitting Set problem) to construct a base set $\Delta_1 \subset [\delta_1, 4\delta_1]$ which intersects with every I_p : in each round we find a point $x \in [\delta_1, 4\delta_1]$ that hits the most I_p 's, include x into Δ_1 , and remove the I_p 's that are hit by x . In each round the number of remaining I_p 's decreases by

$$s := \frac{\min_p \lambda(I_p)}{4\delta_1 - \delta_1} \geq \frac{\varepsilon/\delta_1}{(\log \varepsilon^{-1})^{O(d)}},$$

so the solution size $|\Delta_1|$ is upper-bounded by

$$1 + \log_{1/(1-s)} |P| = O\left(\frac{\log |P|}{s}\right) = \frac{\delta_1}{\varepsilon} (\log \varepsilon^{-1})^{O(d)}.$$

To implement this greedy algorithm, we use standard data structures (for example, segment trees) that support finding x which hits the most intervals, reporting an interval hit by x , removing an interval, all in logarithmic time per operation. The number of operations is bounded by the total number of small intervals, so the running time is at most $\tilde{O}(|P| \cdot \frac{p}{2\delta_1}) = \tilde{O}(\delta_1^{-1} \varepsilon^{-1})$. \blacktriangleleft

The following lemma evenly partitions the base set Δ_1 into r subsets $\Delta_1^{(1)}, \dots, \Delta_1^{(r)}$, and partitions the profit values $P = \{p_1, \dots, p_m\}$ into $P^{(1)} \cup \dots \cup P^{(r)}$, so that $P^{(j)}$ can be approximated by $\Delta_d^{(j)}$ -multiples. An additional requirement is that $P^{(1)}, \dots, P^{(r)}$ should have size $O(|P|/r)$ each.

► Lemma 14. *Let $\delta_1, \dots, \delta_d$ be parameters satisfying condition (1). Let $P = \{p_1, \dots, p_m\} \subset [1, 2]$ with $m = O(\varepsilon^{-1})$. Given a positive integer parameter $r = O(\min\{\frac{\delta_1}{\varepsilon}, m\})$, there exist r base sets $\Delta_1^{(1)}, \Delta_1^{(2)}, \dots, \Delta_1^{(r)}$ each of size $\frac{\delta_1}{\varepsilon r} \cdot (\log \varepsilon^{-1})^{O(d)}$, and a partition of $P = P^{(1)} \cup P^{(2)} \cup \dots \cup P^{(r)}$ each of size $O(m/r)$, such that for every $1 \leq j \leq r$, every $p \in P^{(j)}$ can be approximated by a $\Delta_d^{(j)}$ -multiple with additive error $O(\varepsilon)$, where $\Delta_d^{(j)}$ is the top set generated by $\Delta_1^{(j)}$.*

These r base sets and the partition of P can be computed by a deterministic algorithm in $\tilde{O}(\varepsilon^{-2}/r)$ time.

Proof. First construct the base set Δ_1 of size $\frac{\delta_1}{\varepsilon} (\log \varepsilon^{-1})^{O(d)}$ from Lemma 13 in $\tilde{O}(\delta_1^{-1} \varepsilon^{-1}) = \tilde{O}(\varepsilon^{-2}/r)$ time, and compute the top set Δ_d that it generates. By Proposition 11, $|\Delta_d| \leq 8^{d-1} \frac{\delta_d}{\delta_1} |\Delta_1| \leq \frac{\delta_d}{\varepsilon} (\log \varepsilon^{-1})^{O(d)}$. Generate and sort all Δ_d -multiples in interval $[1, 2]$. For every $p \in P$, use binary search to find the Δ_d -multiple $ky \leq p$ ($y \in \Delta_d$) closest to p , and then add p to the set Q_x , where $x \in \Delta_1$ is an element that generates y . (Q_x is initialized as empty for every $x \in \Delta_1$.) Then remove every x with $Q_x = \emptyset$ from Δ_1 , so that $|\Delta_1| \leq m$, while every $p \in P$ can still be approximated with $O(\varepsilon)$ additive error by a Δ_d -multiple.

Let $D := \max\{r, |\Delta_1|\}$ and let $s := \lceil m/D \rceil$. For every $x \in \Delta_1$ we divide Q_x evenly into small subsets each having size at most s . The total number of these small subsets is

$$\sum_{x \in \Delta_1} \lceil |Q_x|/s \rceil \leq |\Delta_1| + \sum_{x \in \Delta_1} |Q_x|/s = |\Delta_1| + m/s \leq 2D.$$

We merge these small subsets into r groups, each having at most $\lceil 2D/r \rceil$ small subsets. Then, define set $P^{(j)}$ as the union of small subsets from the j -th group, and let base set $\Delta_1^{(j)}$ contain $x \in \Delta_1$ if any of these small subsets comes from Q_x . So $|\Delta_1^{(j)}| \leq \lceil 2D/r \rceil = \frac{2D}{\varepsilon r} (\log \varepsilon^{-1})^{O(d)}$, and $|P^{(j)}| \leq s \cdot \lceil 2D/r \rceil = O(m/D) \cdot O(D/r) = O(m/r)$. \blacktriangleleft

4.2 Approximation using set towers

We first prove a slightly improved version of Corollary 8. The only purpose of this lemma is to get rid of the $(\log \varepsilon^{-1})^{O(\log \log \varepsilon^{-1})}$ factor in the final running time.

► **Lemma 15.** *Let f_1, \dots, f_m be monotone step functions with ranges contained in $[0, B]$ for some $1 \leq B \leq O(\varepsilon^{-1})$. If every f_i is p_i -uniform and pseudo-concave for some $p_i \in [1, 2]$, then we can compute a monotone step function that approximates $\min\{f_1 \oplus \dots \oplus f_m, B\}$ with factor $1 + O(\varepsilon)$ in $\tilde{O}(\varepsilon^{-1}(Bm + \varepsilon^{-1})/B^{0.01})$ time.*

Proof. Using Lemma 13 with parameters $d = 1, \delta_1 = \varepsilon B^{0.01}$, we get $\Delta \subset [\delta_1, 8\delta_1]$ with size $|\Delta| \leq \tilde{O}(\delta_1/\varepsilon) = \tilde{O}(B^{0.01})$, in $\tilde{O}(\varepsilon^{-2}/B^{0.01})$ time. Adjust every p_i down to the nearest Δ -multiple, and adjust f_i accordingly. This introduces at most $1 + O(\varepsilon)$ error factor. Then use Lemma 6 to compute a monotone step function f_H that approximates $\min\{f_1 \oplus \dots \oplus f_m, B\}$ with additive error $e = O(|\Delta|\delta_1) = \tilde{O}(\varepsilon B^{0.02})$, in $\tilde{O}(B^{0.99}m\varepsilon^{-1})$ time.

Let $B_L := e/\varepsilon$, and use Corollary 8 to compute a monotone step function f_L that approximates $\min\{f_1 \oplus \dots \oplus f_m, B_L\}$ with factor $1 + O(\varepsilon)$ in only $\tilde{O}(B_L m \varepsilon^{-1}) = \tilde{O}(B^{0.02} m \varepsilon^{-1})$ time.

Since f_H approximates $\min\{f_1 \oplus \dots \oplus f_m, B\}$ with additive error εB_L , $\max\{f_L, f_H\}$ is a $1 + O(\varepsilon)$ approximation of $\min\{f_1 \oplus \dots \oplus f_m, B\}$. \blacktriangleleft

Now we can approximate the profit function $\min\{B, \bigoplus_{p_k \in P^{(j)}} f_k\}$ for each group $P^{(j)}$, using the multi-level approach described in Section 1.3.

► **Lemma 16.** *Let f_1, \dots, f_m be given monotone step functions with ranges contained in $[0, B]$, and every f_k is p_k -uniform and pseudo-concave for some $p_k \in [1, 2]$. Assume $m = O(\varepsilon^{-1})$, $\Omega(\varepsilon^{-0.01}) \leq B \leq O(\varepsilon^{-1})$. Let r be a given positive integer parameter with $r = O(m), r = o(B)$.*

We can set $d = O(\log \log \varepsilon^{-1})$ and choose d parameters $\delta_1, \dots, \delta_d$ satisfying condition (1), such that the following statement holds:

Let $P^{(1)} \cup \dots \cup P^{(r)}$ be the partition of set $P = \{p_1, \dots, p_m\}$ returned by the algorithm in Lemma 14 with the above parameters. Then for any $1 \leq j \leq r$, using the base set $\Delta_1^{(j)}$ from Lemma 14, we can compute a monotone step function that approximates $\min\{B, \bigoplus_{p_k \in P^{(j)}} f_k\}$ with factor $1 + O(\varepsilon)$, in $(\varepsilon^{-2}/r^{0.01} + m\varepsilon^{-1}B^{1/2}/r^{3/2})(\log \varepsilon^{-1})^{O(d)}$ time.

Proof. We can assume $B \geq 4r$, and define d to be the unique positive integer such that

$$2^{2^{d-1}} \leq \frac{\sqrt{B}}{\sqrt{r}} < 2^{2^d} = 4^{2^{d-1}}.$$

Then $d = O(\log \log \frac{\sqrt{B}}{\sqrt{r}}) = O(\log \log \varepsilon^{-1})$. Pick $\alpha \in [2, 4)$ such that

$$\alpha^{2^{d-1}} = \frac{\sqrt{B}}{\sqrt{r}}. \tag{7}$$

Define

$$\delta_i := \varepsilon \sqrt{Br} / \alpha^{2^{d-i}}, \quad 0 \leq i \leq d. \tag{8}$$

Then

$$\delta_d = \frac{\varepsilon\sqrt{Br}}{\alpha}, \delta_1 = \varepsilon r \quad (9)$$

Note that $\delta_d = \varepsilon\sqrt{B} \cdot O(\sqrt{r}) = \varepsilon\sqrt{B} \cdot o(\sqrt{B}) = \varepsilon \cdot o(B) = o(1)$. Hence the parameters $\delta_1, \dots, \delta_d$ satisfy condition (1) for sufficiently small ε .

The base set $\Delta_1^{(j)}$ from Lemma 14 has size $\frac{\delta_1}{\varepsilon r}(\log \varepsilon^{-1})^{O(d)}$. We compute the generated set tower $\Delta_1^{(j)}, \Delta_2^{(j)}, \dots, \Delta_d^{(j)}$. By Proposition 11, $|\Delta_i^{(j)}| \leq \frac{\delta_i}{\varepsilon r}(\log \varepsilon^{-1})^{O(d)}$. Let

$$t := \max \left\{ \alpha, \max_j |\Delta_i^{(j)}| \left/ \frac{\delta_i}{\varepsilon r} \right. \right\} = (\log \varepsilon^{-1})^{O(d)} \quad (10)$$

and define

$$B_i := Bt / \alpha^{2^{d-i}}, \quad 0 \leq i \leq d. \quad (11)$$

Then $B \leq B_d \leq B \cdot (\log \varepsilon^{-1})^{O(d)}$, and it's easy to verify that

$$|\Delta_i^{(j)}| \cdot \delta_i \leq B_{i-1}\varepsilon, \quad (1 \leq i \leq d). \quad (12)$$

For every $1 \leq i \leq d$, adjust every $p_k \in P^{(j)}$ down to the nearest $\Delta_i^{(j)}$ -multiple and adjust f_k accordingly, which introduces a $1 + O(\varepsilon)$ error factor. Then use Lemma 6 to obtain a monotone step function g_i which approximates $\min\{\bigoplus_{p_k \in P^{(j)}} f_k, B_i\}$ with additive error $O(|\Delta_i^{(j)}|\delta_i) = O(\varepsilon B_{i-1})$ in $\tilde{O}(|P^{(j)}|B_i/\delta_i)$ time.

Then we use Lemma 15 to obtain a monotone step function g_0 which approximates $\min\{\bigoplus_{p_k \in P^{(j)}} f_k, B_0\}$ with $1 + O(\varepsilon)$ factor, in $\tilde{O}(\varepsilon^{-1}(|P^{(j)}|B_0 + \varepsilon^{-1})B_0^{-0.01})$ time. Notice that $B_0 = rt$.

Finally, $\max\{g_0, g_1, g_2, \dots, g_d\}$ is a $1 + O(\varepsilon)$ approximation of $\min\{\bigoplus_{p_k \in P^{(j)}} f_k, B_d\}$, where $B_d \geq B$. Overall running time is

$$\begin{aligned} & \tilde{O}(\varepsilon^{-1}(|P^{(j)}|B_0 + \varepsilon^{-1})B_0^{-0.01}) + \sum_{1 \leq j \leq d} \tilde{O}(|P^{(j)}|B_j/\delta_j) \\ &= \tilde{O}\left(\varepsilon^{-1}\left(\frac{m}{r} \cdot (rt) + \varepsilon^{-1}\right)(rt)^{-0.01}\right) + d \cdot \tilde{O}\left(\frac{m}{r}B_d/\delta_d\right) \\ &= (\varepsilon^{-2}/r^{0.01} + m\varepsilon^{-1}B^{1/2}/r^{3/2})(\log \varepsilon^{-1})^{O(d)}. \end{aligned}$$

◀

Now we merge the results from all r groups, and obtain an approximation of the final result $\min\{f_1 \oplus \dots \oplus f_m, B\}$.

► **Lemma 17.** *Let f_1, \dots, f_m be given monotone step functions with ranges contained in $[0, B]$, and every f_k is p_k -uniform and pseudo-concave for some $p_k \in [1, 2]$. Assume $m = O(1/\varepsilon)$, $\Omega(\varepsilon^{-0.01}) \leq B \leq O(\varepsilon^{-1})$. We can approximate $\min\{f_1 \oplus \dots \oplus f_m, B\}$ with factor $1 + O(\varepsilon)$ in $O(\varepsilon^{-2}B^{1/3}/2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time.*

Proof. Assume $m \geq \varepsilon^{-1}$, by adding zero functions which do not change the answer.

Let $r = o(B)$ be a positive integer parameter to be determined later.

Using Lemma 14 and Lemma 16, we can get a partition of $\{p_1, \dots, p_m\} = P^{(1)} \cup \dots \cup P^{(r)}$ and then get an $1 + O(\varepsilon)$ approximation of $\min\{\bigoplus_{p_k \in P^{(j)}} f_k, B\}$ for every $1 \leq j \leq r$, in $r \cdot (\varepsilon^{-2}/r^{0.01} + m\varepsilon^{-1}B^{1/2}/r^{3/2})(\log \varepsilon^{-1})^{O(d)} = (r^{0.99} + \sqrt{B/r})\varepsilon^{-2}(\log \varepsilon^{-1})^{O(\log \log \varepsilon^{-1})}$ overall time.

Then we use Lemma 3 to merge all these r functions in $\tilde{O}((\frac{1}{\varepsilon})^2 r / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time.

Setting $r = B^{1/3} 2^{c\sqrt{\log(1/\varepsilon)}}$, where $c > 0$ is some small enough constant, the total complexity is

$$O(\varepsilon^{-2} B^{1/3} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}).$$

◀

► **Lemma 18.** *Let I be a set of m items with $p_i \in [1, 2]$ for every $i \in I$, where $\Omega(\varepsilon^{-2/3}) \leq m \leq O(\varepsilon^{-1})$. One can approximate f_I with factor $1 + O(\varepsilon)$ in $O(\varepsilon^{-3/2} m^{3/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time.*

Proof. Let f_1, \dots, f_m denote the profit functions of the m items.

Let $r = o(m^{1/2})$ be a positive integer parameter to be determined later. Obtain a partition of $\{p_1, \dots, p_m\} = P^{(1)} \cup \dots \cup P^{(r)}$ using Lemma 14. Let $B := \max_i \sum_{p \in P^{(i)}} p \leq 2 \max_i |P^{(i)}| = \Theta(m/r)$. Then $r = o(B)$. Use Lemma 16 to get an $1 + O(\varepsilon)$ approximation of $\bigoplus_{p_k \in P^{(j)}} f_k = \min\{\bigoplus_{p_k \in P^{(j)}} f_k, B\}$ for every $1 \leq j \leq r$, in $r \cdot (\varepsilon^{-2}/r^{0.01} + m\varepsilon^{-1} B^{1/2}/r^{3/2})(\log \varepsilon^{-1})^{O(d)} = (\varepsilon^{-2} r^{0.99} + m^{3/2} \varepsilon^{-1}/r)(\log \varepsilon^{-1})^{O(\log \log \varepsilon^{-1})}$ overall time.

Then we use Lemma 3 to merge all these r functions in $\tilde{O}((\frac{1}{\varepsilon})^2 r / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time.

Setting $r = m^{3/4} \varepsilon^{1/2} 2^{c\sqrt{\log(1/\varepsilon)}}$, where $c > 0$ is some small enough constant, the total complexity is

$$O(\varepsilon^{-3/2} m^{3/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}).$$

◀

► **Corollary 19** (restated Theorem 2). *For $n = O(\frac{1}{\varepsilon})$, there is a deterministic $(1 + \varepsilon)$ -approximation algorithm for 0-1 knapsack in $O\left((n^{3/4} (\frac{1}{\varepsilon})^{3/2} + (\frac{1}{\varepsilon})^2) / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}\right)$ time.*

Proof. Divide the items into $O(\log \frac{n}{\varepsilon})$ groups, each containing items with $p_i \in [2^j, 2^{j+1}]$ for some j . Use Lemma 18 to solve each group, and merge them using Lemma 3. ◀

5 Main algorithm

5.1 A greedy lemma

Our improved algorithm uses the following lemma, which gives an upper bound on the total profit of cheap items (with low p_i/w_i) in an optimal knapsack solution.

► **Lemma 20.** *Let H, L be two subsets of items with $p_i \in [1, 2]$. Let $W = \sum_{h \in H} w_h$ and $q = \min_{h \in H} \frac{p_h}{w_h}$. Suppose $\max_{l \in L} \frac{p_l}{w_l} \leq q(1 - \alpha)$ for some $0 < \alpha < 1$. Let $f = f_H \oplus f_L$, $\tilde{f} = f_H \oplus \min\{\frac{2}{\alpha}, f_L\}$. Then for every $x \leq W$, $f(x) = \tilde{f}(x)$.*

Proof. By greedy, $f(W) = \sum_{h \in H} p_h = \tilde{f}(W)$ clearly holds. Now consider $0 \leq x < W$. Suppose $f_L(x') + f_H(x - x')$ achieves its maximum value at $x' = w_L$, i.e., $f(x) = f_L(w_L) + f_H(x - w_L)$. It suffices to prove $f_L(w_L) \leq \frac{2}{\alpha}$.

Let $J \subseteq H$ be a subset of items with total weight $w_J \leq x - w_L$ and total profit achieving optimal value $f_H(x - w_L)$. Let $K \subseteq H \setminus J$ be a subset of items with total weight w_K , such that $w_K \leq w_L$, and $w_K + w_i > w_L$ for every remaining item $i \in H \setminus (J \cup K)$. Such K can be constructed by a simple greedy algorithm.

Since $w_J + w_K \leq (x - w_L) + w_L < W = \sum_{h \in H} w_h$, the remaining set $H \setminus (J \cup K)$ contains at least one item h_0 . Hence, $w_L - w_K < w_{h_0} = p_{h_0} / \frac{p_{h_0}}{w_{h_0}} \leq 2/q$, and equivalently $qw_K > qw_L - 2$.

Since $J \cup K$ is a subset of H with total weight bounded by x , we have $f_H(x) \geq \sum_{k \in K} p_k + \sum_{j \in J} p_j$, and thus $f_H(x) - f_H(x - w_L) = f_H(x) - \sum_{j \in J} p_j \geq \sum_{k \in K} p_k \geq qw_K > qw_L - 2$.

Hence $qw_L - 2 < f_H(x) - f_H(x - w_L) \leq f(x) - f_H(x - w_L) = f_L(w_L) \leq q(1 - \alpha)w_L$, which shows that $q\alpha w_L \leq 2$. So $f_L(w_L) \leq q(1 - \alpha)w_L \leq qw_L \leq 2/\alpha$, which concludes the proof. \blacktriangleleft

5.2 FPTAS for Subset Sum

We will use the efficient FPTAS for the subset sum problem by Kellerer et al. [9] as a subroutine in our algorithm.

► **Lemma 21** ([9], implicit). *Let I be a set of n items and W be a number. We can obtain a list S of $O(\frac{1}{\varepsilon})$ numbers in $O(n + (\frac{1}{\varepsilon})^2 \log \frac{1}{\varepsilon})$ time, such that for every $s \leq W$ that is the subset sum $s = \sum_{j \in J} w_j$ of some subset $J \subseteq I$, there exists $s' \in S$ with $s - \varepsilon W \leq s' \leq s$.*

► **Remark 22.** This result wasn't explicitly stated in [9], but can be easily seen from their analysis of the correctness of the FPTAS.

► **Corollary 23.** *Let I be a set of n items with $p_i \in [1, 2]$ and $p_i = w_i$ for every item $i \in I$. We can approximate f_I with factor $1 + O(\varepsilon)$ in $O(n \log n + \varepsilon^{-2} \log \frac{1}{\varepsilon} \log n)$ time.*

Proof. Notice that approximating s with additive error εW implies approximation factor $1 + O(\varepsilon)$ for $W/2 \leq s \leq W$. So we simply apply Lemma 21 with $W = 2^j$ for $0 \leq j \leq 1 + \log n$, and merge all obtained lists. \blacktriangleleft

5.3 Improved algorithm

► **Lemma 24.** *Let I be a set of n items with $p_i \in [1, 2]$ for every $i \in I$. We can approximate f_I with factor $1 + O(\varepsilon)$ in $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time.*

Proof. Let $B = \lceil \varepsilon^{-1} \rceil$ and assume $n \geq B$ (if $n < B$, we can simply apply Lemma 18). By Lemma 5, we can approximate f_I with additive error $O(\varepsilon B)$ in $O(n \log \frac{1}{\varepsilon})$ time, so we only need to approximate $\min\{f_I, B\}$ with factor $1 + O(\varepsilon)$.

We sort the items by their unit profits p_i/w_i . Let set H contain the top B items with the highest unit profits. Define $q = \min_{h \in H} \frac{p_h}{w_h}$, and let M be the set of remaining items i with $q(1 - \alpha) \leq \frac{p_i}{w_i} \leq q$, where parameter $0 < \alpha < 1$ is to be determined later. Let set L contain the remaining items not included in H or M .

Using Lemma 18, we can compute \tilde{f}_H which approximates f_H with factor $1 + O(\varepsilon)$ in time $O(B^{3/4} \varepsilon^{-3/2} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})}) = O(\varepsilon^{-9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$.

Since $\max_{l \in L} \frac{p_l}{w_l} < q(1 - \alpha)$, Lemma 20 states that $f_H \oplus f_L$ and $f_H \oplus \min\{2/\alpha, f_L\}$ agree when $x \leq W_H = \sum_{h \in H} w_h$. Since $(f_H \oplus f_L)(W_H) = \sum_{h \in H} p_h \geq B$, this implies $\min\{B, f_H \oplus f_L\} = \min\{B, f_H \oplus \min\{2/\alpha, f_L\}\}$. For every item $l \in L$, we round down p_l to a power of $1 + \varepsilon$, so that there are only $\log_{1+\varepsilon} 2 = O(\varepsilon^{-1})$ distinct values. This only multiplies the approximation factor by $1 + \varepsilon$. Then we use Lemma 17 to compute an approximation of $\min\{2/\alpha, f_L\}$ with factor $1 + O(\varepsilon)$ in $\tilde{O}(\varepsilon^{-2} (2/\alpha)^{1/3} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ time. We merge it with \tilde{f}_H and obtain an approximation of $\min\{f_H \oplus f_L, B\}$ with factor $1 + O(\varepsilon)$.

For every $m \in M$, we round down p_m so that the unit profit p_m/w_m becomes a power of $1 + \varepsilon$. After rounding, the approximation factor is only multiplied by $1 + \varepsilon$, and there are at most $\log_{1+\varepsilon} \frac{q}{q(1-\alpha)} = O(\alpha/\varepsilon)$ distinct unit profits in M . Let M_q denote the set of items in M with unit profit q . For each q , we use Lemma 23 to obtain a $1 + \varepsilon$ approximation of

the function f_{M_q} in $O(|M_q| + \varepsilon^{-2})$ time. Then we use Lemma 3 to merge these functions and obtain a $1 + \varepsilon$ approximation of f_M . The total time is $O(|M| \log n) + \tilde{O}(\alpha \varepsilon^{-3})$.

Finally we merge the functions and get an approximation of $\min\{B, f_L \oplus f_H \oplus f_M\}$ with factor $1 + O(\varepsilon)$. The total time is $O(n \log \frac{1}{\varepsilon}) + \tilde{O}(\alpha \varepsilon^{-3} + \varepsilon^{-2} (2/\alpha)^{1/3} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$, which is $O(n \log \frac{1}{\varepsilon} + \varepsilon^{-9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$ if we choose $\alpha = \varepsilon^{3/4} / 2^{c\sqrt{\log(1/\varepsilon)}}$ for a sufficiently small constant c . ◀

► **Corollary 25** (restated Theorem 1). *There is a deterministic $(1 + \varepsilon)$ -approximation algorithm for 0-1 knapsack with running time $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{9/4} / 2^{\Omega(\sqrt{\log(1/\varepsilon)})})$.*

Proof. Divide the items into $O(\log \frac{n}{\varepsilon})$ groups, each containing items with $p_i \in [2^j, 2^{j+1}]$ for some j . Use Lemma 24 to solve each group, and merge them using Lemma 3. ◀

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A Proof of Lemma 12

► **Theorem 26** (Reminder of Lemma 12). *Let T_1, T_2, \dots, T_d be positive real numbers satisfying $T_1 \geq 2$ and $T_{i+1} \geq 2T_i$. There exist at least $T_d/(\log T_d)^{O(d)}$ integers t satisfying the following condition: t can be written as a product of integers $t = n_1 n_2 \cdots n_d$, such that $n_1 n_2 \cdots n_i \in (T_i/2, T_i]$ for every $1 \leq i \leq d$.*

Proof. For every $1 \leq k \leq d$, we say an ordered k -tuple (p_1, p_2, \dots, p_k) is *valid* if every p_i is prime, and $p_1 p_2 \cdots p_i \in (T_i/2, T_i]$ for every $1 \leq i \leq k$. Then the product $t = p_1 p_2 \cdots p_d$ of any valid d -tuple (p_1, \dots, p_d) satisfies our condition. For any integer t , there are at most $d!$ different valid d -tuples with product t (which could be obtained by permuting t ’s prime factors). Let N_k denote the number of valid k -tuples. Then it suffices to show $N_d/d! \geq T_d/(\log T_d)^{O(d)}$.

By the prime number theorem and Bertrand-Chebyshev theorem, there exists a positive constant C such that

$$\pi(x) - \pi(x/2) \geq x/(C \log x), \text{ for all } x \geq 2,$$

where $\pi(x)$ denotes the number of primes less than or equal to x . We will prove $N_k \geq T_k/(C \log T_k)^k$ for all $1 \leq k \leq d$ by induction.

First note that this statement is trivial for $k = 1$. For $k \geq 2$, a valid k -tuple (p_1, \dots, p_k) can be obtained by appending any prime $p_k \in (T_k/(2P), T_k/P]$ to any valid $(k - 1)$ -tuple (p_1, \dots, p_{k-1}) with product $P = p_1 \cdots p_{k-1} \leq T_{k-1}$. The number of such primes p_k is

$$\pi(T_k/P) - \pi(T_k/(2P)) \geq \frac{T_k/P}{C \log(T_k/P)} \geq \frac{T_k/T_{k-1}}{C \log T_k}.$$

Summing over all valid $(k - 1)$ -tuples, we have

$$N_k \geq N_{k-1} \cdot \frac{T_k/T_{k-1}}{C \log T_k} \geq \frac{T_{k-1}}{(C \log T_{k-1})^{k-1}} \cdot \frac{T_k/T_{k-1}}{C \log T_k} \geq \frac{T_k}{(C \log T_k)^k}.$$

Hence, $N_d \geq T_d/(C \log T_d)^d$ by induction. Observe that $T_d \geq 2^d$ and we have

$$\frac{N_d}{d!} \geq \frac{T_d}{(Cd \log T_d)^d} \geq \frac{T_d}{(C \log^2 T_d)^d} \geq \frac{T_d}{(\log T_d)^{O(d)}},$$

which finishes the proof. ◀