

Roundtrip Spanners with $(2k - 1)$ Stretch

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Abstract

A roundtrip spanner of a directed graph G is a subgraph of G preserving roundtrip distances approximately for all pairs of vertices. Despite extensive research, there is still a small stretch gap between roundtrip spanners in directed graphs and undirected spanners. For a directed graph with real edge weights in $[1, W]$, we first propose a new deterministic algorithm that constructs a roundtrip spanner with $(2k - 1)$ stretch and $O(kn^{1+1/k} \log(nW))$ edges for every integer $k \geq 1$, then remove the dependence of size on W to give a roundtrip spanner with $(2k - 1 + o(1))$ stretch and $O(kn^{1+1/k} \log n)$ edges. While keeping the edge size small, our result improves the previous $2k + \epsilon$ stretch roundtrip spanners in directed graphs [Roditty, Thorup, Zwick'02; Zhu, Lam'18], and almost match the undirected $(2k - 1)$ -spanner with $O(kn^{1+1/k})$ edges [Althöfer et al. '93] which is optimal under Erdős conjecture.

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1 Introduction

A t -spanner of a graph G is a subgraph of G in which the distance between every pair of vertices is at most t times their distance in G , where t is called the stretch of the spanner. Sparse spanner is an important choice to implicitly presenting all-pair distances [18], and spanners also have application background in distributed systems and communication networks. For undirected graphs, $(2k - 1)$ -spanner with $O(n^{1+1/k})$ edges is proposed and conjectured to be optimal [2, 16]. However, directed graphs may not have sparse spanners with respect to normal distance measure. For instance, in a bipartite graph with two sides U and V , if there is an directed edge from every vertex in U to every vertex in v , then removing any edge (u, v) in this graph will destroy the reachability from u to v , so its only spanner is itself, which has $O(n^2)$ edges. To circumvent this obstacle, one can approximate the optimal spanner in terms of edge size (e.g. in [3, 9]), or can define directed spanners on different distance measures. This paper will study directed sparse spanners on roundtrip distances.

Roundtrip distance is a natural metric with good property. Cowen and Wagner [7, 8] first introduce it into directed spanners. Formally, roundtrip distance between vertices u, v is defined as $d_G(u \rightleftharpoons v) = d_G(u \rightarrow v) + d_G(v \rightarrow u)$, where $d_G(u \rightarrow v)$ is the length of shortest path from u to v . For a directed graph $G = (V, E)$, a subgraph $G' = (V, E')$ ($E' \subseteq E$) is called a t -roundtrip-spanner of G if $\forall u, v \in G, d_{G'}(u \rightleftharpoons v) \leq t \cdot d_G(u \rightleftharpoons v)$. t is called the stretch of the roundtrip spanner.

In directed graph $G = (V, E)$ ($n = |V|, m = |E|$) with real edge weights in $[1, W]$, Roditty et al. [15] provides a $(2k + \epsilon)$ -spanner with $O(\min\{(k^2/\epsilon)n^{1+1/k} \log(nW), (k/\epsilon)^2 n^{1+1/k} (\log n)^{2-1/k}\})$ edges. Recently, Zhu and Lam [17] derandomizes it and improves the size to $O((k/\epsilon)n^{1+1/k} \log(nW))$ edges, while the stretch is also $2k + \epsilon$. We make a step further based on these works and reduce the stretch to $2k - 1$. Formally, we state our main results in the following theorem.

Theorem 1. *For any directed graph G with maximum edge weight W and integer $k \geq 1$, there exists a $(2k - 1)$ -roundtrip spanner of G with $O(kn^{1+1/k} \log(nW))$ edges, and a $(2k - 1 + O(1/n^c))$ -roundtrip spanner of G with $O(kn^{1+1/k} \log n)$ edges for any constant c . Both spanners can be constructed in $\tilde{O}(kmn \log W)$ time.*

Actually, our result almost matches the lower bound following girth conjecture. The girth conjecture, implicitly mentioned by Erdős [11], says that for any k , there exists a graph with n vertex and $\Omega(n^{1+1/k})$ edges whose girth (minimum cycle) is at least $2k + 2$. This conjecture implies that no algorithm can construct a spanner of $O(n^{1+1/k})$ size and less than $2k - 1$ stretch for all undirected graph with n vertices [16]. This lower bound also holds for roundtrip spanners on directed graphs.

Our approach is based on the scaling constructions of the $(2k + \epsilon)$ -stretch roundtrip spanners in [15, 17]. To reduce the stretch, we construct inward and outward shortest path trees from vertices in a hitting set [1, 10] of size $O(n^{1/k})$, and carefully choose the order to process vertices in order to make the stretch exactly $2k - 1$. To further make the size of the spanner strongly subquadratic, we use a similar approach as in [15] to contract small edges in every scale.

1.1 Related Works

Like many previous works aiming at constructing sparse roundtrip spanners with small stretch, this paper also has $\Omega(nm)$ construction time. Pachoci et al. [13] proposes an algorithm which can construct $O(k \log n)$ -roundtrip-spanner with $O(n^{1+1/k} \log^2 n)$ edges. Its construction time is

$O(mn^{1/k} \log^5 n)$, which breaks the cubic time barrier. Very recently, Chechik et al. [6] give an algorithm which constructs $O(k \log \log n)$ -roundtrip-spanner with $\tilde{O}(n^{1+1/k})$ edges in $\tilde{O}(m^{1+1/k})$ time.

For spanners defined with respect to normal directed distance, researchers aim to approximate the k -spanner with minimum number of edges. Dinitz and Krauthgamer [9] achieve $\tilde{O}(n^{2/3})$ approximation in terms of edge size, and Berman et al. [3] improves the approximation ratio to $\tilde{O}(n^{1/2})$.

Another type of directed spanners is transitive-closure spanner, introduced by Bhattacharyya et al. [5]. In this setting the answer may not be a subgraph of G , but a subgraph of the transitive closure of G . In other words, selecting edges outside the graph is permitted. The tradeoff is between diameter (maximum distance) and edge size. One of Bhattacharyya et al.'s results is spanner with diameter k and $O((n \log n)^{1-1/k})$ approximate to optimal edge size [5], using a combination of linear programming rounding and sampling. Berman et al. [4] improves the approximation ratio to $O(n^{1-1/[k/2]} \log n)$. We refer to Raskhodnikova [14] as a review of transitive-closure spanner.

1.2 Organization

In Section 2, the notations and basic concepts used in this paper will be discussed. In Section 3 we describe the construction of the $(2k - 1)$ -roundtrip spanner with $O(kn^{1+1/k} \log(nW))$ edges, then in Section 4 we improve the size of the spanner to $O(kn^{1+1/k} \log n)$ and the stretch becomes $(2k - 1 + o(1))$.

2 Preliminaries

In this paper we consider a directed graph $G = (V, E)$ with non-negative real edge weights w where $w(e) \in [1, W]$ for all $e \in E$. Denote $G[U]$ to be the subgraph of G induced by $U \subseteq V$, i.e. $G[U] = (U, E \cap (U \times U))$. A roundtrip path between nodes u and v is a cycle (not necessarily simple) passing through u and v . The roundtrip distance between u and v is the minimum length of roundtrip paths between u and v . Denote $d_U(u \rightleftharpoons v)$ to be the roundtrip distance between u and v in $G[U]$. (Sometimes we may also use $d_U(u \rightleftharpoons v)$ to denote a roundtrip shortest path between u, v in $G[U]$.) It satisfies:

- For $u, v \in U$, $d_U(u \rightleftharpoons u) = 0$ and $d_U(u \rightleftharpoons v) = d_U(v \rightleftharpoons u)$.
- If $d_U(u \rightarrow v)$ is the distance from u to v in $G[U]$, then $d_U(u \rightleftharpoons v) = d_U(u \rightarrow v) + d_U(v \rightarrow u)$.
- For $u, v, w \in U$, $d_U(u \rightleftharpoons v) \leq d_U(u \rightleftharpoons w) + d_U(w \rightleftharpoons v)$.

Here $d_U(u \rightarrow v)$ is the one-way distance from u to v in $G[U]$. Of course $d(u \rightleftharpoons v)$ is just the roundtrip distance between u and v in the original graph $G = (V, E)$.

In G , a t -roundtrip spanner of G in a subgraph H of G on the same vertex set V such that the roundtrip distance between any pair of $u, v \in V$ in H is at most $t \cdot d(u \rightleftharpoons v)$. t is called the *stretch* of the spanner.

Given a center $u \in U$ and a radius R , define roundtrip ball $Ball_U(u, R)$ to be the set of vertices whose roundtrip distance on $G[U]$ to center u is strictly smaller than the radius R . Formally, $Ball_U(u, R) = \{v \in U : d_U(u \rightleftharpoons v) < R\}$. Then the size of the ball, denoted by $|Ball_U(u, R)|$, is the number of vertices in it. Similarly we define $\overline{Ball}_U(u, R) = \{v \in U : d_U(u \rightleftharpoons v) \leq R\}$.

Subroutine **InOutTrees**(U, u, R) calculates the edge set of an inward and an outward shortest path tree centered at u spanning vertices in $Ball_U(u, R)$ on $G[U]$. (That is, the shortest path tree from u to all vertices in $Ball_U(u, R)$ and the shortest path tree from all vertices in $Ball_U(u, R)$ to u .) It is easy to see that the shortest path trees will not contain vertices outside $Ball_U(u, R)$:

Lemma 2. *The inward and outward shortest path trees returned by **InOutTrees**(U, u, R) only contains vertices in $Ball_U(u, R)$.*

Proof. For any $v \in Ball_U(u, R)$, let C be a cycle containing u and v such that the length of C is less than R . Then for any vertex $w \in C$, $d_U(u \rightleftharpoons w) < R$, so w must be also in the trees returned by **InOutTrees**(U, u, R). \square

For all notations above, we can omit the subscript V when the roundtrip distance is considered in the original graph $G = (V, E)$. Our algorithm relies on the following well-known theorem to calculate hitting sets deterministically.

Theorem 3. (Cf. Aingworth et al. [1], Dor et al. [10]) *For universe V and its subsets S_1, S_2, \dots, S_n , if $|V| = n$ and the size of each S_i is greater than p , then there exists a hitting set $H \subseteq V$ intersecting all S_i , whose size $|H| \leq n \ln n/p$, and such a set H can be found in $O(np)$ time deterministically.*

3 A $(2k - 1)$ -Roundtrip-Spanner Algorithm

In this section we introduce our main algorithm constructing a $(2k - 1)$ -roundtrip-spanner with $O(kn^{1+1/k} \log(nW))$ edges for any G . We may assume $k \geq 2$ in the following analysis, since the result is trivial for $k = 1$.

Our approach combines the idea of [15] and [17]. In [17], given a length L , we pick an arbitrary vertex u and find the smallest integer h such that $|\overline{Ball}(u, (h + 1)L)| < n^{1/k} |\overline{Ball}(u, h \cdot L)|$, then we include the inward and outward shortest path tree centered at u spanning $\overline{Ball}(u, (h + 1)L)$ and remove vertices in $\overline{Ball}(u, h \cdot L)$ from V . We can see that $h \leq k$, so the stretch is $2k$ for u, v with roundtrip distance L , and by a scaling approach the final stretch is $2k + \epsilon$. We observe that if $h = k - 1$, $|\overline{Ball}(u, (k - 1)L)| \geq n^{(k-1)/k}$, so by Theorem 3 we can preprocess the graph by choosing a hitting set H with size $O(n^{1/k} \log n)$ and construct inward and outward shortest path trees centered at all vertices in H , then we do not need to include the shortest path trees spanning $\overline{Ball}(u, (h + 1)L)$. The stretch can then be decreased to $2k - 1 + \epsilon$. To make the stretch equals $2k - 1$, instead of arbitrarily selecting u each time, we carefully define the order to select u .

3.1 Preprocessing

We first define a radius $R(u)$ for each vertex u . It is crucial for the processing order of vertices.

Definition 4. *For all $u \in V$, we define $R(u)$ to be the maximum length R such that $|Ball(u, R)| < n^{1-1/k}$, that is, if we sort the vertices by their roundtrip distance to u in G by increasing order, $R(u)$ is the roundtrip distance from u to the $\lceil n^{1-1/k} \rceil$ -th vertex.*

For any $u \in V$, $|\overline{Ball}(u, R(u))| \geq n^{1-1/k}$. By Theorem 3, we can find a hitting set H intersecting all sets in $\{\overline{Ball}(u, R(u)) : u \in V\}$, such that $|H| = O(n^{1/k} \log n)$. For all $t \in H$, we build an inward and an outward shortest path tree of G centered at t , and denote the set of edges of these trees by E_0 and include them in the final spanner. This step generates $O(n^{1+1/k} \log n)$ edges in total, and it is easy to obtain the following statement:

Lemma 5. For $u, v \in V$ such that $d(u \rightleftharpoons v) \geq R(u)/(k-1)$, the roundtrip distance between u and v in the graph (V, E_0) is at most $(2k-1)d(u \rightleftharpoons v)$.

Proof. Find the vertex $t \in H$ such that $t \in \overline{Ball}(u, R(u))$, that is, $d(u \rightleftharpoons t) \leq R(u)$. Then the inward and outward shortest path trees from t will include $d(u \rightleftharpoons t)$ and $d(t \rightleftharpoons v)$. By $R(u) \leq (k-1)d(u \rightleftharpoons v)$, we have $d(t \rightleftharpoons v) \leq d(t \rightleftharpoons u) + d(u \rightleftharpoons v) \leq k \cdot d(u \rightleftharpoons v)$. So the roundtrip distance of u and v in E_0 is at most $d(u \rightleftharpoons t) + d(t \rightleftharpoons v) \leq (2k-1)d(u \rightleftharpoons v)$. \square

3.2 Approximating a Length Interval

Instead of approximating all roundtrip distances at once, we start with an easier subproblem of approximating all pairs of vertices whose roundtrip distance is within an interval $[L/(1+\epsilon), L]$. Parameter ϵ is a real number in $(0, 1/(2k-2)]$. The procedure **Cover** (G, k, L, ϵ) described in Algorithm 1 will return a set of edges which gives a $(2k-2+\epsilon)$ -approximation of roundtrip distance $d(u \rightleftharpoons v)$ if $R(u)/(k-1) > d(u \rightleftharpoons v)$ for $d(u \rightleftharpoons v) \in [L/(1+\epsilon), L]$.

Algorithm 1 **Cover** $(G(V, E), k, L, \epsilon)$

```

1:  $U \leftarrow V, \hat{E} = \emptyset$ 
2: while  $U \neq \emptyset$  do
3:    $u \leftarrow \arg \max_{u \in U} R(u)$ 
4:    $step \leftarrow \min\{R(u)/(k-1), L\}$ 
5:    $h \leftarrow$  minimum positive integer satisfying  $|Ball_U(u, h \cdot step)| < n^{h/k}$ 
6:   Add INOUTTREES $(U, u, h \cdot step)$  to  $\hat{E}$ 
7:   Remove  $Ball_U(u, (h-1)step)$  from  $U$ 
8: end while
9: return  $\hat{E}$ 

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Note that in this algorithm, initially $U = V$ and the balls are considered in $G[U] = G$. In the end of every iteration we remove a ball from U , and the following balls are based on the roundtrip distances in $G[U]$. However, $R(u)$ does not need to change during the algorithm, which can still be based on roundtrip distances in the original graph G . The analysis for the size of the returned set \hat{E} and the stretch are as follows.

Lemma 6. The returned edge set of **Cover** (G, k, L, ϵ) has $O(n^{1+1/k})$ size.

Proof. When processing a vertex u , by the selection of h in line 5, $|Ball_U(u, h \cdot step)| < n^{h/k}$ and $|Ball_U(u, (h-1)step)| \geq n^{(h-1)/k}$. When $h \geq 2$ it is because of h 's minimality, and when $h = 1$ it is because $u \in Ball_U(u, 0)$. So each time **INOUTTREES** is called, the size of ball to build shortest path trees is no more than $n^{1/k}$ times the size of ball to remove. During an execution of **Cover** (G, k, L, ϵ) , each vertex is removed once from U . Therefore the total size of edges added in \hat{E} is $O(n^{1+1/k})$. \square

We can also see that if the procedure **Cover** $(G[U], k, L, \epsilon)$ is run on a subgraph $G[U]$ induced on a subset $U \subseteq V$, then the size of \hat{E} is bounded by $O(|U|n^{1/k})$. It is also easy to see that h is at most $k-1$ by our choice of u :

Lemma 7. The h selected at line 5 in **Cover** (G, k, L, ϵ) satisfies $h \leq k-1$.

Proof. In $G[U]$, the ball $Ball_U(u, (k-1)step)$ must have size no greater than $Ball(u, (k-1)step)$ since the distance in $G[U]$ cannot be decreased and some vertices may be removed. Since $|Ball(u, R(u))| < n^{1-1/k}$ and $step \leq R(u)/(k-1)$, we get $|Ball_U(u, (k-1)step)| \leq |Ball(u, (k-1)step)| < n^{1-1/k}$, thus $h \leq k-1$. \square

Next we analysis the roundtrip distance stretch in \hat{E} . Note that in order to make the final stretch $2k-1$, for the roundtrip distance approximated by edges in \hat{E} we can make the stretch $(2k-2)(1+\epsilon)$, but for the roundtrip distance approximated by E_0 we need to make the stretch at most $2k-1$ as E_0 stays the same.

Lemma 8. *For any pair of vertices u, v such that $d(u \rightleftharpoons v) \in [L/(1+\epsilon), L]$, either $\text{Cover}(G, k, L, \epsilon)$'s returned edge set \hat{E} can form a cycle passing through u, v with length at most $(2k-2)(1+\epsilon)d(u \rightleftharpoons v)$, or $R(u) < (k-1)d(u \rightleftharpoons v)$, in which case the E_0 built in Section 3.1 can form a detour cycle with length at most $(2k-1)d(u \rightleftharpoons v)$ by Lemma 5.*

Proof. Consider any pair of vertices u, v with roundtrip distance $d = d(u \rightleftharpoons v) \in [L/(1+\epsilon), L]$, and a shortest cycle P going through u, v with length d .

During $\text{Cover}(G, k, L, \epsilon)$, consider the vertices on P that are first removed from U . Suppose w is one of the first removed vertices, and w is removed as a member of $Ball_{U_c}(c, (h_c-1)step_c)$ centered at c . This is to say $d_{U_c}(c \rightleftharpoons w) < (h_c-1)step_c$.

Case 1: $step_c \geq d$. Then

$$d_{U_c}(c \rightleftharpoons u) \leq d_{U_c}(c \rightleftharpoons w) + d_{U_c}(w \rightleftharpoons u) < (h_c-1)step_c + d \leq h_c step_c,$$

and $u \in Ball_{U_c}(c, h_c step_c)$. The second inequality holds because U_c is the remaining vertex set before removing w , so by definition of w , all vertices on P are in U_c . Symmetrically $v \in Ball_{U_c}(c, h_c step_c)$. $\text{InOutTrees}(U_c, c, h_c step_c)$ builds a detour cycle passing u, v with length $< 2h_c step_c$. By Lemma 7, we have $h_c \leq k-1$. Also $step_c \leq L \leq (1+\epsilon)d$, therefore we build a detour of length $< 2(k-1)step_c \leq (2k-2)(1+\epsilon)d$ in \hat{E} .

Case 2: $step_c < d$. Because $d \leq L$, this case can only occur when $step_c = R(c)/(k-1)$. Because c is chosen before u , $R(u) \leq R(c) = (k-1)step_c < (k-1)d$. By Lemma 5, E_0 can give a $(2k-1)$ -approximation of d . \square

3.3 Main Construction

Now we can proceed to prove the first part of the main theorem based on a scaling on lengths of the cycles from 1 to $2nW$.

Theorem 9. *For any directed graph G with real edge weights in $[1, W]$, there exists a polynomial time constructible $(2k-1)$ -roundtrip-spanner of G with $O(kn^{1+1/k} \log(nW))$ edges.*

Proof. Note that the roundtrip distance between any pair of vertices must be in the range $[1, 2(n-1)W]$. First do the preprocessing in Section 3.1. Then divide the range of roundtrip distance $[1, 2nW]$ into intervals $((1+\epsilon)^{p-1}, (1+\epsilon)^p]$, where $\epsilon = 1/(2k-2)$. Call $\text{Cover}(G, k, (1+\epsilon)^p, \epsilon)$ for $p = 0, \dots, \lceil \log_{1+\epsilon}(2nW) \rceil$, and merge all returned edges to form a spanner.

First we prove that the edge size is $O(kn^{1+1/k} \log(nW))$. Preprocessing adds $O(n^{1+1/k} \log n)$ edges. $\text{Cover}(G, k, (1+\epsilon)^p, \epsilon)$ is called for $\log_{1+1/(2k-2)}(2nW) = O(k \log(nW))$ times. By Lemma 6,

each call generates $O(n^{1+1/k})$ edges. So the total number of edges in the roundtrip spanner is $O(kn^{1+1/k} \log(nW))$.

Next we prove the stretch is $2k - 1$. For any pair of vertices u, v with roundtrip distance d , let $p = \lceil \log_{1+\epsilon} d \rceil \in [0, \log_{1+\epsilon}(2nW)]$, then $d \in ((1+\epsilon)^{p-1}, (1+\epsilon)^p]$. By Lemma 8, either the returned edge set of $\text{Cover}(G, k, (1+\epsilon)^p, \epsilon)$ can form a detour cycle passing through u, v of length at most $(2k-2)(1+\epsilon)d = (2k-1)d$, or the edges in E_0 can form a detour cycle passing through u, v of length at most $(2k-1)d$.

In conclusion this algorithm can construct a $(2k-1)$ -roundtrip-spanner with $O(kn^{1+1/k} \log(nW))$ edges. \square

3.4 Construction Time

The running time of the algorithm in the proof of Theorem 9 is $O(kn(m + n \log n) \log(nW))$, in which the bottleneck is the $\tilde{O}(n)$ single-source shortest paths calls. It is also easy to see that the algorithm is deterministic. Next we analysis construction time in detail.

In preprocessing, for any $u \in V$, $R(u)$ can be calculated by running Dijkstra searches with Fibonacci heap [12] starting at u , so the calculating $R(\cdot)$ takes $O(n(m + n \log n))$ time. Finding H takes $O(n^{2-1/k})$ time by Theorem 3. Building E_0 takes $O(n^{1/k} \log n \cdot (m + n \log n))$ time.

A **Cover** call's while loop runs at most n times since each time at least one node is removed. In a loop, u can be found in $O(n)$, and all other operations regarding roundtrip balls can be done in linear time by Dijkstra searches starting at u on $G[U]$. Therefore a **Cover** call takes $O(n(m + n \log n))$ time.

Cover is called $O(k \log(nW))$ times. Combined with the preprocessing time, the total construction time is $O(kn(m + n \log n) \log(nW))$.

4 Removing the Dependence on W

The size of the roundtrip spanner in Section 3 is dependent on maximum edge weight W . In this section we can remove this dependence by designing the scaling approach more carefully, then the stretch will be $(2k - 1 + O(1/n^c))$ for any constant c .

Our idea is similar to that in [15]. When we consider the roundtrip distances between $[L/(1+\epsilon), L]$, all cycles with length $\leq L/n^3$ have little affect so we can contract them into one node, and all edges with length $> (2k-1)L$ cannot be in any $(2k-1)L$ detour cycles, so it can be deleted. Thus, an edge with length l can only be in the $O(\log_{1+\epsilon} n)$ iterations for L between $l/(2k-1)$ and $l \cdot n^3$ (based on the girth of this edge). We can also show that the number of strongly connected components in the graphs considered in all iterations is bounded by $O(kn \log n)$, thus we can get an $O(kn^{1+1/k} \log n)$ -size spanner. However, to get the $(2k-1+o(1))$ -stretch in the presence of contracted nodes, we need a careful analysis.

First we define the girth of an edge, and study its properties:

Definition 10. We define the girth of an edge e in G to be the length of minimum directed cycle containing e , and denote it by $g(e)$.

It is easy to see that for $e = (u, v)$, $d(u \rightsquigarrow v) \leq g(e)$. In $O(n(m + n \log n))$ time we can compute $g(e)$ for all edges e in G [12]. We can get the following observation:

Lemma 11. *Given a length L , the edge set \hat{E} returned by $\text{Cover}(G, k, L, \epsilon)$ does not include edges e with $g(e) > (k - 1)L$.*

Proof. Since in $\text{Cover}(G, k, L, \epsilon)$ we include in-out-trees (in a subgraph of G) from u with radius $h \cdot \text{step}$, where $h \leq k - 1$ by Lemma 7 and $\text{step} \leq L$, so the radius is at most $(k - 1)L$. Any edge with girth greater than $(k - 1)L$ cannot be in such in-out-trees. \square

The main algorithm is shown in Algorithm 2.

Algorithm 2 $\text{Spanner}(G(V, E), k)$

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1: Do the preprocessing in Section 3.1. Let  $E_0$  be the added edges
2:  $\hat{E} \leftarrow E_0$ 
3:  $\epsilon \leftarrow 1/(2k - 2)$ 
4: for  $p \leftarrow 0$  to  $\lceil \log_{1+\epsilon}(2nW) \rceil$  do
5:    $L \leftarrow (1 + \epsilon)^p$ 
6:   Contract all edges  $e$  with  $g(e) \leq L/n^3$  in  $G$  to form a graph  $G_p$ 
7:   (Delete edges  $e$  with  $g(e) > 2(k - 1)L$  from  $G_p$ )
8:   For all edges  $e = (u, v)$  in  $G_p$ , if  $v$  is contracted from  $q$  original vertices in  $G$ , let  $w(u, v) \leftarrow$ 
      $w(u, v) + (q - 1) \cdot L/n^3$  in  $G_p$ 
9:   Recalculate  $R(\cdot)$  with respect to  $G_p$ , and call it  $R_p(\cdot)$ 
10:   $\hat{E} \leftarrow \hat{E} \cup \text{Cover}(G_p, k, L, \epsilon)$ 
11: end for
12: return  $H(V, \hat{E})$ 

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We can check that Lemma 8 still holds when calling $\text{Cover}(G_p, k, L, \epsilon)$ w.r.t. $R_p(\cdot)$ and the weights adjusted in line 8 in the current graph G_p . We adjust the weight in G_p in order to give an upper bound of the roundtrip distance in the original graph G :

Lemma 12. *In the graph G_p during the algorithm, if a contracted vertex v in G_p contains original vertices $U = \{u_1, u_2, \dots, u_q\}$ of G , then every $u_i \in U$ is in a cycle in the subgraph $G[U]$ of length at most $(1 + \epsilon)^p/n^3$. So the roundtrip distances between vertices in U are at most $(q - 1) \cdot (1 + \epsilon)^p/n^3$.*

Proof. If an edge e is contracted in G_p , then $g(e) \leq (1 + \epsilon)^p/n^3$. Every edge f in the cycle with length $g(e)$ must have $g(f) \leq g(e)$, so such edges are also contracted to the same vertex in G_p . It is also easy to see that the roundtrip distance between $u_i, u_j \in U$ is at most $(q - 1) \cdot (1 + \epsilon)^p/n^3$ since every contracted edge corresponds to a cycle of length $(1 + \epsilon)^p/n^3$. \square

Lemma 13. *For u, v in the original graph, let the contracted vertices in G_p containing them are u', v' , respectively. If u' and v' are different and strongly connected in G_p , then their roundtrip distance d' in the weight adjusted graph G_p satisfies $d(u \rightleftharpoons v) \leq d' \leq d(u \rightleftharpoons v) + (1 + \epsilon)^p/n^2$.*

Proof. In the minimum cycle containing u and v in the original graph G , for the vertices in the cycle which is contracted to a vertex w' in G_p , find the first and last vertices contracted to w' in the cycle, and call them w_1 and w_2 . Then by Lemma 12, $d(w_1 \rightleftharpoons w_2) \leq (q - 1) \cdot (1 + \epsilon)^p/n^3$ if w' contains q original vertices. When we contract w_1 and w_2 to w' in G_p and adjust the weights in line 8, we add $(q - 1) \cdot (1 + \epsilon)^p/n^3$ to every edge pointing to w' , which is at least the distance from w_1 to w_2 , thus we have $d' \geq d(u \rightleftharpoons v)$. Also in every roundtrip distance in G_p , the amount of adjustment is at most $(1 + \epsilon)^p/n^2$, so $d' \leq d(u \rightleftharpoons v) + (1 + \epsilon)^p/n^2$. \square

Since $G_{p'}$ can be seen as a graph contracted from G_p if $p' > p$, we have the following corollary:

Corollary 14. *During the algorithm $\text{Spanner}(G, k)$, when more and more edges are contracted, the roundtrip distance between every pair of vertices cannot decrease in the weight adjusted G_p .*

By Lemma 11, $\text{Cover}(G_p, k, L, \epsilon)$ will not include edges with girth longer than $(k-1)L$ in the weight adjusted G_p , thus will not include edges with girth longer than $2(k-1)L$ in G . We include line 7 for our analysis, but it is not necessary.

Lemma 15. *The subgraph returned by algorithm $\text{Spanner}(G, k)$ has $O(kn^{1+1/k} \log n)$ edges.*

Proof. Preprocessing adds $O(n^{1+1/k} \log n)$ edges as in Section 3.1. Next we count $\text{Cover}(G, k, L, \epsilon)$'s returned edges.

We remove the directions of all edges in G to get an undirected graph G' , and remove the directions of all edges in every G_p to get an undirected graph G'_p , but define the weight of an edge e in G' and every G'_p to be the girth $g(e)$ in G . Let F be a minimum spanning forest of G' w.r.t. the girth $g(e)$. We can see that in iteration p , if we remove edges in F with $g(e) > 2(k-1)(1+\epsilon)^p$ and contract edges e with $g(e) \leq (1+\epsilon)^p/n^3$ in F , then the connected components in F will just be the connected components in G'_p , which are the strongly connected components in G_p . This is because of the cycle property of MST: If an edge $e = (u, v)$ in G'_p has $g(e) \leq (1+\epsilon)^p/n^3$, then in F all edges in the path connecting u, v have girth $\leq (1+\epsilon)^p/n^3$, thus u, v are already contracted; If an edge $e = (u, v)$ in G'_p has $g(e) \leq 2(k-1)(1+\epsilon)^p$, then in F all edges in the path connecting u, v have girth $\leq 2(k-1)(1+\epsilon)^p$, so u, v are in the same component in F .

So the total size of connected components $\{C : |C| \geq 2\}$ in G'_p is at most 2 times the number of edges e in F with $(1+\epsilon)^p/n^3 < g(e) \leq 2(k-1)(1+\epsilon)^p$, and every edge in F can be in at most $\log_{1+\epsilon} 2(k-1)n^3 = O(k \log n)$ number of different G'_p . Thus, the total number of connected components with size at least 2 in all G'_p is bounded by $O(kn \log n)$. By Lemma 6, after calling $\text{Cover}(G_p, k, L, \epsilon)$, \hat{E} will have $|C|n^{1/k}$ new edges for every connected component C with $|C| \geq 2$ in G'_p . Thus the total number of edges in the subgraph returned by $\text{Spanner}(G, k)$ is bounded by $O(kn^{1+1/k} \log n)$. \square

We can now prove the stretch of this roundtrip spanner. Note that in the algorithm $L = (1+\epsilon)^p$ is an upper bound for the roundtrip distance d' in G_p , but not for the original distance in G , so we need to analyze it carefully.

Lemma 16. *Algorithm $\text{Spanner}(G, k)$ constructs a $(2k-1 + O(1/n))$ -roundtrip-spanner of G .*

Proof. We prove it by the induction on p . When p is small, there is no contract vertices in G_p , then it is the same as Lemma 8, so we get the $(2k-1)$ -stretch. Next we assume that the distances between vertices of G contracted in the same vertex in G_p have a $(2k-1)$ -approximation in \hat{E} . (Thus, they are strongly connected in \hat{E} .)

For any pair of vertices u, v with roundtrip distance $d = d(u \rightleftharpoons v)$ on G . By Lemma 13 and Corollary 14, there exists a p such that in G_p the roundtrip distance d' between u' and v' satisfies $(1+\epsilon)^{p-1} < d' \leq (1+\epsilon)^p$, where u', v' are the contracted vertices in G_p containing u, v , respectively. (Suppose $p' = \lceil \log_{1+\epsilon} d \rceil$, then $d \in ((1+\epsilon)^{p'-1}, (1+\epsilon)^{p'}]$, and $(1+\epsilon)^{p'-1} < d \leq d_{G_{p'}}(u \rightleftharpoons v) \leq d + (1+\epsilon)^{p'}/n^2 \leq (1+1/n^2)(1+\epsilon)^{p'}$. And in $G_{p'+1}$ the roundtrip distance cannot be smaller than in $G_{p'}$ and $d_{G_{p'+1}}(u \rightleftharpoons v) \leq d + (1+\epsilon)^{p'+1}/n^2 \leq (1 + (1+\epsilon)/n^2)(1+\epsilon)^{p'} < (1+\epsilon)^{p'+1}$, when $\epsilon \gg 1/n^2$. So p can be p' or $p' + 1$.)

By Lemma 8, either $\text{Cover}(G_p, k, (1+\epsilon)^p, \epsilon)$'s returned edge set \hat{E} can form a detour cycle with length at most $(2k-2)(1+\epsilon)d'$ on graph G_p , or $R_p(u') < (k-1)d'$. In the latter case since in weight adjusted G_p distance cannot be decreased, and some vertices are contracted in G_p , $R(u)$ in the original graph G must also be smaller than $(k-1)d'$, by Lemma 5, in E_0 the roundtrip distance between u, v is at most $(2k-2)d' + d \leq (2k-1 + O(k/n^2))d \leq (2k-1 + O(1/n))d$. (This is because we have $d' \leq d + (1+\epsilon)^p/n^2$ by Lemma 13.) In the former case, by inductive hypothesis, the original vertices contracted in one vertex in G_p are strongly connected in \hat{E} , so the roundtrip distance d'' between u and v in final \hat{E} is at most

$$(2k-2)(1+\epsilon)d' + (1+\epsilon)^p/n^2 \leq (2k-1 + O(1/n^2))d' \leq (2k-1 + O(1/n))d$$

($k \geq 2$ and n is not too small.) □

Note that we can make the stretch to be $2k-1 + O(1/n^c)$ for any constant c if we contract edges with girth $\leq L/n^{c+2}$ in line 6.

4.1 Construction Time

The analysis of **Spanner**'s running time is similar to Section 3.4. Compared with the primary version, **Spanner** adds operations of building G_p and recalculating weights and $R(\cdot)$ before each **Cover** call. We also need to calculate $g(\cdot)$ in preprocessing, which can be done by n Dijkstra searches. G_p can be built in $O(m)$ time. Recalculating $R(\cdot)$ takes $O(n(m + n \log n))$ and recalculating weights takes $O(m)$ time. **Cover** is called $\log_{1+\epsilon}(2nW) = O(k \log(nW))$ times. Therefore the total construction time is still $O(kn(m + n \log n) \log(nW))$.

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