

Long-term Causal Inference Under Persistent Confounding via Data Combination

Guido Imbens^{1*}; Nathan Kallus^{2*}; Xiaojie Mao^{3*}; Yuhao Wang^{4*}

¹Stanford University; ²Cornell University;

³Tsinghua University; ⁴Tsinghua University and Shanghai Qi Zhi Institute.

Abstract

We study the identification and estimation of long-term treatment effects when both experimental and observational data are available. Since the long-term outcome is observed only after a long delay, it is not measured in the experimental data, but only recorded in the observational data. However, both types of data include observations of some short-term outcomes. In this paper, we uniquely tackle the challenge of persistent unmeasured confounders, *i.e.*, some unmeasured confounders that can simultaneously affect the treatment, short-term outcomes and the long-term outcome, noting that they invalidate identification strategies in previous literature. To address this challenge, we exploit the sequential structure of multiple short-term outcomes, and develop three novel identification strategies for the average long-term treatment effect. We further propose three corresponding estimators and prove their asymptotic consistency and asymptotic normality. We finally apply our methods to estimate the effect of a job training program on long-term employment using semi-synthetic data. We numerically show that our proposals outperform existing methods that fail to handle persistent confounders.

1 Introduction

Empirical researchers and decision-makers are often interested in learning the long-term treatment effects of interventions. For example, economists are interested in the effect of early childhood education on lifetime earnings [Chetty et al., 2011], marketing practitioners are interested in the effects of incentives on customers' long-term behaviors [Yang et al., 2020a], IT companies are interested in the effects of webpage designs on users' long-term behaviors [Hohnhold et al., 2015]. Since a long-term effect can be quite different from short-term effects [Kohavi et al., 2012], accurately evaluating the long-term effect is crucial for comprehensively understanding the intervention of interest.

However, learning long-term treatment effects is very challenging in practice, since long-term outcomes are often observed only after a long delay. This is especially a problem for randomized experiments, because experiments usually have relatively short durations due to cost considerations. For example, industry online controlled experiments (*i.e.*, A/B testing) usually last for only a few weeks, so practitioners commonly recognize the inference of long-term effects as a top challenge [Gupta et al., 2019]. In contrast, observational data are often easier and cheaper to acquire, so they are more likely to include long-term outcome observations. Nevertheless, observational data are very susceptible to unmeasured confounding, which can lead to severely biased treatment effect

*Alphabetical order.

estimates. Therefore, long-term causal inference is very challenging with only a single type of data, either due to missing long-term outcome (in experimental data) or unmeasured confounding (in observational data).

In this paper, we study the identification and estimation of long-term treatment effects when *both* experimental and observational data are available. By combining these two different types of data, we hope to leverage their complementary strengths, *i.e.*, the randomized treatment assignments in the experimental data, and the long-term outcome observations in the observational data. In particular, we aim to tackle complex *persistent confounding* in the observational data. That is, we allow some unmeasured confounders to have persistent effects, in the sense that they can affect *not only* the short-term outcomes, *but also* the long-term outcome. Persistent confounders are prevalent in long-term studies. For example, in the study of early childhood education and lifetime earnings, students' intelligence or capabilities can affect both short-term and long-term earnings. Our setup is summarized in the causal diagrams in Figure 1.

A few previous works also consider data combination for long-term causal inference, but they impose strong restrictions on the unmeasured confounders. Athey et al. [2019] rely on a surrogate criterion first proposed by Prentice [1989]. This surrogate criterion precludes any unobserved variable that can simultaneously affect the short-term and long-term outcomes (see Section 3.2). Athey et al. [2020] assume a latent confoundedness condition, requiring unobserved confounders to have no direct effects on the long-term outcome but only indirect effects through the short-term outcomes. Neither of these works can handle persistent confounders. By permitting the existence of persistent confounders, our setup is substantially more general than these existing literature.

In this paper, we propose the *first* identification strategies for the long-term treatment effect in presence of persistent confounders. These identification strategies are based on *multiple* short-term outcomes. Since these short-term outcomes can provide rich information about unmeasured confounding, we view them as *proxy variables* for the persistent confounders. Importantly, we discover that the *sequential structure* of these short-term outcomes is very useful in identifying the long-term treatment effect. To the best of our knowledge, this is the first time that the internal structure of short-term outcomes is used to address unmeasured confounding in long-term causal inference. Indeed, although Athey et al. [2019, 2020] also advocate multiple short-term outcomes, they view them as a whole, without leveraging their internal structure at all. Therefore, our identification strategies contribute to new insights on the role of short-term outcomes in long-term causal inference.

Our contributions are summarized as follows:

- We propose three novel identification strategies for the average long-term treatment effect when persistent confounders are present. These identification strategies rely on three groups of short-term outcomes where two groups of them can be considered as informative proxy variables for the unmeasured confounders (Assumption 5). These short-term outcomes, together with the long-term outcome, follow a sequential structure encapsulated in a conditional independence condition (Assumption 4). Our identification strategies are more general than those in the previous literature. In particular, when there are no persistent confounders, one of our identification strategies can recover the identification strategy in Athey et al. [2020].
- Based on each of the three identification strategies, we propose a corresponding average long-term treatment effect estimator. These estimators involve fitting two nuisance functions that are defined as solutions to two conditional moment equations. Our estimation procedures accommodate any nuisance estimator among many existing ones. We provide high level conditions for the asymptotic consistency and asymptotic normality of our estimators.

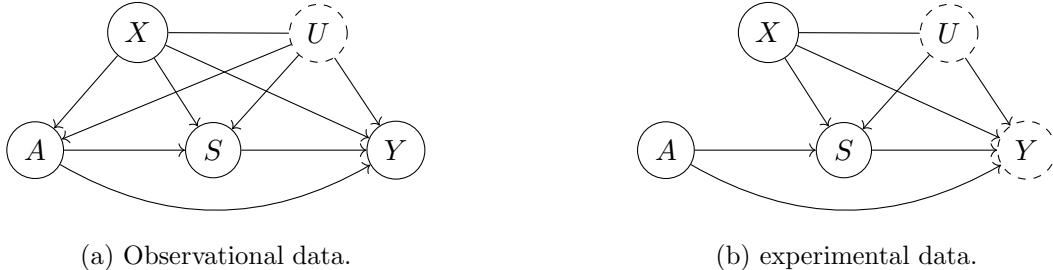


Figure 1: Causal diagrams for observational and experimental datas with persistent confounders. Here A denotes the treatment, S denotes (multiple) short-term outcomes, Y denotes the long-term outcome, X denotes covariates, and U denotes unmeasured confounders. Confounders U in both samples and the long-term outcome Y in the experimental data are unobserved, so they are indicated by dashed circles. Note that unmeasured confounders U can simultaneously affect short-term outcomes S and the long-term outcome Y .

- We evaluate the performance of our proposed estimators based on large-scale experimental data for a job-training program with long-term employment observations. We combine part of the experimental data and some semi-synthetic observational data with realistic persistent confounding. We demonstrate that our proposed estimators can substantially outperform benchmarks that fail to handle persistent confounding, including the state-of-the-art estimator proposed by Athey et al. [2020].

The rest of this paper is organized as follows. We first review the related literature in Section 2 and set up our problem in Section 3. Then we discuss our identification strategies in Section 4, where each subsection features one different identification strategy. In Section 5, we present our long-term treatment effect estimators and analyze their asymptotic properties. We further discuss some extensions in Section 6. In Section 7, we illustrate the performance of methods in a semi-synthetic experiment. We finally conclude this paper in Section 8.

2 Related Literature

2.1 Surrogates

Our paper is related to a large body of biostatistics literature on surrogate outcomes; see reviews in Weir and Walley [2006], VanderWeele [2013], Joffe and Greene [2009].

These literature consider using the causal effect of an intervention on a surrogate outcome (*e.g.*, patients’ short-term health) as a proxy for its treatment effect on the outcome of primary interest (*e.g.*, long-term health). To this end, many criteria have been proposed to ensure the validity of the surrogate outcome. Examples include the statistical surrogate criterion [Prentice, 1989], principal surrogate criterion [Frangakis and Rubin, 2002], consistent surrogate criterion [Chen et al., 2007], among many others. However, these criteria can easily run into logic paradox [Chen et al., 2007] or rely on unidentifiable quantities, showing the challenge of causal inference when the primary outcome is completely missing. When multiple surrogates are available, Wang et al. [2020], Price et al. [2018] consider transforming these surrogates to optimally approximate the primary outcome. Their approaches can avoid the surrogate paradox discussed in Chen et al. [2007]. Nevertheless, learning surrogate transformations requires experimental data with long-term outcome observations in the first place, which can be very demanding in practice.

In contrast, our paper does not need long-term outcome observations in the experimental data but only need them in observational data. Moreover, our paper does not view short-term outcomes as proxies for the long-term outcome, so we avoid these previous surrogate criteria. Instead, we view them as proxies for unmeasured confounders to correct for confounding bias. See also discussions in Section 2.3.

2.2 Data Combination for Long-term Causal Inference and Decision-Making

Following Athey et al. [2019], some recent literature also combine experimental and observational data, and rely on the statistical surrogate criterion, either to estimate cumulative treatment effects in dynamic settings [Battocchi et al., 2021] or learn long-term optimal treatment policies [Yang et al., 2020a, Cai et al., 2021b]. Chen and Ritzwoller [2021] derive the efficiency lower bound for average long-term treatment effect in settings of Athey et al. [2019] and Athey et al. [2020]. Singh [2021, 2022] further develop debiased long-term treatment effect estimators based on machine learning nuisance estimation. In contrast, Kallus and Mao [2020], Cai et al. [2021a] combine two datasets that both satisfy unconfoundedness. Still, all of these literature rule out persistent confounding, which is the main problem tackled in this paper.

A concurrent and independent work by Ghassami et al. [2022] uses alternative conditions or additional variables to alleviate latent confounding in long-term causal inference. They focus on causal effect identification rather than estimation and propose two identification approaches. The first is based on an equi-confounding assumption analogous to the parallel-trend assumption in difference-in-differences methods [Card and Krueger, 1994]. The second requires auxiliary proxy variables satisfying certain generic conditions (in addition to the short-term outcomes) and is closest to our approach in Sections 4.1 and 4.2 and appendix C.1. Our work specifically leverages the special sequential structure of multiple short-term outcomes, and shows how such short-term outcomes can proxy the confounders. This obviates the need to search for external proxy variables and allows us to understand the different types of confounders and which need to be controlled (see appendix D.2). Importantly, we also develop estimation methods. In particular, we develop a doubly robust identification strategy in Section 4.3, which leads to the best-performing estimator in our numerical experiments. We further extend this doubly robust strategy to a setting with covariate shift in Section 6. We offer additional discussion and comparison below Corollary 1 in Appendix C.1.

Broadly speaking, there is a growing interest in combining experimental and observational data to improve causal inference [*e.g.*, Chen et al., 2021, Cheng and Cai, 2021, Yang et al., 2020b,c, Colnet et al., 2020, Kallus et al., 2018, Rosenman et al., 2022, 2020]. In these literature, the outcomes of interest can be observed in both types of data, so causal effect identification is already guaranteed by the experimental data. The goal of data combination in these literature is to improve causal effect estimation. In contrast, in our setting, data combination is crucial for causal identification since any one data set alone cannot identify the long-term treatment effect.

2.3 Proximal Causal Inference

Our identification proposals are related to the proximal causal inference literature [Tchetgen Tchetgen et al., 2020]. This line of literature proposes to deal with unmeasured confounding by proxy variables. The seminal work of Miao et al. [2016] first prove the identification of treatment effects with two different types of proxy variables for unmeasured confounders: one type of proxy variables (called negative control outcomes) are not affected by the treatment, and other type of proxy variables (called negative control treatments) do not affect the outcome. Since then, a series of

works have proposed a variety of different estimation methods for proximal causal inference [Kallus et al., 2021, Ghassami et al., 2021b, Deaner, 2021, Singh, 2020, Miao and Tchetgen, 2018, Shi et al., 2020, Mastouri et al., 2021, Cui et al., 2020]. The proximal causal inference framework has been also extended to longitudinal data analysis [Imbens et al., 2021, Ying et al., 2021, Shi et al., 2021], mediation analysis [Dukes et al., 2021, Ghassami et al., 2021a], and off-policy evaluation and learning [Bennett and Kallus, 2021, Tennenholtz et al., 2020, Qi et al., 2021, Xu et al., 2021].

The existing proximal causal inference literature focus on a single observational dataset. In contrast, in this paper we consider combining observational and experimental data. We view short-term outcomes as proxy variables for persistent unmeasured confounders. However, all of these short-term outcomes can be affected by the treatment (see Figure 3 below), so they do not satisfy the proxy conditions in Miao et al. [2016]. In this paper, we establish novel identification strategies that leverage the additional experimental data. See also discussions in Remark 2.

3 Problem Setup

We consider a binary treatment variable $A \in \mathcal{A} = \{0, 1\}$ where $A = 1$ stands for the treated group and $A = 0$ stands for the control group. We are interested in the treatment effect on a long-term outcome. Based on the potential outcome framework [Rubin, 1974], we postulate potential long-term outcomes $Y(0) \in \mathcal{Y} \subseteq \mathbb{R}$ and $Y(1) \in \mathcal{Y} \subseteq \mathbb{R}$, which would be realized were the treatment assignment equal 0 and 1 respectively. In reality, we can at most observe the potential outcome corresponding to the actual treatment assignment, *i.e.*, $Y = Y(A)$. However, these long-term outcomes may take too long time to realize so they may be difficult to observe, especially in randomized experiments that usually have short durations. Nevertheless, it is usually still possible to observe some short-term outcomes. We postulate potential short-term outcomes $S(1) \in \mathcal{S}$, $S(0) \in \mathcal{S}$, and denote the observable actual short-term outcomes as $S = S(A)$. In this paper, we consider *multiple* short-term outcomes, so by default S is a vector. The structure of these short-term outcomes will be described in section 3.3. Additionally, we can observe some pre-treatment covariates denoted as $X \in \mathcal{X}$.

We have access to two samples: an observational (O) sample with n_O units and an experimental (E) sample with n_E units. We suppose that the observational sample is a random sample from the population of interest, where for each unit i we can observe independently and identically distributed (X_i, A_i, S_i, Y_i) . The experimental sample may be a selective sample from the same population, where for each unit i we only observe (X_i, A_i, S_i) but *not* the long-term outcome. We use a binary indicator $G_i \in \{E, O\}$ to denote which sample a unit i belongs to. Without loss of generality, we can consider a combined i.i.d sample of size $n = n_O + n_E$ from an artificial super-population, namely, $\mathcal{D} = \{(G_i, X_i, A_i, S_i, Y_i \mathbb{I}[G_i = O]) : i = 1, \dots, n_O + n_E\}$. We use \mathbb{P} and \mathbb{E} to denote the probability measure and expectation with respect to this super-population, and use $p(\cdot)$ to denote the associated probability density function or probability mass function. We also denote the observational and experimental subsamples as \mathcal{D}_O and \mathcal{D}_E respectively.

In this paper, we aim to combine the observational and experimental samples to learn the average long-term treatment effect for the observational data population:

$$\tau = \mu(1) - \mu(0), \tag{1}$$

where $\mu(a) = \mathbb{E}[Y(a) \mid G = O]$.

3.1 Basic Assumptions for Observational and experimental datas

We now describe some basic assumptions for the experimental and observational data, which, unless otherwise stated, are maintained throughout this paper.

First, we consider confounded treatment assignments in the observational data:

$$(Y(a), S(a)) \not\perp A \mid X, G = O. \quad (2)$$

Confounding is common in observational studies, as the treatment assignments are not controlled, and there often exist some unmeasured confounders $U \in \mathcal{U}$.

Assumption 1 (Observational data). For $a \in \{0, 1\}$,

$$(Y(a), S(a)) \perp A \mid U, X, G = O, \quad (3)$$

and $0 < \mathbb{P}(A = 1 \mid U, X, G = O) < 1$ almost surely.

Here Equations (2) and (3) mean that U and X together account for all confounding in the observational data, but the observed covariates X alone are not enough. Moreover, we impose the overlap condition $0 < \mathbb{P}(A = 1 \mid U, X, G = O) < 1$, which is a standard assumption in causal inference literature. Because of the unmeasured confounders U , the observational data alone is not enough to identify the treatment effect parameter τ in eq. (1).

In contrast, the treatment assignments are assigned completely at random in the experimental data.

Assumption 2 (Experimental Data). For $a \in \{0, 1\}$,

$$(Y(a), S(a), U, X) \perp A \mid G = E, \quad (4)$$

and $0 < \mathbb{P}(A = 1 \mid G = E) < 1$ almost surely.

Although unconfounded, the experimental data do not contain long-term outcome observations, so the experimental data alone is not enough to identify the treatment effect either. This motivates us to combine the observational and experimental data. To this end, we further assume the following assumption.

Assumption 3 (Data Combination). For any $a \in \{0, 1\}$,

$$(S(a), U, X) \perp G, \quad (5)$$

and almost surely,

$$\frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} < \infty. \quad (6)$$

Equation (5) in Assumption 3 means that the experimental data has *external validity*, in that the distribution of $(S(a), U, X)$ in the experimental data is the same as that in the observational data (*i.e.*, the population of interest). Similar assumptions also appear in previous literature that attempt to combine different samples [*e.g.*, Athey et al., 2020, 2019, Kallus and Mao, 2020]. Equation (6) means that the conditional distributions of $(U, X) \mid A$ on the experimental and observational data have enough overlap, which is also a common assumption in missing data literature [Tsiatis, 2007]. Assumption 3 ensures that the two samples have enough commonality, so it is meaningful to combine them.

Note that Equation (5) still allows the distributions of potential long-term outcome $Y(a)$ in the experimental and observational data to be different, so that the long-term treatment effect on the experimental population can be different from our target. Moreover, in Section 6 we further relax eq. (5) to allow the distributions of covariates X to be different in the two samples.

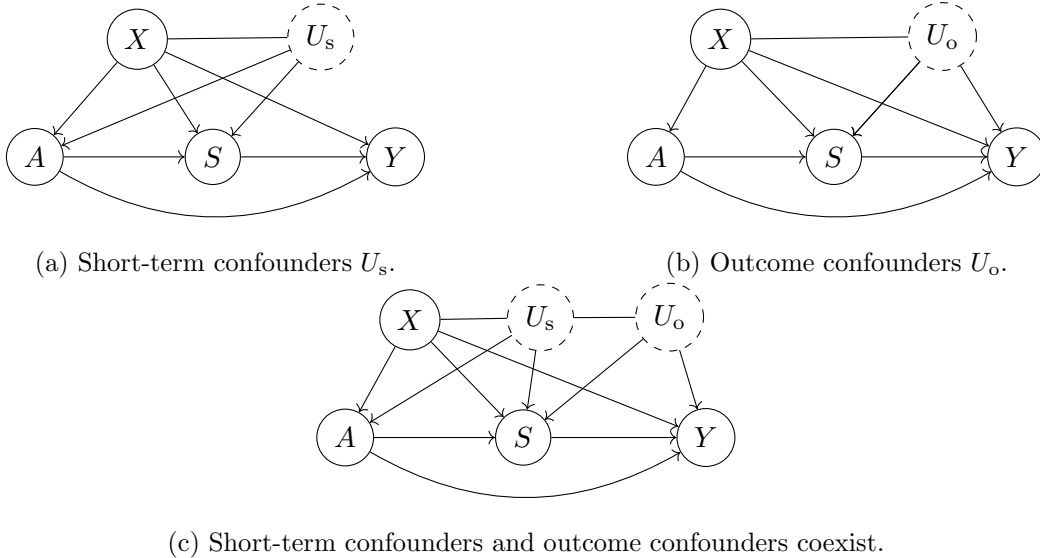


Figure 2: Short-term confounders and outcome confounders in the observational data.

3.2 Persistent Confounding

To identify the average long-term treatment effect, existing literature need to impose strong restrictions for unmeasured confounders. Athey et al. [2020] assume $Y(a) \perp A \mid S(a), X, G = O$. This assumption means that short-term outcomes can fully mediate the unmeasured confounding effects on the the long-term outcome. This is possible only when all unmeasured confounders are *short-term confounders*, *i.e.*, confounders that can affect *only* short-term outcomes, but *not* the long-term outcome (see Figure 2a). Athey et al. [2019] assume the surrogate criterion¹ proposed by Prentice [1989], requiring long-term outcome to be conditionally independent with the treatment given short-term outcomes.

However, these previous assumptions are fragile. Consider some unmeasured *outcome confounders*, *i.e.*, variables that *simultaneously* affect the short-term and long-term outcomes (see Figure 2b). Outcome confounders capture common causes for the outcomes. For example, intelligence can be an unmeasured outcome confounder for people’s short-term and long-term earnings. When unmeasured *outcome confounders* coexist with unmeasured *short-term confounders*, short-term outcomes become colliders between them (see Figure 2c), invalidating assumptions in both Athey et al. [2020] and Athey et al. [2019].

In this paper, we tackle the challenge of *persistent confounding*: we allow unmeasured confounders U to affect not only the treatment and short-term outcomes, but also the long-term outcome (see Figure 1a). This obviously accomodates coexsiting short-term confounders and outcome confounders, *i.e.*, U_s and U_o in Figure 2c respectively, as we can simply consider $U = (U_s, U_o)$. Our persistent confounding model is also far more general: it allows unmeasured variables to simultaneously affect all of treatment, short-term and long-term outcomes, instead of only a subset of them.

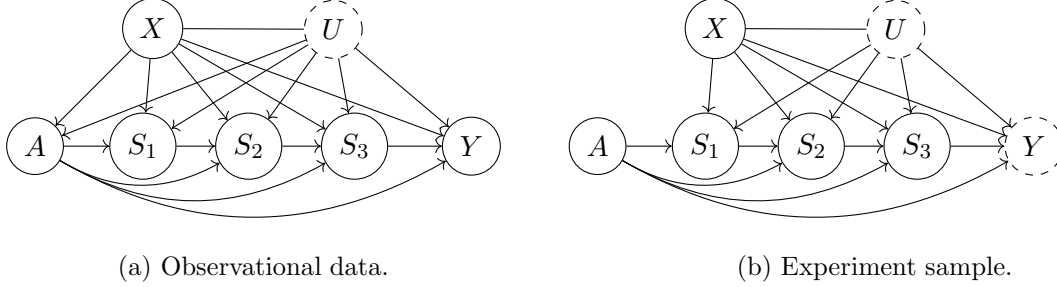


Figure 3: Sequential structure of three groups of short-term outcomes.

3.3 Three Groups of Short-term Outcomes

To address general persistent confounding, we must need additional information. In this paper, we consider leveraging *multiple short-term outcomes*. In particular, we consider *three groups* of potential short-term outcomes $S(a) = (S_1(a), S_2(a), S_3(a)) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$, and their actual counterparts $S_i = (S_1, S_2, S_3)$, both sorted in a temporal order. We assume the following conditional independence assumption for the potential short-term and long-term outcomes.

Assumption 4 (Sequential Outcomes). *For $a \in \{0, 1\}$,*

$$(Y(a), S_3(a)) \perp S_1(a) \mid S_2(a), U, X, G = O, \quad (7)$$

Assumption 4 holds when the short-term and long-term outcomes can be directly affected by *only* outcomes immediately preceding them, but *not* outcomes further in the past (see Figure 3). This captures the sequential structure of the short-term and long-term outcomes. For example, it holds when the potential outcomes follow autoregressive structural equations of suitable orders (see Example 2 below for a simple instance). Note that previous literature with multiple short-term outcomes usually view them as a whole and do not distinguish each individual outcome [*e.g.*, Athey et al., 2019, Kallus and Mao, 2020] nor leverage their internal structure. In contrast, our paper exploits their sequential structure, which is encapsulated in the conditional independence in Assumption 4. As we will show in Section 4, this sequential structure turns out to be very useful in addressing the persistent confounding challenge.

Moreover, we assume that short-term outcomes (S_1, S_3) contain enough information for the unmeasured confounders U , formalized in the following completeness conditions.

Assumption 5 (Completeness Conditions). *For any $s_2 \in \mathcal{S}_2$, $a \in \{0, 1\}$, $x \in \mathcal{X}$,*

1. *If $\mathbb{E}[g(U) \mid S_3, S_2 = s_2, A = a, X = x, G = O] = 0$ holds almost surely, then $g(U) = 0$ almost surely.*
2. *If $\mathbb{E}[g(U) \mid S_1, S_2 = s_2, A = a, X = x, G = O] = 0$ holds almost surely, then $g(U) = 0$ almost surely.*

These completeness conditions require that the short-term outcomes (S_1, S_3) are strongly dependent with the unmeasured confounders, and they have sufficient variability relative to the unmeasured confounders U . Under these conditions, (S_1, S_3) can be viewed as strong proxy variables²

¹The surrogate criterion in Athey et al. [2019] also forbids the treatment to have direct effects on the long-term outcome unmediated by the short-term outcomes (*i.e.*, no direct edge from A to Y in Figure 2a). Our paper does not need this restriction.

²Note that we do not require S_2 to be strong proxy variables. Instead, we only require S_2 to block the path between S_1 and S_3 as depicted in Figure 3, so that Assumption 4 is plausible.

for the unmeasured confounders U . Completeness conditions are widely assumed in recent proximal causal inference literature [Miao et al., 2016, Shi et al., 2020, Miao and Tchetgen, 2018, Cui et al., 2020]. However, these literature require proxy variables that are not causally affected by the treatment, termed as negative controls. In contrast, here both of (S_1, S_3) can be affected by the treatment and thus do not directly fit into these previous literature.

4 Identification

In this section, we establish three novel identification strategies for the average long-term treatment effect in presence of general persistent confounding.

4.1 Identification via Outcome Bridge Function

We first introduce the concept of outcome bridge function, which will play an important role in our first identification strategy.

Assumption 6 (Outcome Bridge Function). *There exists an outcome bridge function $h_0 : \mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$ defined as follows:*

$$\mathbb{E}[Y \mid S_2, A, U, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O]. \quad (8)$$

According to Equation (8), an outcome bridge function h_0 gives a transformation of short-term outcomes (S_3, S_2) , treatment A , and covariates X , such that the confounding effects of the unmeasured variables U on this transformation can reproduce those on the long-term outcome Y . So we can expect outcome bridge functions to be useful in tackling unmeasured confounding.

In general nonparametric models, the existence of an outcome bridge function can be ensured by the completeness conditions in Assumption 5 condition 1 and some additional technical conditions. See Appendix B for details. This means that an outcome bridge function exists when the short-term outcomes S_3 are sufficiently informative for the unmeasured confounders U . In some simple cases below, we provide more specialized existence conditions that have similar qualitative implications.

Example 1 (Discrete Setting). Suppose that $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = \{s_{(j)} : j = 1, \dots, M_s\}$ and $\mathcal{U} = \{u_{(k)} : k = 1, \dots, M_u\}$. For any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$, let $\mathbb{E}[Y \mid s_2, a, \mathbf{U}, x] \in \mathbb{R}^{M_u}$ denote the vector whose k th element is $\mathbb{E}[Y \mid S_2 = s_2, A = a, U = u_{(k)}, X = x, G = O]$, and $P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x) \in \mathbb{R}^{M_s \times M_u}$ the matrix whose (j, k) th element is

$$\mathbb{P}(S_3 = s_{(j)} \mid S_2 = s_2, A = a, U = u_{(k)}, X = x, G = O).$$

The existence of an outcome bridge function is equivalent to the existence of a solution $z \in \mathbb{R}^{M_s}$ to the following linear equation system for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$:

$$[P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x)]^\top z = \mathbb{E}[Y \mid s_2, a, \mathbf{U}, x] \quad (9)$$

This holds if the matrix $P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x)$ has a full column rank for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$. This full column rank condition means that S_3 is strongly dependent with U and the number of possible values of S_3 (*i.e.*, M_s) is no smaller than the number of possible values of U (*i.e.*, M_u). In other words, S_3 carries enough information about U .

Example 2 (Linear Model). Suppose that (Y, S_3, S_2, S_1) are generated from the following linear structural equation system:

$$\begin{aligned} Y &= \tau_y A + \alpha_y^\top S_3 + \beta_y^\top X + \gamma_y^\top U + \epsilon_y, \\ S_j &= \tau_j A + \alpha_j S_{j-1} + \beta_j X + \gamma_j U + \epsilon_j, \quad j \in \{3, 2\} \\ S_1 &= \tau_1 A + \beta_1 X + \gamma_1 U + \epsilon_1, \end{aligned}$$

where $\tau_y, (\tau_j, \alpha_y, \beta_y, \gamma_y), (\alpha_j, \beta_j, \gamma_j)$ are scalars, vectors, and matrices of conformable sizes respectively, and ϵ_y, ϵ_j are independent mean-zero noise terms such that $\epsilon_y \perp (S, A, U, X)$ and $\epsilon_j \perp (S_{j-1}, \dots, S_1, A, U, X)$. If γ_3 has a full column rank, then for any solution ω to the linear equation $\gamma_3^\top \omega = \gamma_y$, it can be easily shown that a valid outcome bridge function is

$$h_0(s_3, s_2, a, x) = \theta_3^\top s_3 + \theta_2^\top s_2 + \theta_1 a + \theta_0^\top x,$$

where $\theta_3 = \omega + \alpha_y, \theta_2 = -\alpha_3^\top \omega, \theta_1 = \tau_y - \tau_3^\top \omega, \theta_0 = \beta_y - \beta_3^\top \omega$. Full-column-rank γ_3 again means that S_3 is sufficiently informative for the unmeasured confounders U .

Note that outcome bridge functions in Equation (8) are defined in terms of unmeasured confounders, so we cannot directly use this definition to learn outcome bridge functions from observed data. In the following lemma, we give an alternative characterization of outcome bridge functions, only in terms of distributions of observed data.

Lemma 1. *Under Assumptions 1 to 4, the completeness condition in Assumption 5 condition 2 and Assumption 6, any function h_0 that satisfies*

$$\begin{aligned} \mathbb{E}[Y \mid S_2, S_1, A, X, G = O] \\ = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O] \end{aligned} \quad (10)$$

is also a valid outcome bridge function in the sense of Equation (8).

In Lemma 1, we assume the completeness condition in Assumption 5 condition 2, which requires the short-term outcomes S_1 to be informative enough for the unmeasured confounders U . Under this additional assumption, outcome bridge functions can be equivalently characterized by the conditional moment equation in Equation (10). Note that Equation (10) simply replaces the unmeasured confounders U in Equation (8) by the observed short-term outcomes S_1 . The resulting conditional moment equation only depends on observed variables.

We finally establish the identification of the average long-term treatment effect in the following theorem.

Theorem 1. *Under conditions in Lemma 1, the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (10),*

$$\begin{aligned} \tau &= \mathbb{E}[h_0(S_3, S_2, A, X) \mid A = 1, G = E] \\ &\quad - \mathbb{E}[h_0(S_3, S_2, A, X) \mid A = 0, G = E]. \end{aligned} \quad (11)$$

Theorem 1 states that the average long-term treatment effect can be recovered by marginalizing *any* outcome bridge function (which is defined on the observational data distribution) over the experimental data distribution. This shows how observational and experimental data can be combined together to identify the long-term treatment effect.

Remark 1 (Connection to Athey et al. [2020]). The proposed identification strategy in Equation (11) can be viewed as a generalization of that in Athey et al. [2020]. When there only exist short-term confounders, Athey et al. [2020] shows that we only need a single group of short-term outcomes. We can let $S_1 = S_3 = \emptyset$ and $S = S_2$, then $h_0(S_2, A, X) = \mathbb{E}[Y | S, A, X, G = O]$ is the unique solution to Equation (10), and it can be plugged into Equation (11) to identify the average long-term treatment effect. This recovers the identification strategy in Theorem 1 of Athey et al. [2020] when specialized to Assumption 3. In Appendix C.1, we further relax Assumption 3 and our identification strategy there can exactly recover the strategy of Athey et al. [2020]. See discussions below Corollary 1 in Appendix C.1.

4.2 Identification via Selection Bridge Function

The second identification strategy involves an alternative function called selection bridge function.

Assumption 7 (Selection Bridge Function). *There exists a selection bridge function $q_0 : \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$ defined as follows:*

$$\frac{p(S_2, U, X | A, G = E)}{p(S_2, U, X | A, G = O)} = \mathbb{E}[q_0(S_2, S_1, A, X) | S_2, A, U, X, G = O]. \quad (12)$$

According to Equation (12), a selection bridge function q_0 gives a transformation of short-term outcomes (S_2, S_1) , treatment A , and covariates X , which can adjust for distributional differences between the experimental and observational data. In Appendix F.1 Lemma 6, we prove that under assumption 3, the density ratio in left hand side of Equation (12) is almost surely finite, so Equation (12) is well-defined.

In general nonparametric models, the existence of a selection bridge function can be ensured by the completeness condition in Assumption 5 condition 2 and some additional technical conditions. See Appendix B for details. This means that a selection bridge function exists when the short-term outcomes S_1 are sufficiently informative for the unmeasured confounders U . We can also derive more specialized existence conditions for Examples 1 and 2 (see Appendix A).

Again, selection bridge functions in Equation (12) are defined in terms of unmeasured confounders. Below, we derive alternative characterizations in terms of distributions of observed variables.

Lemma 2. *Under assumptions 1 to 4, the completeness condition in Assumption 5 condition 1, and Assumption 7, any function q_0 that satisfies*

$$\frac{p(S_3, S_2, X | A, G = E)}{p(S_3, S_2, X | A, G = O)} = \mathbb{E}[q_0(S_2, S_1, A, X) | S_3, S_2, A, X, G = O] \quad (13)$$

or equivalently,

$$\mathbb{E}\left[\mathbb{I}[G = O] \left(\frac{\mathbb{P}(A, G = E)}{\mathbb{P}(A, G = O)} q_0(S_2, S_1, A, X) + 1 \right) | S_2, S_1, A, X\right] = 1 \quad (14)$$

is also a valid selection bridge function in the sense of Equation (12).

In Lemma 2, we assume the completeness condition in Assumption 5 condition 1, which requires the short-term outcomes S_3 to be informative enough for the unmeasured confounders U . Under this additional assumption, selection bridge functions can be equivalently characterized by the conditional moment equations in Equations (13) and (14), both involving only observed variables. Here Equation (13) is a direct analogue to Equation (12), replacing U in Equation (12) by S_3 in Equation (13). Equation (14) is a more convenient formulation for estimation as it does not involve any conditional density function.

Theorem 2. *Under conditions in Lemma 2, the average long-term treatment effect is identifiable: for any function q_0 that satisfies Equation (13) or Equation (14),*

$$\begin{aligned} \tau = & \mathbb{E} [q_0(S_2, S_1, A, X) Y \mid A = 1, G = O] \\ & - \mathbb{E} [q_0(S_2, S_1, A, X) Y \mid A = 0, G = O]. \end{aligned} \quad (15)$$

Theorem 2 states that the average long-term treatment effect can be also identified by *any* selection bridge function. This provides an alternative to the identification strategy based on outcome bridge functions in Theorem 1.

Remark 2 (Comparison with Proximal Causal Inference). As we discussed in Section 2.3, our identification strategies are related to identification in the proximal causal inference literature. Indeed, we also take a proxy variable perspective, viewing short-term outcomes (S_1, S_3) as proxy variables for the unmeasured confounders U . Moreover, the characterization for outcome bridge function h_0 given in Equation (10) has an analogue in Miao and Tchetgen [2018].

Nevertheless, our setting is substantially different from the existing proximal causal inference literature. The short-term outcomes (S_1, S_3) are both affected by the treatment, so they do not satisfy the proxy conditions in Miao et al. [2016]. Our identification strategies also feature a novel use of the experimental data. This is crucial in our setting but not considered in the previous proximal causal inference literature. Notably, our identification strategy in Theorem 2 relies on a new selection bridge function. This bridge function is specialized to our data combination setting, without any analogue in the existing proximal causal inference literature.

4.3 Doubly Robust Identification

In Sections 4.1 and 4.2, we present two different identification strategies, based on outcome bridge functions and selection bridge functions respectively. Now we combine them into a doubly robust identification strategy.

Theorem 3. *Fix functions $h : \mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$ and $q : \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$. If either conditions in Theorem 1 hold and $h = h_0$ satisfies Equation (10), or conditions in Theorem 2 hold and $q = q_0$ satisfies Equation (13) or Equation (14), then the average long-term treatment effect can be identified according to the following:*

$$\begin{aligned} \tau = & \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E] \\ & + \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} [q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid A = a, G = O]. \end{aligned} \quad (16)$$

Theorem 3 shows that Equation (16) identifies the average long-term treatment effect when it uses either a valid outcome bridge function or a valid selection bridge function. But it does not need both bridge functions to be valid. This is why it is called doubly robust.

5 Estimation

In this section, we provide three different estimators for the average long-term treat effect, corresponding to the three different identification strategies in Section 4 respectively. This involves combining two samples, so we assume that as $n \rightarrow \infty$, $n_E/n_O \rightarrow \lambda$ where $0 < \lambda < \infty$. This is a

common assumption in the data combination literature [e.g., Angrist and Krueger, 1992, Graham et al., 2016].

Apparently, before estimating the average long-term treatment effect, we need to first estimate the outcome and/or selection bridge functions. Following previous literature [e.g., Cui et al., 2020, Miao and Tchetgen, 2018], we assume that bridge functions uniquely exist (see also Remark 4 below).

Assumption 8 (Unique Bridge Functions). *1. There exists a unique outcome bridge function h_0 that satisfies Equation (8) in Assumption 6.*

2. There exists a unique selection bridge function q_0 that satisfies Equation (12) in Assumption 7.

Estimating bridge functions amounts to solving the conditional moment equations in Equations (10) and (14) based on finite-sample data. This can be realized by a variety of estimators, which we review in Remark 3 below. In this section, we consider generic bridge function estimators, which can be any from those in Remark 3, and discuss different ways to construct the long-term treatment effect estimator.

Below, we define three different estimators for the counterfactual mean parameter $\mu(a)$, $a \in \mathcal{A}$. They all use the cross-fitting technique when constructing bridge function estimators. This technique has been widely used to accommodate complex nuisance function estimators while preserving strong asymptotic guarantees [e.g., Chernozhukov et al., 2019, Zheng and Laan, 2011].

Definition 1 (Cross-fitted Counterfactual Mean Estimator). *Fix $a \in \mathcal{A}$ and an integer $K \geq 2$.*

- 1. Randomly split the observational data \mathcal{D}_O into K (approximately) even folds, denoted as $\mathcal{D}_{O,1}, \dots, \mathcal{D}_{O,K}$ respectively.*
- 2. For $k = 1, \dots, K$, use all observational data other than the k th fold, i.e., $\mathcal{D}_{O,-k} := \cup_{j \neq k} \mathcal{D}_{O,j}$, to construct the outcome bridge function estimator based on Equation (10) and/or the selection bridge function estimator based on Equation (14). Denote them as $\hat{h}_k(S_3, S_2, A, X)$ and $\hat{q}_k(S_2, S_1, A, X)$ respectively.*
- 3. Use any of the following counterfactual mean estimators:*

$$\begin{aligned} \hat{\mu}_{OTC}(a) &= \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_E^{(a)}} \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a] \hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) \right], \\ \hat{\mu}_{SEL}(a) &= \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a] \hat{q}_k(S_{2,i}, S_{1,i}, A_i, X_i) Y_i \right], \\ \hat{\mu}_{DR}(a) &= \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_E^{(a)}} \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a] \hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) \right] \\ &\quad + \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a] \hat{q}_k(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - \hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) \right) \right], \end{aligned}$$

where $n_E^{(a)} = \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a]$ and $n_{O,k}^{(a)} = \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a]$ are the numbers of units with treatment level a in the experimental data \mathcal{D}_E and the k -th fold of observational data $\mathcal{D}_{O,k}$ respectively.

Based on the counterfactual mean estimators in Definition 1, we can easily construct the average long-term treatment effect estimators:

$$\hat{\tau}_{\text{OTC}} = \hat{\mu}_{\text{OTC}}(1) - \hat{\mu}_{\text{OTC}}(0), \quad \hat{\tau}_{\text{SEL}} = \hat{\mu}_{\text{SEL}}(1) - \hat{\mu}_{\text{SEL}}(0), \quad \hat{\tau}_{\text{DR}} = \hat{\mu}_{\text{DR}}(1) - \hat{\mu}_{\text{DR}}(0).$$

To analyze the asymptotic properties of these treatment effect estimators, we need to impose some high level conditions on the bridge function estimators.

Assumption 9 (Error Rates of Bridge Function Estimators). *1. Suppose that there exists $\tilde{h} \in \mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$ and a sequence $\rho_{h,n} \rightarrow 0$ such that*

$$\|\hat{h}_k - \tilde{h}\|_{\mathcal{L}_2(\mathbb{P})} = O_{\mathbb{P}}(\rho_{h,n}), \quad \forall k \in \{1, \dots, K\}.$$

2. Suppose that there exists $\tilde{q} \in \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$ and a sequence $\rho_{q,n} \rightarrow 0$ such that

$$\|\hat{q}_k - \tilde{q}\|_{\mathcal{L}_2(\mathbb{P})} = O_{\mathbb{P}}(\rho_{q,n}), \quad \forall k \in \{1, \dots, K\}.$$

Assumption 9 specifies that the outcome bridge function estimator and selection bridge function estimator converge to some limits \tilde{h} and \tilde{q} at rates of $\rho_{h,n}$ and $\rho_{q,n}$ respectively. Note that we do not necessarily require these estimators to be consistent, *i.e.*, we allow $\tilde{h} \neq h_0$ or $\tilde{q} \neq q_0$, as we show in the following theorem.

Theorem 4 (Estimation Consistency). *1. If conditions in Theorem 1, Assumption 8 condition 1, and Assumption 9 condition 1 hold, and $\tilde{h} = h_0$, then $\hat{\tau}_{\text{OTC}}$ is consistent.*

2. If conditions in Theorem 2, Assumption 8 condition 2, and Assumption 9 condition 2 hold, and $\tilde{q} = q_0$, then $\hat{\tau}_{\text{SEL}}$ is consistent.

3. If conditions in either of the two statements above hold, then $\hat{\tau}_{\text{DR}}$ is consistent.

Theorem 4 shows that if the outcome bridge function estimator is consistent (*i.e.*, $\tilde{h} = h_0$), then the corresponding treatment effect estimator $\hat{\tau}_{\text{OTC}}$ is consistent. Similarly, if the selection bridge function estimator is consistent (*i.e.*, $\tilde{q} = q_0$), then the corresponding treatment effect estimator $\hat{\tau}_{\text{SEL}}$ is also consistent. In contrast, the estimator $\hat{\tau}_{\text{DR}}$ is more robust, in that it is consistent if either of the two bridge function estimators is consistent.

Theorem 4 establishes the consistency of treatment effect estimators given only high level conditions on the bridge function estimators, regardless of how they are actually constructed. However, the actual ways to construct bridge function estimators generally do impact the asymptotic distributions of treatment effect estimators $\hat{\tau}_{\text{OTC}}$ and $\hat{\tau}_{\text{SEL}}$. So we only focus on the asymptotic distribution of estimator $\hat{\tau}_{\text{DR}}$, which can be still derived under generic high level conditions.

Theorem 5 (Asymptotic Distribution of Doubly Robust Estimator). *Suppose that conditions in both Theorem 4 statement 1 and Theorem 4 statement 2 hold and $\rho_{h,n}\rho_{q,n} = o(n^{-1/2})$. Then as $n \rightarrow \infty$,*

$$\sqrt{n}(\hat{\tau}_{\text{DR}} - \tau) \rightsquigarrow \mathcal{N}(0, \sigma^2),$$

where

$$\begin{aligned} \sigma^2 = & \frac{1 + \lambda}{\lambda} \mathbb{E} \left[\left(\frac{A - \mathbb{P}(A = 1 | G = E)}{\mathbb{P}(A = 1 | G = E)} (h_0(S_3, S_2, A, X) - \mu(A)) \right)^2 \mid G = E \right] \\ & + (1 + \lambda) \mathbb{E} \left[\left(\frac{A - \mathbb{P}(A = 1 | G = E)}{\mathbb{P}(A = 1 | G = E)} q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) \right)^2 \mid G = O \right]. \end{aligned}$$

Theorem 5 shows that if both bridge function estimators are consistent (*i.e.*, $\tilde{h} = h_0$ and $\tilde{q} = q_0$), and the product of their convergence rates is $o(n^{-1/2})$, then the doubly robust treatment effect estimator $\hat{\tau}_{\text{DR}}$ is asymptotically normal with a closed-form asymptotic variance. One can easily estimate this asymptotic variance by plugging estimates into all unknowns therein. In the following theorem, we show that this asymptotic variance actually attains the efficiency lower bound.

Theorem 6 (Asymptotic Efficiency). *Let \mathbb{P} be a distribution instance that satisfies Assumptions 1 to 4 and 6 to 8. Then, the efficiency lower bound for the average long-term treatment effect τ locally at the distribution \mathbb{P} is equal to σ^2 given in Theorem 5.*

Theorem 6 implies that under the asserted assumptions, treatment effect estimator $\hat{\tau}_{\text{DR}}$ is asymptotically optimal, in the sense that it achieves the smallest asymptotic variance among all regular and asymptotically linear estimators [Van der Vaart, 2000].

Remark 3 (Bridge Function Estimators). Estimating bridge functions amounts to estimating roots to conditional moment equations in Equations (10) and (14). This can be implemented by a variety of methods. Examples include Generalized Method of Moments (GMM) [*e.g.*, Miao and Tchetgen, 2018, Cui et al., 2020, Hansen, 1982], sieve methods [*e.g.*, Ai and Chen, 2003, Newey and Powell, 2003, Hall and Horowitz, 2005], kernel density estimators [*e.g.*, Darolles et al., 2010, Hall and Horowitz, 2005], Reproducing Kernel Hilbert Space methods [*e.g.*, Singh et al., 2019, Ghassami et al., 2021b], neural network methods [*e.g.*, Hartford et al., 2017, Bennett et al., 2019], and more generally, adversarial learning methods [*e.g.*, Dikkala et al., 2020, Kallus et al., 2021]. We can use any of these to estimate the bridge functions. Convergence rates of the resulting bridge function estimators (in the sense of Assumption 9) can be analyzed according to these literature.

Remark 4 (Unique Bridge Function Assumption). In this section, we assume that bridge functions uniquely exist in Assumption 8. As we discussed in Sections 4.1 and 4.2, bridge functions exist if the short-term outcomes S_1 and S_3 are sufficiently informative for the unmeasured confounders. But when they are more informative than necessary, there may exist more than one bridge functions. For example, in Example 1, when the matrix $P(\mathbf{S}_3 \mid s_2, a, \mathbf{U}, x)$ has full column rank and S_3 has more values than the unmeasured confounders U , Equation (9) admits many solutions z and each of them corresponds to one different outcome bridge function. Therefore, Assumption 8 may not be true, especially when there are a large number of short-term outcomes S_1 and S_3 .

When Assumption 8 is violated and bridge functions are non-unique, it is still possible to prove consistency of the treatment effect estimators [Kallus et al., 2021, Ghassami et al., 2021b]. Nevertheless, analyzing the asymptotic distributions of these estimators becomes very challenging, as bridge function estimators may not converge to any fixed limit. This is why almost all previous proximal causal inference literature assume unique bridge functions when studying statistical inference. One exception is the regularized GMM approach proposed in Imbens et al. [2021], but it only applies when bridge functions are parametric. Therefore, we leave the statistical inference with nonunique bridge functions for future study, and only focus on unique bridge functions in this paper.

6 Extensions

In this section, we extend our identification results by relaxing Assumptions 2 and 3. In particular, we relax Assumption 3 by allowing the covariate distribution to be different in the experimental and observational data. This is an important extension because these two types of data are often

collected from different environments, where the covariate distributions are very likely to be different. For example, because observational data are usually easier to collect and have larger scale than experimental data, its covariate distribution may be more representative of the entire population of interest; while experimental data may only correspond to a selective sub-population. Therefore, we consider the following assumption to allow for different covariate distributions in two types of data. This assumption is also called *conditional external validity* in Athey et al. [2020].

Assumption 10 (Data Combination, Modified). *Suppose that for any $a \in \{0, 1\}$,*

$$(S(a), U) \perp G \mid X, \quad (17)$$

and Equation (6) holds almost surely.

Moreover, we relax Assumption 2 by allowing the treatment assignment in the experimental data to depend on covariates X , instead of being completely at random. This accommodates not only completely randomized designs but also stratified randomized designs for the experimental data.

Assumption 11 (Experimental Data, Modified). *Suppose that for any $a \in \{0, 1\}$,*

$$(Y(a), S(a), U) \perp A \mid X, G = E, \quad (18)$$

and $0 < \mathbb{P}(A = 1 \mid X, G = E) < 1$ almost surely.

Below we extend the doubly robust identification strategy in Theorem 3, which shows that the long-term average treatment effect is still identifiable under the weaker Assumptions 10 and 11.

Theorem 7. *Fix functions $h : \mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$, $q : \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$, and denote $\bar{h}_E(a, x) = \mathbb{E}[h(S_3, S_2, A, X) \mid A = a, X = x, G = E]$. Suppose Assumptions 1, 4, 10 and 11 hold, and either of the following two conditions holds:*

1. *The completeness condition in Assumption 5 condition 2 and Assumption 6 hold, and $h = h_0$ satisfies Equation (10);*
2. *The completeness condition in Assumption 5 condition 1 and Assumption 7 hold, and $q = q_0$ satisfies Equation (13) or Equation (14).*

Then the average long-term treatment effect is identifiable:

$$\begin{aligned} \tau = & \sum_{a \in \{0, 1\}} (-1)^{1-a} \left\{ \mathbb{E}[\bar{h}_E(a, X) \mid G = O] \right. \\ & + \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} (h(S_3, S_2, A, X) - \bar{h}_E(A, X)) \mid G = E \right] \\ & \left. + \mathbb{E} \left[\frac{\mathbb{P}(G = E \mid A = a) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid G = O \right] \right\} \end{aligned}$$

Theorem 7 shows that even under weaker Assumptions 10 and 11, outcome and selection bridge functions can be still used to identify the average long-term treatment effect. This again has the doubly robust property in that it only requires one of the bridge functions to be correct rather than both. Compared to Theorem 3, Theorem 7 additionally incorporates the ratios $\mathbb{P}(G = O \mid X) / \mathbb{P}(G = E \mid X)$ to adjust for the covariate distribution discrepancy in the two types

of data (Assumption 10). It also uses the propensity score $\mathbb{P}(A = a \mid X, G = E)$ to account for the dependence between treatment A and covariates X in the experimental data (Assumption 11).

In Appendix C.1, we further show that by setting $q = 0, h = h_0$ or $h = 0, q = q_0$ in Theorem 7, we can obtain direct analogues of Theorems 1 and 2 that involve only a single bridge function. In Appendix C.2, we prove that the estimating equation based on the doubly robust identification strategy in Theorem 7 satisfies the *Neyman orthogonality* property [Chernozhukov et al., 2019], and show that the resulting treatment effect estimator has appealing asymptotic properties. Moreover, in Appendix D, we present some additional extensions. In Appendix D.1, we extend our identification strategies to the setting where pre-treatment outcomes are available. In Appendix D.2, we show that it is possible to relax completeness conditions in Assumption 5, so that short-term outcomes need to be rich enough for only part of the unmeasured confounders rather than all of them.

7 Numerical Studies

In this section, we illustrate the performance of our proposed estimators using data for the Greater Avenues to Independence (GAIN) job training program in California. GAIN is a job assistance program in the late 1980s designed for low-income population. To evaluate its real impacts on employment, MDRC conducted a randomized experiment in 6 counties of California. We use the dataset analyzed in Athey et al. [2019], which contains MDRC’s experimental results from 4 out of the 6 counties in California: Alameda, Los Angeles, and San Diego and Riverside. For each experiment participant, this dataset records a binary treatment variable indicating whether being treated by the GAIN program or not, quarterly job employment information after treatment assignments, and other covariate information (*e.g.*, age, education, marriage). See Hotz et al. [2006], Athey et al. [2019] for more information about the GAIN program.

In our numerical studies, we consider the San Diego data as our experimental dataset \mathcal{D}_E , and construct an observational dataset \mathcal{D}_O based on the Riverside data via *biased subsampling*. Then we apply our proposed estimators $\hat{\tau}_{\text{OTC}}$ and $\hat{\tau}_{\text{DR}}$ to estimate the average treatment effect of the GAIN program on the long-term employment. Since the original data are obtained from randomized experiments, we consider the average treatment effect thereof as the “ground truth”, and use it to evaluate the errors of different estimators.

7.1 Data Preparation

We need to prepare for an experimental dataset \mathcal{D}_E and an observational dataset \mathcal{D}_O . For the experimental dataset, we directly use data from San Diego, which includes $n_E^{(1)} = 6978$ people in the treatment group and $n_E^{(0)} = 1154$ people in the control group. For the observational dataset, we consider inject synthetic confounding into the Riverside data (which originally include $N_1 = 4405$ people in the treatment group and $N_0 = 1040$ people in the control group).

We construct the observational dataset using a biased subsampling procedure. We randomly subsample units from the Riverside data according to a sampling probability function $\pi(A, U) \in (0, 1)$, where $A \in \{0, 1\}$ is the treatment assignment and $U \in \{0, 1, 2, 3\}$ is the highest education level (“0” means below 9-th grade, “1” means 9-th to 11-th grade, “2” means 12-th grade, and “3” means above 12-th grade). This creates dependence between the treatment assignment and the education level for the units subsampled into \mathcal{D}_O . We choose education because it is quite likely to have *persistent* effects on participants’ potential employment in all quarters following the treatment. Then we remove the education level U from \mathcal{D}_O (and also \mathcal{D}_E). As a result, the education level

becomes a plausible persistent unmeasured confounder in \mathcal{D}_O .

To quantify the strength of unmeasured confounding in \mathcal{D}_O , we index the sampling probability function $\pi(A, U)$ by a positive parameter η . We set the sampling probability for control units as $\pi(0, U) = \max\{1 - \eta U/3, 0.2\}$ and the sampling probability for treated units as $\pi(1, U)$ that satisfies the following equation:

$$\frac{N_0}{N_0 + N_1} \pi(0, U) + \frac{N_1}{N_0 + N_1} \pi(1, U) \equiv \frac{N_1}{N_1 + N_0} + \frac{N_0}{N_1 + N_0} \max\{1 - \eta, 0.2\}.$$

It is easy to show that as η grows, the discrepancy between $\pi(0, U)$ and $\pi(1, U)$ also grows. This implies stronger dependence between U and A in the observational dataset \mathcal{D}_O , thus stronger unmeasured confounding. In Appendix E Proposition 3, we prove that with this choice of $\pi(1, U)$, the subsampling procedure does not shift the distribution of education level U , so that it is not against Assumption 3. Moreover, the subsampling procedure does not influence Assumptions 1 and 2 since the sampling probability function only depends on A, U .

In our numerical studies, we consider the short-term outcomes (S_1, S_2, S_3) as the employment status in the first two quarters, in the third and fourth quarters, and in the fifth and sixth quarters after the treatment respectively. We consider the long-term outcome Y as the 20-th quarter employment. These are all binary variables indicating whether the participants are employed in the corresponding quarters after the treatment assignments.

7.2 Results

Table 1 reports the performance of different estimators over 1000 replications of the data subsampling. Each replication results in a different observational dataset \mathcal{D}_O with different number of treated units $n_O^{(1)} < N_1$ and different number of control units $n_O^{(0)} < N_0$. For evaluation we consider two criterion over the 1000 replications: the Mean Absolute Error (MAE) and the Median of Absolute Errors (MedAE).

In Table 1, we compare the performance of our proposed estimators $\hat{\tau}_{\text{OTC}}$, $\hat{\tau}_{\text{SEL}}$ and $\hat{\tau}_{\text{DR}}$ in Section 5 with two benchmarks: the naive difference-in-mean estimator that uses only the observational dataset, and the imputation estimator proposed in Section 4.1 of Athey et al. [2020] that uses both datasets and information of all short-term outcomes $S = (S_1, S_2, S_3)$. The naive estimator completely ignores confounding, and the estimator in Athey et al. [2020] can account for only short-term confounding but not persistent confounding. To evaluate the performance of our estimators and Athey et al. [2020], we consider the percentage decrease of the Mean Absolute Error (MAE) and the Median of Absolute Errors (MedAE) of these estimators relative to the naive estimator. Here, a positive value of percentage decrease means performance improvement over the naive estimator so larger value of percentage decrease indicates better performance; a negative value of percentage decrease means higher estimation error than even the naive estimator.

In our estimators and the imputation estimator in Athey et al. [2020], we need to first estimate some nuisance functions. We specify the outcome bridge function in our estimators and the imputation function in Athey et al. [2020] to be linear functions, and specify the selection bridge function in our estimators to be of the form $q(s_2, s_1, a, x) = \exp(\beta_{2,a}^\top s_2 + \beta_{1,a}^\top s_1 + \beta_{0,a}^\top x + \gamma_a)$. Since these are all simple parametric functions, we do not need the cross-fitting technique described in Section 5, but instead use the same data for nuisance estimation and the final plug-in estimation. To estimate the bridge functions, we employ the generalized method of moment (GMM) approach in Cui et al. [2020]. We consider a standard GMM approach and the approach with additional ridge regularization, *i.e.*, regularizing the L_2 norms of bridge function coefficients in the GMM objectives, as suggested by Imbens et al. [2021]. When we estimate the bridge function corresponding

		$\hat{\tau}_{\text{OTC}}$				$\hat{\tau}_{\text{SEL}}$				$\hat{\tau}_{\text{DR}}$				Athey et al.		Naive
η		0	.33	.67	1	0	.33	.67	1	0	.33	.67	1	NR	CV	
0	MAE	67	89	84	82	81	81	80	80	71	95	90	88	11	17	0.053
	Med	67	89	84	82	81	81	80	80	71	95	90	88	11	17	0.053
0.2	MAE	18	84	80	78	79	79	79	79	24	89	86	85	19	15	0.059
	Med	61	84	80	78	79	79	79	79	65	90	87	85	18	15	0.059
0.4	MAE	17	79	75	74	76	76	76	76	23	84	82	80	25	13	0.067
	Med	62	79	76	74	77	76	76	76	65	85	83	82	25	13	0.067
0.6	MAE	10	73	70	69	72	72	72	72	17	79	77	76	31	11	0.076
	Med	60	74	71	69	73	73	72	72	63	80	78	77	31	10	0.076
0.8	MAE	-25	66	64	62	67	67	67	67	-11	72	71	70	33	8	0.088
	Med	57	66	64	62	68	67	67	67	59	73	72	71	32	8	0.088
1	MAE	24	65	63	62	68	68	67	67	32	72	71	70	35	6	0.095
	Med	57	65	63	62	69	68	68	68	60	73	72	71	36	6	0.095
1.2	MAE	-267	64	62	61	68	68	68	67	-323	72	70	70	37	5	0.104
	Med	56	65	62	61	70	69	69	68	59	74	72	71	38	5	0.104
1.4	MAE	-13	62	59	58	69	68	67	67	-12	71	70	69	38	4	0.115
	Med	51	63	60	58	72	71	71	70	53	75	74	73	38	4	0.115
1.6	MAE	5	61	58	56	68	68	67	66	10	71	70	68	40	4	0.124
	Med	49	61	58	56	71	71	70	68	52	74	73	72	40	3	0.124

Table 1: Percent improvement in error over the naive estimate for different estimators, including our proposed estimators $\hat{\tau}_{\text{OTC}}$, $\hat{\tau}_{\text{SEL}}$ and $\hat{\tau}_{\text{DR}}$, the estimator proposed in Athey et al. [2020], and the naive difference-in-mean estimator. Here for our estimator we implemented using no regularization (the “0” column), ridge regularization with $\lambda = 0.33/n_O^{(a)}$, $0.67/n_O^{(a)}$ and $1/n_O^{(a)}$ (the “.33”, “.67” and “1” columns) respectively. For Athey et al. [2020], we considered using no regularization (“NR”) and using ridge regularization where the regularization parameter is selected by cross validation (“CV”). For our methods and Athey et al. [2020], we choose percentage decrease of MAE and MedAE relative to the naive difference-in-mean estimator as our evaluation criterion. In particular, larger percentage decrease means better performance. The negative percentage decrease means that the performance is worse than the naive estimator. For reference, we also added the MAE and MedAE of the naive difference-in-mean estimator.

to the treatment level $a \in \{0, 1\}$, we set the regularization tuning parameter as $\lambda = \lambda_0(n_O^{(a)})^{-1}$ for $\lambda_0 = 0, 0.33, 0.67, 1$ (note that here $\lambda_0 = 0$ corresponds to no ridge regularization). For the imputation function of Athey et al. [2020], we implement it using ordinary least squares estimation and ridge regularization with the regularization parameter chosen by cross-validation. For the cross-validated ridge regression, we choose the default option in the R package `glmnet` [Simon et al., 2011].

From Table 1, we observe that with $\lambda_0 \neq 0$, the performance of our proposed estimators $\hat{\tau}_{\text{OTC}}$, $\hat{\tau}_{\text{SEL}}$, $\hat{\tau}_{\text{DR}}$ is stable. They consistently outperform the benchmarks, in terms of both criteria. In particular, the doubly robust estimator $\hat{\tau}_{\text{DR}}$ performs the best, reducing the estimation errors of benchmark methods by large margins. Notably, although the benchmark estimator proposed by Athey et al. [2020] improves upon the naive estimator, it is always outperformed by our proposed estimators. This may be due to the fact that the estimator in Athey et al. [2020] cannot handle persistent confounding. We also observe that as the unmeasured confounding becomes stronger (*i.e.*, as η grows), all estimators have higher estimation errors, especially the naive estimator. This

validates our ways of introducing unmeasured confounding described in Section 7.1.

As mentioned before, we also implemented our estimators *without* ridge regularization (namely $\lambda_0 = 0$). We observe that the mean absolute error (MAE) of the resulting estimators $\hat{\tau}_{\text{OTC}}$, $\hat{\tau}_{\text{SEL}}$ and $\hat{\tau}_{\text{DR}}$ may be worse than Athey et al. [2020], and sometimes even worse than the naive estimator. This is because unregularized estimators sometimes can be unstable and MAE is sensitive to outlier estimates. Indeed, estimating bridge functions requires solving inverse problems defined by conditional moment equations, which can be intrinsically difficult. This problem is not specific to our setting, but also rather common for proximal causal inference [Imbens et al., 2021]. Nevertheless, median absolute errors (MedAE) is robust to outlier estimates so our estimators still outperform the benchmarks in terms of MedAE. This shows that our proposed estimators, regularized or not, always effectively reduce the confounding bias.

8 Conclusion

In this paper, we consider combining experimental and observational data for long-term causal inference. We are particularly interested in the challenge of potentially persistent confounding, *i.e.*, unmeasured confounders that can affect both the short-term and the long-term outcomes. To overcome this challenge, we leverage the sequential structure of multiple short-term outcomes and use part of them as proxy variables for the unmeasured confounders. We propose three novel identification strategies for the average long-term treatment effect. Based on each of them, we design flexible treatment effect estimators and provide asymptotic guarantees. Our results show that the long-term treatment effect can be identified and estimated under much more general conditions than before.

Beyond the specific results we present here, our work also reveals an interesting role of the structure of multiple short-term outcomes in long-term causal inference. To the best of our knowledge, the structure of repeated outcome measurements is largely unexplored in the long-term causal inference literature. We hope that our work will inspire other researchers to study plausible structures of short-term outcomes and their benefits for long-term causal inference.

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A Selection Bridge Functions in Special Examples

A.1 Discrete Setting

Recall that in Example 1, we consider $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = \{s_{(j)} : j = 1, \dots, M_s\}$ and $\mathcal{U} = \{u_{(k)} : k = 1, \dots, M_u\}$. For any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$, let $P(\mathbf{S}_1 | s_2, a, \mathbf{U}, x) \in \mathbb{R}^{M_s \times M_u}$ denote the matrix whose (j, k) th element is

$$\mathbb{P}(S_1 = s_{(j)} | S_2 = s_2, A = a, U = u_{(k)}, X = x, G = O),$$

and $r(s_2, \mathbf{U}, x; a) \in \mathbb{R}^{M_u}$ denote the vector whose k th element is

$$p(s_2, u_{(k)}, x | a, G = E) / p(s_2, u_{(k)}, x | a, G = O).$$

The existence of a selection bridge function is equivalent to the existence of a solution $z \in \mathbb{R}^{M_s}$ to the following linear equation system for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$:

$$[P(\mathbf{S}_1 | s_2, a, \mathbf{U}, x)]^\top z = r(s_2, \mathbf{U}, x; a).$$

This holds if the matrix $P(\mathbf{S}_1 | s_2, a, \mathbf{U}, x)$ has full column rank for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{X}$. This full column rank condition means that S_1 is strongly informative for U .

A.2 Linear Models

Recall that in Example 2, (Y, S_3, S_2, S_1) are generated from the following linear structural equation system:

$$\begin{aligned} Y &= \tau_y A + \alpha_y^\top S_3 + \beta_y^\top X + \gamma_y^\top U + \epsilon_y, \\ S_j &= \tau_j A + \alpha_j S_{j-1} + \beta_j X + \gamma_j U + \epsilon_j, \quad j \in \{3, 2\} \\ S_1 &= \tau_1 A + \beta_1 X + \gamma_1 U + \epsilon_1, \end{aligned}$$

where $\tau_y, (\tau_j, \alpha_y, \beta_y, \gamma_y), (\alpha_j, \beta_j, \gamma_j)$ are scalars, vectors, and matrices of conformable sizes respectively, and ϵ_y, ϵ_j are independent mean-zero noise terms such that $\epsilon_y \perp (S, A, U, X)$ and $\epsilon_j \perp (S_{j-1}, \dots, S_1, A, U, X)$.

We further assume $\mathbb{P}(A = 1 | U, X, G = E) = 1/2$ and $\mathbb{P}(A = 1 | U, X, G = O) = (1 + \exp(\kappa_1^\top U + \kappa_2^\top X))^{-1}$. We also assume that $(\epsilon_3, \epsilon_2, \epsilon_1)$ follows a joint Gaussian distribution with zero mean and a diagonal covariance matrix. Denote the covariance matrix for ϵ_j as $\sigma_j^2 I_j$ for $j = 1, \dots, 3$ where I_j is an identity matrix of formable size.

Proposition 1. *Given the data generating process described above $S_1 | S_2, A, U, X, G = O$ follows a Gaussian distribution with conditional expectation*

$$\mathbb{E}[S_1 | S_2, A, U, X, G = O] = \lambda_1 S_2 + \lambda_2 A + \lambda_3 X + \lambda_4 U,$$

where

$$\begin{aligned} \lambda_1 &= \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1}, \\ \lambda_2 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \tau_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \tau_2 \\ \lambda_3 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \beta_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \beta_2 \\ \lambda_4 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \gamma_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \gamma_2. \end{aligned}$$

When λ_4 has full column rank, then for any $\tilde{\theta}_1$ such that $\tilde{\theta}_1^\top \lambda_4 = \tilde{\kappa}_2^\top$ and $a \in \mathcal{A}$, there exists a selection bridge function of the following form for some matrices $\tilde{\theta}_2, \tilde{\theta}_0$ of conformable sizes and some constants $c_{1,a}, c_{0,a}$:

$$q_0(S_2, S_1, a, X) = c_{1,a} \exp\left((-1)^a \left(\tilde{\theta}_2^\top S_2 + \tilde{\theta}_1^\top S_1 + \tilde{\theta}_0^\top X\right)\right) + c_{0,a}.$$

B Completeness Conditions and Existence of Bridge Functions

The conditional moment equations in Equations (8) and (12) that define outcome bridge functions and selection bridge functions are Fredholm integral equations of the first kind. Following Miao et al. [2016], we characterize the existence of their solutions (*i.e.*, the outcome and selection bridge functions) by singular value decomposition of compact operators [Carrasco et al., 2007].

Let $L_2(p(z))$ denote the space of all square integrable functions of z with respect to the distribution $p(z)$. It is a Hilbert space with inner product $\langle f_1, f_2 \rangle = \int f_1(z) f_2(z) p(z) dz$. Consider linear operators $\mathcal{T}_{s_2, a, x} : L_2(p(s_3 | s_2, a, x)) \rightarrow L_2(p(u | s_2, a, x))$, $\mathcal{T}'_{s_2, a, x} : L_2(p(s_1 | s_2, a, x)) \rightarrow L_2(p(u | s_2, a, x))$ defined as follows:

$$\begin{aligned} [\mathcal{T}_{s_2, a, x} h](s_2, a, u, x) &= \mathbb{E}[h(S_3, S_2, A, X) | S_2 = s_2, A = a, U = u, X = x, G = O], \\ [\mathcal{T}'_{s_2, a, x} q](s_2, a, u, x) &= \mathbb{E}[q(S_2, S_1, A, X) | S_2 = s_2, A = a, U = u, X = x, G = O]. \end{aligned}$$

Assumption 12. *For any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{A}$,*

1. $\iint p(s_3 | s_2, a, u, x) p(u | s_3, s_2, a, x) ds_3 du < \infty$.
2. $\iint p(s_1 | s_2, a, u, x) p(u | s_1, s_2, a, x) ds_1 du < \infty$.

According to Example 2.3 in Carrasco et al. [2007], Assumption 12 ensures that for any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{A}$, $\mathcal{T}_{s_2, a, x}, \mathcal{T}'_{s_2, a, x}$ are both compact operators. Then by Theorem 2.41 in Carrasco et al. [2007], both of them admit singular value decomposition. Namely, there exist $(\lambda_{s_2, a, x, j}, \psi_{s_2, a, x, j}, \phi_{s_2, a, x, j})_{j=1}^\infty$ and $(\lambda'_{s_2, a, x, j}, \psi'_{s_2, a, x, j}, \phi'_{s_2, a, x, j})_{j=1}^\infty$ such that for any j ,

$$\begin{aligned} \mathcal{T}_{s_2, a, x} \psi_{s_2, a, x, j} &= \lambda_{s_2, a, x, j} \phi_{s_2, a, x, j} \\ \mathcal{T}'_{s_2, a, x} \psi'_{s_2, a, x, j} &= \lambda'_{s_2, a, x, j} \phi'_{s_2, a, x, j}. \end{aligned}$$

Assumption 13. For any $s_2 \in \mathcal{S}_2, a \in \mathcal{A}, x \in \mathcal{A}$,

1. $\mathbb{E}[Y \mid s_2, a, u, x, G = O]$ and $\frac{p(s_2, u, x | a, G = E)}{p(s_2, u, x | a, G = O)}$ both belong to $L_2(p(u \mid s_2, a, x))$.
2. $\sum_{j=1}^n \lambda_{s_2, a, x, j}^{-2} |\langle \mathbb{E}[Y \mid s_2, a, u, x, G = O], \phi_{s_2, a, x, j} \rangle|^2 < \infty$.
3. $\sum_{j=1}^n \lambda_{s_2, a, x, j}'^{-2} \left| \left\langle \frac{p(s_2, u, x | a, G = E)}{p(s_2, u, x | a, G = O)}, \phi'_{s_2, a, x, j} \right\rangle \right|^2 < \infty$.

Under regularity conditions in Assumptions 12 and 13, it can be shown that completeness conditions in Assumption 5 guarantee the existence of bridge functions.

Proposition 2 (Existence of Bridge Functions). *Suppose that Assumptions 12 and 13 hold.*

1. *If the completeness condition in Assumption 5 condition 1 holds, then there exists an outcome bridge function h_0 satisfying Equation (8).*
2. *If the completeness condition in Assumption 5 condition 2 holds, then there exists an outcome bridge function q_0 satisfying Equation (12).*

Proposition 2 can be proved by Picard's Theorem [Kress et al., 1989, Theorem 15.18]. See Lemma 2 in Miao et al. [2016] or Lemma 13 and 14 in Kallus et al. [2021] for details.

C Relaxing Assumptions 2 and 3

In this section, we present additional identification results under Assumptions 10 and 11 instead of the stronger conditions in Assumptions 2 and 3, and discuss their relations to the existing literature. We also discuss how to estimate the average long-term treatment effect in this case, based on the doubly robust identification strategy in Theorem 7.

C.1 Identification

In Theorem 7, we consider extending the doubly robust identification strategy in Theorem 3, which involves both outcome and selection bridge functions. We now show that based on Theorem 7, we can also extend Theorems 1 and 2.

Corollary 1. *Suppose Assumptions 1, 4, 10 and 11 holds.*

1. *If further the completeness condition in Assumption 5 condition 2 and Assumption 6 hold, then the average long-term treatment effect can be identified by any function h_0 that satisfies Equation (10):*

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E}[\mathbb{E}[h_0(S_3, S_2, A, X) \mid A = a, X, G = E] \mid G = O]. \quad (19)$$

2. *If further the completeness condition in Assumption 5 condition 1 and Assumption 7 hold, then the average long-term treatment effect can be identified by any function q_0 that satisfies Equation (13) or Equation (14):*

$$\tau = \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\frac{\mathbb{P}(G = E \mid A = a) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} \times q_0(S_2, S_1, A, X) Y \mid G = O \right] \quad (20)$$

Proof for Corollary 1. Obviously, Equation (19) can be proved by setting $h = h_0, q = 0$ in Theorem 7 and Equation (20) can be proved by setting $q = q_0, h = 0$ in Theorem 7. \square

We note that the two identification strategies in Corollary 1 are closely related to those in Athey et al. [2020], Ghassami et al. [2022]. As we discussed in Remark 1, when there is no persistent confounder, we can let $S_1 = S_3 = \emptyset$ and $S = S_2$. Then $h_0(S_2, A, X) = \mathbb{E}[Y | S, A, X, G = O]$ is the unique solution to Equation (10). As a result, the identification strategy in Equation (19) exactly recovers the identification strategy in Theorem 1 in Athey et al. [2020]. Moreover, in the setup in Ghassami et al. [2022], if we use S_3 as their short-term outcomes, S_1 as their auxiliary proxies, and condition on S_2 appropriately, then the identification strategies in their Theorems 9 and 10 coincide with ours in Equations (19) and (20) respectively. Compared to Ghassami et al. [2022], we only require short-term outcomes without needing to search for additional external proxies. Furthermore, we provide a more general doubly robust identification strategy in Theorem 7 that has no analogue in Ghassami et al. [2022]. See also discussions in Section 2.2 for additional comparisons.

Under the weaker conditions in Assumptions 10 and 11, Corollary 1 and theorem 7 shows that we need more complex identification strategies for the average long-term treatment effect over the observational data distribution. Actually, even in this case, the simpler identification strategies in Sections 4.1 to 4.3 are still useful. Below we show that under an additional assumption, they can identify average long-term treatment effect over the experimental data distribution.

Corollary 2. *Suppose Assumptions 1, 4 to 7, 10 and 11 hold and $Y(a) \perp G | S(a), U, X$. Then Equation (11) in Theorem 1, Equation (15) in Theorem 2 and Equation (16) in Theorem 3 all identify the average long-term treatment effect over the experimental data distribution, i.e.,*

$$\tau_E = \mathbb{E}[Y(1) - Y(0) | G = E],$$

In Corollary 2, we still assume the weaker conditions in Assumptions 10 and 11. But we additionally require that the experimental and observational data share a common conditional distribution of the potential long-term outcome. This additional assumption ensures that the bridge functions defined in terms of the observational data distribution can also be used to identify the average long-term treatment effect over the experimental data distribution.

C.2 Estimation

We can again leverage the doubly robust identification strategy in Theorem 7 to estimate the average long-term treatment effect. This involves some nuisance functions/parameters $\eta^* = (\eta_1^*, \eta_2^*, \dots, \eta_7^*)$:

$$\begin{aligned} \eta_1^*(S_3, S_2, A, X) &= h_0(S_3, S_2, A, X), \quad \eta_2^*(X) = \{\mathbb{E}[h_0(S_3, S_2, A, X) | A = a, X, G = E] : a = 0, 1\}, \\ \eta_3^*(S_2, S_1, A, X) &= q_0(S_2, S_1, A, X), \quad \eta_4^*(X) = \{\mathbb{P}(A = a | X, G = E) : a = 0, 1\}, \\ \eta_5^*(X) &= \frac{\mathbb{P}(G = O | X)}{\mathbb{P}(G = E | X)}, \quad \eta_6^* = \frac{\mathbb{P}(G = E | A = a)}{\mathbb{P}(G = O | A = a)}, \quad \eta_7^* = \frac{\mathbb{P}(G = E)}{\mathbb{P}(G = O)}. \end{aligned}$$

According to Theorem 7, once we know these nuisance functions/parameters, we immediately have

$$\begin{aligned} \tau &= \sum_{a \in \{0, 1\}} (-1)^{1-a} \{ \mathbb{E}[\phi_1(Y, S, a, X; \eta^*) | G = O] \\ &\quad + \mathbb{E}[\phi_2(Y, S, a, X; \eta^*) | G = E] + \mathbb{E}[\phi_3(Y, S, a, X; \eta^*) | G = O] \}, \end{aligned} \quad (21)$$

where

$$\begin{aligned}
\phi_1(Y, S, a, X; \eta^*) &= \bar{h}_{E,0}(a, X) = \mathbb{E}[h_0(S_3, S_2, a, X) \mid A = a, X, G = E] \\
\phi_2(Y, S, a, X; \eta^*) &= \frac{\mathbb{P}(G = E) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} (h_0(S_3, S_2, a, X) - \bar{h}_{E,0}(a, X)) \\
\phi_3(Y, S, a, X; \eta^*) &= \frac{\mathbb{P}(G = E \mid A = a) \mathbb{P}(G = O \mid X)}{\mathbb{P}(G = O \mid A = a) \mathbb{P}(G = E \mid X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a \mid X, G = E)} \\
&\quad \times q_0(S_2, S_1, a, X) (Y - h_0(S_3, S_2, A, X)).
\end{aligned}$$

Again, we can follow Section 5 to construct an average long-term treatment effect estimator by plugging estimates in place of unknown nuisances above. In particular, we can use a cross-fitting procedure similar to Definition 1. Namely, we first split the two datasets into multiple folds, and for each fold, we estimate the treatment effect with plug-in nuisance estimates trained on the out-of-fold data, and finally average all treatment effect estimates across different folds. We denote the resulting cross-fitted treatment effect estimator as $\hat{\tau}$.

In the following lemma, we prove that the doubly robust equation above satisfies the so-called *Neyman orthogonality* property, which means that the corresponding treatment effect estimator is insensitive to estimation errors of the nuisance functions/parameters.

Lemma 3. *The estimating equation implied by Equation (21) satisfies the Neyman Orthogonality property, namely, the pathwise derivative of the following map at η^* along any feasible direction is equal to 0:*

$$\begin{aligned}
\eta \mapsto \sum_{a \in \{0,1\}} (-1)^{1-a} \{ &\mathbb{E}[\phi_1(Y, S, a, X; \eta) \mid G = O] \\
&+ \mathbb{E}[\phi_2(Y, S, a, X; \eta) \mid G = E] + \mathbb{E}[\phi_3(Y, S, a, X; \eta) \mid G = O] \}. \quad (22)
\end{aligned}$$

The Neyman orthogonality property plays a central role in the recent debiased machine learning literature [e.g., Chernozhukov et al., 2019]. It protects the estimator of primary parameters from the errors in estimating nuisance parameters, so that even when the nuisance estimators have slow convergence rates (e.g., nonparametric machine learning estimators), the final estimator is still \sqrt{n} -consistent and asymptotically normal. In the following theorem, we show that this is the case for our cross-fitted average long-term treatment effect estimator.

Theorem 8. *Suppose Assumptions 1, 4 to 8, 10 and 11 hold, and the nuisance estimator for every function in η^* converges to the truth at $o_{\mathbb{P}}(n^{-1/4})$ rate in terms of its root mean squared error. Then*

$$\sqrt{n}(\hat{\tau} - \tau) \rightsquigarrow \mathcal{N}(0, \sigma^2),$$

where

$$\begin{aligned}
\sigma^2 &= (1 + \lambda) \mathbb{E} \left[(\phi_1(Y, S, 1, X; \eta^*) - \phi_1(Y, S, 0, X; \eta^*) - \tau)^2 \mid G = O \right] \\
&\quad + \frac{1 + \lambda}{\lambda} \mathbb{E} \left[(\phi_2(Y, S, 1, X; \eta^*) - \phi_2(Y, S, 0, X; \eta^*))^2 \mid G = E \right] \\
&\quad + (1 + \lambda) \mathbb{E} \left[(\phi_3(Y, S, 1, X; \eta^*) - \phi_3(Y, S, 0, X; \eta^*) - \tau)^2 \mid G = O \right].
\end{aligned}$$

In Theorem 8, we show that as long as the nuisance functions are consistently estimated at $o_{\mathbb{P}}(n^{-1/4})$ rate, the cross-fitted treatment effect estimator $\hat{\tau}$ is \sqrt{n} -consistent and has an asymptotic normal distribution.

D Additional Extensions

In this section, we extend our identification results to more settings. For simplicity, we focus on extending the first identification strategy in Theorem 1.

D.1 Pre-treatment Outcomes

In the main text, the short-term outcomes $S = (S_1, S_2, S_3)$ are all post-treatment outcomes. In this part, we let part of the outcomes be pre-treatment.

We first consider the setting where S_1 is pre-treatment but S_2, S_3 are post-treatment. Below we modify Assumptions 1 to 4 accordingly.

Assumption 14 (Pre-treatment S_1). *Suppose the following hold for $a \in \{0, 1\}$:*

1. *On the observational data, we have $(Y(a), S_3(a), S_2(a)) \perp A \mid S_1, U, X, G = O$ and $0 < \mathbb{P}(A = 1 \mid S_1, U, X, G = O) < 1$ almost surely.*
2. *On the experimental data, we have $(Y(a), S_3(a), S_2(a), U) \perp A \mid S_1, X, G = E$ and $0 < \mathbb{P}(A = 1 \mid S_1, X, G = O) < 1$ almost surely.*
3. *The external validity $(S_3(a), S_2(a), U) \perp G \mid S_1, X$ and overlap*

$$\frac{p(S_1, U, X \mid A = a, G = E)}{p(S_1, U, X \mid A = a, G = O)} < \infty, \quad \text{almost surely.}$$

4. *The sequential structure $(Y(a), S_3(a)) \perp S_1 \mid S_2(a), U, X, G = O$.*

Note that Assumption 14 consider the most general setting: we allow the treatment assignments in the observational and experimental data to depend on pre-treatment outcomes S_1 , and also allow the distribution of S_1 to be different on the two datasets. Now we extend our identification strategy to this setting.

Corollary 3. *Suppose conditions in Assumption 14, the completeness condition in Assumption 5 condition 2 and Assumption 6 hold. Then the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (10),*

$$\begin{aligned} \tau = & \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_1, A = 1, X, G = E] \mid G = O] \\ & - \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_1, A = 0, X, G = E] \mid G = O]. \end{aligned} \quad (23)$$

The identification strategy in Equation (23) is very similar to Equation (19). Equation (23) essentially augments the covariates X with the pre-treatment outcomes S_1 .

Similarly, we can also consider the setting where both S_1 and S_2 are pre-treatment.

Assumption 15 (Pre-treatment (S_1, S_2)). *Suppose the following hold for $a \in \{0, 1\}$:*

1. *On the observational data, we have $(Y(a), S_3(a)) \perp A \mid S_2, S_1, U, X, G = O$ and $0 < \mathbb{P}(A = 1 \mid S_2, S_1, U, X, G = O) < 1$ almost surely.*
2. *On the experimental data, we have $(Y(a), S_3(a), U) \perp A \mid S_2, S_1, X, G = E$ and $0 < \mathbb{P}(A = 1 \mid S_2, S_1, X, G = O) < 1$ almost surely.*

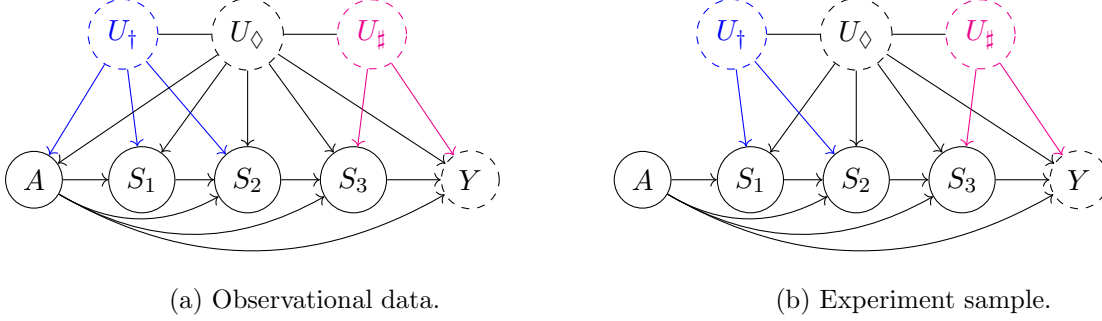


Figure 4: Unmeasured confounders ($U_{\dagger}, U_{\ddagger}$) that can be ignored. Additional covariates X can be present but we do not draw them to avoid cluttering the graphs.

3. *The external validity* ($S_3(a), U \perp G \mid S_2, S_1, X$ and overlap

$$\frac{p(S_2, S_1, U, X \mid A = a, G = E)}{p(S_2, S_1, U, X \mid A = a, G = O)} < \infty, \quad \text{almost surely.}$$

4. *The sequential structure* ($Y(a), S_3(a) \perp S_1 \mid S_2, U, X, G = O$).

We can analogously identify the long-term average treatment effect

Corollary 4. *Suppose conditions in Assumption 15, the completeness condition in Assumption 5 condition 2 and Assumption 6 hold. Then the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (10),*

$$\begin{aligned} \tau = & \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_2, S_1, A = 1, X, G = E] \mid G = O] \\ & - \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) \mid S_2, S_1, A = 0, X, G = E] \mid G = O]. \end{aligned} \quad (24)$$

D.2 Partial Confounding Adjustments

In the main text, the unobserved variables U stand for all unmeasured confounders that can possibly affect the treatment, the short-term outcomes, the long-term outcome, or any subset of them (see Figure 3). The identification strategies in Section 4 require the short-term outcomes (S_1, S_3) to be sufficiently rich relative to all of the unmeasured confounders. In this part, we show that actually we do not need to use the short-term outcomes to handle all such unmeasured confounders. Instead, we can achieve identification under lower requirements for the short-term outcomes, still using the same identification strategies.

In Figure 4, we plot three different types of unmeasured confounders: confounders U_{\diamond} can affect any of (Y, S_3, S_2, S_1, A), confounders U_{\dagger} can affect (S_2, S_1, A) but not (S_3, Y), while confounders U_{\ddagger} can affect (S_3, Y) but not (S_2, S_1, A). Naively, one can view $U = (U_{\diamond}, U_{\dagger}, U_{\ddagger})$ and argue identifiability following any of Theorems 1 to 3. This would require the short-term outcomes (S_1, S_3) to be rich enough relative to all of ($U_{\diamond}, U_{\dagger}, U_{\ddagger}$). Now we show that this is not necessary. Instead, we need (S_1, S_3) to be rich enough relative to *only* U_{\diamond} , but *not necessarily* ($U_{\dagger}, U_{\ddagger}$).

We first extend Assumptions 1 to 4 to the current setting, by substituting U_{\diamond} for U in these previous assumptions.

Assumption 16. *Assume the following conditions hold for any $a \in \{0, 1\}$:*

1. ($Y(a), S_3(a) \perp A \mid S_2(a), U_{\diamond}, X, G = O$ and $0 < \mathbb{P}(A = 1 \mid U_{\diamond}, X, G = O) < 1$ almost surely.

2. $(S_2(a), U_\diamond) \perp A \mid X, G = E$ and $0 < \mathbb{P}(A = 1 \mid X, G = E) < 1$ almost surely.

3. $(S_3(a), S_2(a), U_\diamond) \perp G \mid X$, and

$$\frac{p(U_\diamond, X \mid A = a, G = E)}{p(U_\diamond, X \mid A = a, G = O)} < \infty.$$

4. $(Y(a), S_3(a)) \perp S_1(a) \mid S_2(a), U_\diamond, X, G = O$.

It is easy to verify that the current setting depicted in Figure 4 can satisfy Assumption 16. Moreover, below we modify the completeness condition in Assumption 5 condition 2 and the outcome bridge function assumption in Assumption 6.

Assumption 17. 1. For any $s_2 \in \mathcal{S}_2$, $a \in \{0, 1\}$, $x \in \mathcal{X}$,

$$\text{if } \mathbb{E}[g(U_\diamond) \mid S_1, S_2 = s_2, A = a, X = x, G = O] = 0 \text{ holds almost surely,}$$

then $g(U_\diamond) = 0$ almost surely.

2. There exists an outcome bridge function $h_0 : \mathcal{S}_3 \times \mathcal{S}_2 \times \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$ such that

$$\mathbb{E}[Y \mid S_2, A, U_\diamond, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, A, U_\diamond, X, G = O]. \quad (25)$$

In Assumption 17(a), we assume a partial completeness condition, which only require the short-term outcomes S_1 to be rich enough relative to U_\diamond . In Assumption 17(b), we only require the bridge function to capture the unmeasured confounding due to U_\diamond . This is possible when the short-term outcomes S_3 are rich enough relative to U_\diamond . Importantly, we do not need S_1, S_3 to be rich enough relative to $(U_\diamond, U_\dagger, U_\ddagger)$ together.

Then we show that the long-term average treatment effect can be identified according to the equation we derived in Corollary 1. This means actually the same identification strategy still works under lower requirements on the short-term outcomes.

Corollary 5. Suppose Assumptions 16 and 17 hold. Then the average long-term treatment effect is identifiable: for any function h_0 that satisfies Equation (10), Equation (19) in Corollary 1 holds.

E Additional Results for Numerical Studies

In the following proposition, we justify the sampling probability function described in Section 7.1.

Proposition 3. Let (Z_1, Z_2, A) be a random vector with $(Z_1, Z_2) \perp A$ and $A \in \{0, 1\}$. Let $G \in \{0, 1\}$ be a binary random variable such that $G \perp Z_2 \mid Z_1$ and

$$\mathbb{P}(G = 1 \mid Z_1, A = 1) \mathbb{P}(A = 1) + \mathbb{P}(G = 1 \mid Z_1, A = 0) \mathbb{P}(A = 0) \equiv C,$$

where C is a positive constant. Then the probability density of (Z_1, Z_2) satisfies that

$$p(z_1, z_2 \mid G = 1) \equiv p(z_1, z_2), \quad \forall z_1, z_2.$$

We can let Z_1 be the education level U , Z_2 be other covariates and the potential short-term outcomes, A be the GAIN treatment assignment, and G be the indicator for whether being selected into the observational dataset \mathcal{D}_O . Then Proposition 3 means that the subsampling procedure does not change the distribution of latent confounders, covariates, and potential short-term outcomes. This explains why the subsampling is not against Assumption 3.

F Proofs

F.1 Supporting Lemmas

Lemma 4. *Under Assumptions 1 to 3, we have*

$$(S_3, S_2) \perp G \mid A, U, X. \quad (26)$$

Proof. For any $a \in \mathcal{A}$, $s_3 \in \mathcal{S}_3$, $s_2 \in \mathcal{S}_2$ and $g \in \{E, O\}$, we have

$$\begin{aligned} p_{S_3, S_2}(s_3, s_2 \mid U, X, A = a, G = g) &= p_{S_3(a), S_2(a)}(s_3, s_2 \mid U, X, A = a, G = g) \\ &= p_{S_3(a), S_2(a)}(s_3, s_2 \mid U, X, G = g) \\ &= p_{S_3(a), S_2(a)}(s_3, s_2 \mid U, X) \\ &= p_{S_3, S_2}(s_3, s_2 \mid U, X, A = a), \end{aligned}$$

where the second equation follows from Assumptions 1 and 2 and the third equation follows from Assumption 3. \square

Lemma 5. *Under Assumptions 1 and 4, we have*

$$(Y, S_3) \perp S_1 \mid S_2, A, U, X, G = O. \quad (27)$$

Proof. For any $a \in \mathcal{A}$, $s \in \mathcal{S}_2$ and any bounded continuous functions $f : \mathcal{Y} \times \mathcal{S}_3 \rightarrow \mathbb{R}$ and $g : \mathcal{S}_1 \rightarrow \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{E}[f(Y, S_3)g(S_1) \mid S_2 = s, U, X, A = a, G = O] \\ &= \mathbb{E}[f(Y(a), S_3(a))g(S_1(a)) \mid S_2(a) = s, U, X, A = a, G = O] \\ &= \mathbb{E}[f(Y(a), S_3(a))g(S_1(a)) \mid S_2(a) = s, U, X, G = O] \\ &= \mathbb{E}[f(Y(a), S_3(a)) \mid S_2(a) = s, U, X, G = O] \mathbb{E}[g(S_1(a)) \mid S_2(a) = s, U, X, G = O] \\ &= \mathbb{E}[f(Y, S_3) \mid S_2 = s, U, X, A = a, G = O] \mathbb{E}[g(S_1) \mid S_2 = s, U, X, A = a, G = O], \end{aligned}$$

where the second equation follows from Assumption 1, the third equation follows from Equation (7) in Assumption 4, and the fourth equation again follows from Assumption 1. \square

Lemma 6. *Under Assumption 3, for any $a \in \mathcal{A}$, the following holds almost surely:*

$$\frac{p(S_2, U, X \mid A = a, G = E)}{p(S_2, U, X \mid A = a, G = O)} = \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} < \infty \quad (28)$$

Proof. This is proved by noting that

$$\begin{aligned} \frac{p(S_2, U, X \mid A = a, G = E)}{p(S_2, U, X \mid A = a, G = O)} &= \frac{p(S_2(a), U, X \mid A = a, G = E)}{p(S_2(a), U, X \mid A = a, G = O)} \\ &= \frac{p(S_2(a) \mid U, X, A = a, G = E)}{p(S_2(a) \mid U, X, A = a, G = O)} \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} \\ &= \frac{p(U, X \mid A = a, G = E)}{p(U, X \mid A = a, G = O)} < \infty. \end{aligned}$$

where the last equation follows from Equation (5) in Assumption 3. \square

F.2 Proofs for Section 4.1

Proof for lemma 1. In lemma 5, we already proved that Assumptions 1 and 4 imply

$$(Y, S_3) \perp S_1 \mid S_2, A, U, X, G = O.$$

Therefore, for any function $h_0(S_3, S_2, A, X)$, we have

$$\begin{aligned} \mathbb{E}[Y \mid S_2, S_1, A, G = O] &= \mathbb{E}[\mathbb{E}[Y \mid S_2, S_1, A, U, X, G = O] \mid S_2, S_1, A, X, G = O] \\ &= \mathbb{E}[\mathbb{E}[Y \mid S_2, A, U, X, G = O] \mid S_2, S_1, A, X, G = O], \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O] \\ &= \mathbb{E}[\mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, S_1, A, U, X, G = O] \mid S_2, S_1, A, X, G = O] \\ &= \mathbb{E}[\mathbb{E}[h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O] \mid S_2, S_1, A, X, G = O]. \end{aligned}$$

Therefore, for any $h_0(S_3, S_2, A, X)$ that satisfies eq. (8), we have

$$\begin{aligned} 0 &= \mathbb{E}[Y - h_0(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O] \\ &= \mathbb{E}[\mathbb{E}[Y - h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O] \mid S_2, S_1, A, X, G = O]. \end{aligned}$$

It follows from the completeness condition in Assumption 5 condition 2 that

$$\mathbb{E}[Y - h_0(S_3, S_2, A, X) \mid S_2, A, U, X, G = O] = 0,$$

Namely, any function $h_0(S_3, S_2, A, X)$ that satisfies eq. (10) is a valid outcome bridge function satisfying eq. (8). \square

Proof for theorem 1. According to Lemma 1, any function h_0 that solves Equation (10) also satisfies Equation (8). Thus we only need to show that for any function h_0 that solves Equation (8), we have $\mu(a) = \mathbb{E}[h_0(S_3, S_2, A, X) \mid A = a, G = E]$. This is proved as follows:

$$\begin{aligned} &\mathbb{E}[h_0(S_3, S_2, A, X) \mid A = a, G = E] \\ &= \mathbb{E}[\mathbb{E}[h_0(S_3, S_2, a, X) \mid S_2, A = a, U, X, G = E] \mid A = a, G = E] \\ &= \mathbb{E}[\mathbb{E}[h_0(S_3, S_2, a, X) \mid S_2, A = a, U, X, G = O] \mid A = a, G = E] \\ &= \mathbb{E}[\mathbb{E}[Y \mid S_2, A = a, U, X, G = O] \mid A = a, G = E] \\ &= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = E] \\ &= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid G = E] \\ &= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid G = O] = \mathbb{E}[Y(a) \mid G = O] = \mu(a), \end{aligned}$$

where the second equation uses the fact that $G \perp S_3 \mid S_2, A = a, U, X$ (see Equation (26) in Lemma 4) and Equation (28) in Lemma 6, the third equation uses the definition of the outcome bridge function, the fourth equation uses the fact that $Y(a) \perp A \mid S_2(a), U, X, G = O$ according to Assumption 1, the fifth uses the fact that $(S_2(a), U, X) \perp A \mid G = E$ according to Assumption 2, and the sixth equation holds because $G \perp (S_2(a), U, X)$ in Assumption 3 and Equation (28) in Lemma 6. \square

F.3 Proofs for Section 4.2

Proof for Lemma 2. First note that

$$\begin{aligned}
& p(S_3, S_2, X \mid A, G = E) \\
&= \int p(S_3 \mid S_2, A, U = u, X, G = E) p(S_2, u, X \mid A, G = E) du \\
&= \int p(S_3 \mid S_2, A, U = u, X, G = O) p(S_2, u, X \mid A, G = E) du,
\end{aligned}$$

where the second equation follows from $S_3 \perp G \mid S_2, A, U, X$ that we prove in Lemma 4.

Next, note that

$$\begin{aligned}
& p(S_3, S_2, X \mid A, G = O) \mathbb{E}[q_0(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O] \\
&= p(S_3, S_2, X \mid A, G = O) \int p(u \mid S_3, S_2, A, X, G = O) \mathbb{E}[q_0(S_2, S_1, A, X) \mid S_3, S_2, A, U = u, X, G = O] du \\
&= p(S_3, S_2, X \mid A, G = O) \int p(u \mid S_3, S_2, A, X, G = O) \mathbb{E}[q_0(S_2, S_1, A, X) \mid S_2, A, U = u, X, G = O] du \\
&= \int p(S_3, S_2, u, X \mid A, G = O) \mathbb{E}[q_0(S_2, S_1, A, X) \mid S_2, A, U = u, X, G = O] du \\
&= \int p(S_3 \mid S_2, A, U = u, X, G = O) p(S_2, u, X \mid A, G = O) \mathbb{E}[q_0(S_2, S_1, A, X) \mid S_2, A, U = u, X, G = O] du,
\end{aligned}$$

where the second equation follows from $S_1 \perp S_3 \mid S_2, A, U, X, G = O$ that we prove in Lemma 5.

Therefore, any function q_0 that satisfies Equation (13) must satisfy

$$\int p(S_3 \mid S_2, A, U = u, X, G = O) \Delta(S_2, A, u, X) du = 0,$$

where

$$\Delta(S_2, A, U, X) = p(S_2, U, X \mid A, G = E) - p(S_2, U, X \mid A, G = O) \mathbb{E}[q_0(S_2, S_1, A, X) \mid S_2, A, U, X, G = O].$$

By Bayes rule, this is equivalent to

$$P(S_3 \mid S_2, A, X, G = O) \mathbb{E}\left[\frac{\Delta(S_2, A, U, X)}{p(U \mid S_2, A, X, G = O)} \mid S_3, S_2, A, X, G = O\right] = 0.$$

According to assumption 5 condition 1, we have $\Delta(S_2, A, U, X) = 0$ almost surely. In other words, if q_0 satisfies Equation (13), then it must also satisfy Equation (12).

Finally, we prove that Equation (13) is equivalent to Equation (14). Note that Equation (13) is equivalent to

$$\begin{aligned}
\mathbb{E}[q_0(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O] &= \frac{p(S_3, S_2, X \mid A, G = E)}{p(S_3, S_2, X \mid A, G = O)} \\
&= \frac{\mathbb{P}(G = E \mid S_3, S_2, A, X) \mathbb{P}(A, G = O)}{\mathbb{P}(G = O \mid S_3, S_2, A, X) \mathbb{P}(A, G = E)} \\
&= \frac{(1 - \mathbb{P}(G = O \mid S_3, S_2, A, X)) \mathbb{P}(A, G = O)}{\mathbb{P}(G = O \mid S_3, S_2, A, X) \mathbb{P}(A, G = E)}.
\end{aligned}$$

It is equivalent to

$$\begin{aligned} & \mathbb{P}(G = O \mid S_3, S_2, A, X) \mathbb{E} \left[\frac{\mathbb{P}(A, G = E)}{\mathbb{P}(A, G = O)} q_0(S_2, S_1, A, X) \mid S_3, S_2, A, X, G = O \right] \\ &= 1 - \mathbb{P}(G = O \mid S_3, S_2, A, X), \end{aligned}$$

or

$$\mathbb{P}(G = O \mid S_3, S_2, A, X) \mathbb{E} \left[\frac{\mathbb{P}(A, G = E)}{\mathbb{P}(A, G = O)} q_0(S_2, S_1, A, X) + 1 \mid S_3, S_2, A, G = O \right] = 1.$$

The conclusion then follows straightforwardly. \square

Proof for Theorem 2. According to Lemma 2, any function q_0 that solves Equation (13) or Equation (14) must also satisfy Equation (12). Thus we only need to show that for any q_0 that solves Equation (12), we have

$$\mu(a) = \mathbb{E}[q(S_2, S_1, A, X) Y \mid A = a, G = O].$$

This is proved as follows:

$$\begin{aligned} & \mathbb{E}[q_0(S_2, S_1, A, X) Y \mid A = a, G = O] \\ &= \mathbb{E}[\mathbb{E}[q_0(S_2, S_1, A, X) Y \mid S_2, A, U, X, G = O] \mid A = a, G = O] \\ &= \mathbb{E}[\mathbb{E}[q_0(S_2, S_1, A, X) \mid S_2, A, U, X, G = O] \mathbb{E}[Y \mid S_2, A, U, X, G = O] \mid A = a, G = O] \\ &= \mathbb{E} \left[\mathbb{E}[q_0(S_2(a), S_1(a), A, X) \mid S_2(a), A = a, U, X, G = O] \right. \\ & \quad \left. \times \mathbb{E}[Y(a) \mid S_2(a), A = a, U, X, G = O] \mid A = a, G = O \right] \\ &= \mathbb{E}[\mathbb{E}[q_0(S_2, S_1, A, X) \mid S_2, A, U, X, G = O] \mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = O] \\ &= \mathbb{E} \left[\frac{p(S_2, U, X \mid A, G = E)}{p(S_2, U, X \mid A, G = O)} \mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = O \right] \\ &= \mathbb{E} \left[\frac{p(S_2(a), U, X \mid A = a, G = E)}{p(S_2(a), U, X \mid A = a, G = O)} \mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = O \right] \\ &= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = E] \\ &= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid G = E] \\ &= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid G = O] \\ &= \mathbb{E}[Y(a) \mid G = O] = \mu(a). \end{aligned}$$

Here the second equation uses the fact that $Y \perp S_1 \mid S_2, A, U, X, G = O$ that we prove in Lemma 5, the fourth equation uses the fact that $Y(a) \perp A \mid S_2(a), U, X, G = O$ according to Assumption 1, the fifth equation uses the definition of the selection bridge function $q_0(S_2, S_1, A, X)$, the seventh equation uses change of measure, the eighth equation uses the fact that $A \perp (S_2(a), U, X) \mid G = E$ according to Assumption 2, and the ninth equation uses the fact that $G \perp (S_2(a), U, X)$ according to Assumption 3. \square

F.4 Proofs for Section 4.3

Proof for Theorem 3. If conditions in Theorem 1 hold and $h = h_0$ satisfies Equation (10), then

$$\begin{aligned}
& \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E] + \mathbb{E} [q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid A = a, G = O] \\
&= \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E] \\
&\quad + \mathbb{E} [q(S_2, S_1, A, X) \mathbb{E} [Y - h(S_3, S_2, A, X) \mid S_2, S_1, A, X, G = O] \mid A = a, G = O] \\
&= \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E] \\
&= \mu(a),
\end{aligned}$$

where the second equation follows from Equation (10) and the third equation follows from Theorem 1.

If conditions in Theorem 2 hold and $q = q_0$ satisfies Equation (13) or Equation (14), then

$$\begin{aligned}
& \mathbb{E} [h(S_3, S_2, A, X) \mid A = a, G = E] + \mathbb{E} [q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid A = a, G = O] \\
&= \mathbb{E} [q(S_2, S_1, A, X) Y \mid A = a, G = O] \\
&\quad - \mathbb{E} \left[h(S_3, S_2, A, X) \mathbb{E} \left[q(S_2, S_1, A, X) - \frac{p(S_3, S_2, X \mid A, G = E)}{p(S_3, S_2, X \mid A, G = O)} \mid S_3, S_2, A, X, G = O \right] \mid A = a, G = O \right] \\
&= \mathbb{E} [q(S_2, S_1, A, X) Y \mid A = a, G = O] \\
&= \mu(a),
\end{aligned}$$

where the second equation follows from Equation (13) and the third equation follows from Theorem 2. \square

F.5 Proofs for Section 5

Proof for Theorem 4. We first prove statement (2). We define

$$\tilde{\mu}_{\text{SEL}}(a) = \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a] \tilde{q}(S_{2,i}, S_{1,i}, A_i, X_i) Y_i \right].$$

Since we assume $\tilde{q} = q_0$, as $n \rightarrow \infty$, it follows from Law of Large Number and Theorem 2 that

$$\tilde{\mu}_{\text{SEL}}(a) \rightarrow \mathbb{E} [q_0(S_2, S_1, A, X) Y \mid A = a, G = O] = \mu(a).$$

Now we only need to show that $\hat{\mu}_{\text{SEL}}(a) - \tilde{\mu}_{\text{SEL}}(a) = o_p(1)$, as this would imply that $\hat{\mu}_{\text{SEL}}(a) = \mu(a) + o_p(1)$, so that $\hat{\tau}_{\text{SEL}}$ is a consistent estimator for τ . To prove this, note that

$$\hat{\mu}_{\text{SEL}}(a) - \tilde{\mu}_{\text{SEL}}(a) = \frac{1}{K} \sum_{k=1}^K \frac{n_{O,k}}{n_{O,k}^{(a)}} \Delta_{\text{SEL},k}.$$

where

$$\Delta_{\text{SEL},k} = \frac{1}{n_{O,k}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a] (\hat{q}_k(S_{2,i}, S_{1,i}, A_i, X_i) - q_0(S_{2,i}, S_{1,i}, A_i, X_i)) Y_i.$$

Then by Cauchy-Schwartz inequality, for any $k \in \{1, \dots, K\}$, we have

$$\begin{aligned} \text{Var}(\Delta_{\text{SEL},k} \mid \mathcal{D}_{O,-k}) &= \frac{1}{n_{O,k}} \text{Var}(\mathbb{I}[A=a](\hat{q}_k(S_2, S_1, A, X) - q_0(S_2, S_1, A, X))Y \mid \mathcal{D}_{O,-k}) \\ &\leq \frac{1}{n_{O,k}} \mathbb{E}\left[\left(\mathbb{I}[A=a](\hat{q}_k(S_2, S_1, A, X) - q_0(S_2, S_1, A, X))Y\right)^2 \mid \mathcal{D}_{O,-k}\right] \\ &\lesssim \frac{1}{n_{O,k}} \|\hat{q}_k - q_0\|_{\mathcal{L}_2(\mathbb{P})}^2 \leq \frac{\rho_{q,n}^2}{n_{O,k}}. \end{aligned}$$

By Markov inequality, we then have that

$$|\Delta_{\text{SEL},k}| = \mathbb{E}[|\Delta_{\text{SEL},k}| \mid \mathcal{D}_{O,-k}] + O_p\left(\frac{\rho_{q,n}}{\sqrt{n_{O,k}}}\right).$$

Here

$$\mathbb{E}[|\Delta_{\text{SEL},k}| \mid \mathcal{D}_{O,-k}] \lesssim \|\hat{q}_k - q_0\|_{\mathcal{L}_2(\mathbb{P})} = \rho_{q,n}.$$

Therefore

$$\begin{aligned} \hat{\mu}_{\text{SEL}}(a) - \tilde{\mu}_{\text{SEL}}(a) &= \frac{1}{K} \sum_{k=1}^K \frac{n_{O,k}}{n_{O,k}^{(a)}} \Delta_{\text{SEL},k} = \frac{1}{K} \sum_{k=1}^K \frac{1}{\mathbb{P}(A=a \mid G=O)} O_p\left(\rho_{q,n} + \frac{\rho_{q,n}}{\sqrt{n_{O,k}}}\right) \\ &= o_p(1). \end{aligned}$$

Similarly, we can prove that $\hat{\mu}_{\text{OTC}}(a) = \mu(a) + o_p(1)$ so that $\hat{\tau}_{\text{OTC}}$ is a consistent estimator for τ , *i.e.*, statement (1) is true.

Finally, we can similarly prove that $\hat{\mu}_{\text{DR}}(a) - \tilde{\mu}_{\text{DR}}(a) = o_p(1)$, where

$$\begin{aligned} \tilde{\mu}_{\text{DR}}(a) &= \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_E^{(a)}} \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i=a] \tilde{h}(S_{3,i}, S_{2,i}, A_i, X_i) \right] \\ &\quad + \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i=a] \tilde{q}(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - \tilde{h}(S_{3,i}, S_{2,i}, A_i, X_i) \right) \right]. \end{aligned}$$

By Law of Large Number, the limit of $\tilde{\mu}_{\text{DR}}(a)$ is

$$\mathbb{E}\left[\tilde{h}(S_3, S_2, A, X) \mid A=a, G=E\right] + \mathbb{E}\left[\tilde{q}(S_2, S_1, A, X) \left(Y - \tilde{h}(S_3, S_2, A, X)\right) \mid A=a, G=O\right].$$

According to Theorem 3, this is equal to $\mu(a)$ if either $\tilde{q} = q_0$ or $\tilde{h} = h_0$. Thus if either $\tilde{q} = q_0$ or $\tilde{h} = h_0$, $\hat{\mu}_{\text{DR}}(a) - \mu(a) = o_p(1)$ so that $\hat{\tau}_{\text{DR}}$ is a consistent estimator for τ . This proves statement (3). \square

Proof for Theorem 5. By simple algebra, we can show that

$$\begin{aligned} \hat{\mu}_{\text{DR}}(a) - \tilde{\mu}_{\text{DR}}(a) &= \frac{1}{K} \sum_{k=1}^K \frac{n_{O,k}}{n_{O,k}^{(a)}} \Delta_{\text{DR},k}^O + \frac{n_E}{n_E^{(a)}} \Delta_{\text{DR},k}^E \\ &= \frac{1}{K} \sum_{k=1}^K \frac{1}{\mathbb{P}(A=a \mid G=O) + o_p(1)} \Delta_{\text{DR},k}^O + \frac{1}{\mathbb{P}(A=a \mid G=E) + o_p(1)} \Delta_{\text{DR},k}^E \end{aligned}$$

where

$$\Delta_{\text{DR},k}^O = \frac{1}{n_{O,k}} \sum_{i \in \mathcal{D}_{O,k}} \left[\mathbb{I}[A_i = a] \hat{q}_k(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - \hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) \right) - \mathbb{I}[A_i = a] q_0(S_{2,i}, S_{1,i}, A_i, X_i) \left(Y_i - h_0(S_{3,i}, S_{2,i}, A_i, X_i) \right) \right],$$

and

$$\Delta_{\text{DR},k}^E = \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a] \left(\hat{h}_k(S_{3,i}, S_{2,i}, A_i, X_i) - h_0(S_{3,i}, S_{2,i}, A_i, X_i) \right).$$

By following the proof for Theorem 4, we can show that

$$\Delta_{\text{DR},k}^O = \mathbb{E} [\Delta_{\text{DR},k}^O | \mathcal{D}_{O,-k}] + O_p \left(\frac{\max\{\rho_{q,n}, \rho_{h,n}\}}{\sqrt{n_{O,k}}} \right) = \mathbb{E} [\Delta_{\text{DR},k}^O | \mathcal{D}_{O,-k}] + o_p \left(n^{-1/2} \right)$$

and

$$\Delta_{\text{DR},k}^E = \mathbb{E} [\Delta_{\text{DR},k}^E | \mathcal{D}_{O,-k}] + O_p \left(\frac{\rho_{h,n}}{\sqrt{n_E}} \right) = \mathbb{E} [\Delta_{\text{DR},k}^E | \mathcal{D}_{O,-k}] + o_p \left(n^{-1/2} \right).$$

Moreover, have

$$\begin{aligned} & \left| \frac{1}{\mathbb{P}(A = a | G = O)} \mathbb{E} [\Delta_{\text{DR},k}^O | \mathcal{D}_{O,-k}] + \frac{1}{\mathbb{P}(A = a | G = E)} \mathbb{E} [\Delta_{\text{DR},k}^E | \mathcal{D}_{O,-k}] \right| \\ &= \left| \mathbb{E} \left[\hat{h}_k(S_3, S_2, A, X) - h_0(S_3, S_2, A, X) \mid A = a, G = E, \mathcal{D}_{O,-k} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\hat{q}_k(S_2, S_1, A, X) \left(Y - \hat{h}_k(S_3, S_2, A, X) \right) \mid A = a, G = O, \mathcal{D}_{O,-k} \right] \right. \\ & \quad \left. - \mathbb{E} \left[q_0(S_2, S_1, A, X) \left(Y - h_0(S_3, S_2, A, X) \right) \mid A = a, G = O, \mathcal{D}_{O,-k} \right] \right| \\ &= |\mathcal{R}_{k,1} + \mathcal{R}_{k,2} + \mathcal{R}_{k,3}|. \end{aligned} \tag{29}$$

Here

$$\begin{aligned} \mathcal{R}_{k,1} &= \mathbb{E} \left[q_0(S_2, S_1, A, X) \left(\hat{h}_k(S_3, S_2, A, X) - h_0(S_3, S_2, A, X) \right) \mid A = a, G = O, \mathcal{D}_{O,-k} \right] \\ \mathcal{R}_{k,2} &= \mathbb{E} \left[\hat{q}_k(S_2, S_1, A, X) \left(h_0(S_3, S_2, A, X) - \hat{h}_k(S_3, S_2, A, X) \right) \mid A = a, G = O, \mathcal{D}_{O,-k} \right] \\ \mathcal{R}_{k,3} &= 0. \end{aligned}$$

Thus

$$\begin{aligned} \text{Equation (29)} &= \left| \mathbb{E} \left[(q_0 - \hat{q}_k) (\hat{h}_k - h_0) \mid A = a, G = O, \mathcal{D}_{O,-k} \right] \right| \\ &= \left| \mathbb{E} \left[\frac{\mathbb{I}[A = a, G = O]}{\mathbb{P}(A = a, G = O)} (q_0 - \hat{q}_k) (\hat{h}_k - h_0) \mid \mathcal{D}_{O,-k} \right] \right| \\ &\lesssim \|\hat{q}_k - q_0\|_{\mathcal{L}_2(\mathbb{P})} \|\hat{h}_k - h_0\|_{\mathcal{L}_2(\mathbb{P})} \leq \rho_{q,n} \rho_{h,n} = o(n^{-1/2}). \end{aligned}$$

It follows that $\hat{\mu}_{\text{DR}}(a) - \tilde{\mu}_{\text{DR}}(a) = o_p(n^{-1/2})$.

Furthermore,

$$\begin{aligned}
& \tilde{\mu}_{\text{DR}}(a) - \mu(a) \\
&= \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_E^{(a)}} \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a] (h_0(S_{3,i}, S_{2,i}, A_i, X_i) - \mu(a)) \right] \\
&+ \frac{1}{K} \sum_{k=1}^K \left[\frac{1}{n_{O,k}^{(a)}} \sum_{i \in \mathcal{D}_{O,k}} \mathbb{I}[A_i = a] q_0(S_{2,i}, S_{1,i}, A_i, X_i) (Y_i - h_0(S_{3,i}, S_{2,i}, A_i, X_i)) \right] \\
&= \frac{1}{\mathbb{P}(A = a | G = E) n_E} \sum_{i \in \mathcal{D}_E} \mathbb{I}[A_i = a] (h_0(S_{3,i}, S_{2,i}, A_i, X_i) - \mu(a)) \\
&+ \frac{1}{\mathbb{P}(A = a | G = O) n_O} \sum_{i \in \mathcal{D}_O} \mathbb{I}[A_i = a] q_0(S_{2,i}, S_{1,i}, A_i, X_i) (Y_i - h_0(S_{3,i}, S_{2,i}, A_i, X_i)) + o_p(n^{-1/2})
\end{aligned}$$

Combine the results above, we have

$$\begin{aligned}
\hat{\tau}_{\text{DR}} - \tau &= \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} \left[\frac{A_i - \mathbb{P}(A_i = 1 | G_i = E)}{\mathbb{P}(A_i = 1 | G_i = E) (1 - \mathbb{P}(A_i = 1 | G_i = E))} (h_0(S_{3,i}, S_{2,i}, A_i, X_i) - \mu(A_i)) \right] \\
&+ \frac{1}{n_O} \sum_{i \in \mathcal{D}_O} \left[\frac{A_i - \mathbb{P}(A_i = 1 | G_i = O)}{\mathbb{P}(A_i = 1 | G_i = O)} q_0(S_{2,i}, S_{1,i}, A_i, X_i) (Y_i - h_0(S_{3,i}, S_{2,i}, A_i, X_i)) \right] + o_p(n^{-1/2}).
\end{aligned}$$

Then the asserted conclusion follows from Central Limit Theorem. \square

Proof for theorem 6. First, consider a regular parametric submodel indexed by a parameter t : $\mathcal{P}_t = \{p_t(y, s, a, x, g) : t \in \mathbb{R}\}$ where $p_0(y, s, a, x, g)$ equals the true density $p(y, s, a, x, g)$. The associated score function is denoted as $\text{SC}(y, s, a, x, g) = \partial_t \log p_t(y, s, a, x, g)|_{t=0}$. The expectation w.r.t the distribution $p_t(y, s, a, x, g)$ is denoted by \mathbb{E}_t .

We consider the nonparametric model \mathcal{M}_{np} with the maximal tangent space:

$$\mathcal{S} = \{\text{SC}(Y, S, A, X, G) \in L_2(Y, S, A, X, G) : \mathbb{E}[\text{SC}(Y, S, A, X, G)] = 0\}. \quad (30)$$

Now we analyze the path differentiability of the counterfactual mean parameter $\mu_t(a)$ under a submodel distribution with parameter value t . According to Theorem 1, we have

$$\mu_t(a) = \mathbb{E}_t[h_t(S_3, S_2, A, X) | A = a, G = E],$$

where $h_t(S_3, S_2, A, X)$ is the corresponding outcome bridge function defined by

$$\mathbb{E}_t[Y - h_t(S_3, S_2, A, X) | S_2, S_1, A, X, G = O] = 0.$$

Note that we have

$$\begin{aligned}
\frac{\partial}{\partial t} \mu_t(a) |_{t=0} &= \frac{\partial}{\partial t} \mathbb{E}_t[h_t(S_3, S_2, A, X) | A = a, G = E] |_{t=0} \\
&= \mathbb{E}[h_0(S_3, S_2, A, X) \text{SC}(S_3, S_2, X | A, G) | A = a, G = E] \quad (31)
\end{aligned}$$

$$+ \frac{\partial}{\partial t} \mathbb{E}[h_t(S_3, S_2, A, X) | A = a, G = E] |_{t=0}. \quad (32)$$

We first analyze the term in Equation (31).

$$\begin{aligned}
& \mathbb{E}[h_0(S_3, S_2, A, X) \text{SC}(S_3, S_2, X | A, G) | A = a, G = E] \\
&= \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) \text{SC}(S_3, S_2, X | A, G) | A = a, G = E] \\
&= \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) \text{SC}(S_3, S_2, A, X, G) | A = a, G = E] \\
&= \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) \text{SC}(Y, S_3, S_2, S_1, A, X, G) | A = a, G = E] \\
&= \mathbb{E}\left[\frac{\mathbb{I}[A = a, G = E]}{\mathbb{P}(A = a, G = E)} (h_0(S_3, S_2, A, X) - \mu(a)) \text{SC}(Y, S_3, S_2, S_1, A, X, G)\right]
\end{aligned} \tag{33}$$

where the second equation holds because

$$\begin{aligned}
& \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) \text{SC}(A, G) | A = a, G = E] \\
&= \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) | A = a, G = E] \text{SC}(A = a, G = E) = 0,
\end{aligned}$$

and the third equation holds because

$$\begin{aligned}
& \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) \text{SC}(Y, S_1 | S_3, S_2, A, X, G) | A = a, G = E] \\
&= \mathbb{E}[(h_0(S_3, S_2, A, X) - \mu(a)) \mathbb{E}[\text{SC}(Y, S_1 | S_3, S_2, A, X, G) | S_3, S_2, A, X, G] | A = a, G = E] = 0.
\end{aligned}$$

Next we analyze the term in Equation (32).

$$\begin{aligned}
& \frac{\partial}{\partial t} \mathbb{E}[h_t(S_3, S_2, A, X) | A = a, G = E] |_{t=0} \\
&= \frac{\partial}{\partial t} \mathbb{E}\left[\frac{p(S_3, S_2, X | A, G = E)}{p(S_3, S_2, X | A, G = O)} h_t(S_3, S_2, A, X) | A = a, G = O\right] |_{t=0} \\
&= \frac{\partial}{\partial t} \mathbb{E}[q_0(S_2, S_1, A, X) h_t(S_3, S_2, A, X) | A = a, G = O] |_{t=0} \\
&= \mathbb{E}\left[q_0(S_2, S_1, A, X) \frac{\partial}{\partial t} \mathbb{E}[h_t(S_3, S_2, A, X) | S_2, S_1, A, X, G = O] |_{t=0} | A = a, G = O\right],
\end{aligned}$$

where the second equation holds because of Equation (13).

Furthermore, by taking the derivative of the left hand side w.r.t t at $t = 0$, we have

$$\begin{aligned}
& \frac{\partial}{\partial t} \mathbb{E}[h_t(S_3, S_2, A, X) | S_2, S_1, A, X, G = O] |_{t=0} \\
&= \mathbb{E}[(Y - h_0(S_3, S_2, A, X)) \text{SC}(Y, S_3 | S_2, S_1, A, X, G) | S_2, S_1, A, X, G = O] = 0.
\end{aligned} \tag{34}$$

It follows that

$$\begin{aligned}
& \frac{\partial}{\partial t} \mathbb{E}[h_t(S_3, S_2, A, X) | A = a, G = E] |_{t=0} \\
&= \mathbb{E}[q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) \text{SC}(Y, S_3 | S_2, S_1, A, X, G) | A = a, G = O] \\
&= \mathbb{E}[q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) \text{SC}(Y, S_3, S_2, S_1, A, X, G) | A = a, G = O] \\
&= \mathbb{E}\left[\frac{\mathbb{I}[A = a, G = O]}{\mathbb{P}(A = a, G = O)} q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) \text{SC}(Y, S_3, S_2, S_1, A, X, G)\right],
\end{aligned} \tag{35}$$

where the second equation holds because

$$\begin{aligned}
& \mathbb{E}[q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)) \text{SC}(S_2, S_1, A, X, G) | A = a, G = O] \\
&= \mathbb{E}[q_0(S_2, S_1, A, X) \mathbb{E}[Y - h_0(S_3, S_2, A, X) | S_2, S_1, A, X, G = O] \\
&\quad \times \text{SC}(S_2, S_1, A, X, G = O) | A = a, G = O] = 0.
\end{aligned}$$

Combining Equations (33) and (35), we have

$$\frac{\partial}{\partial t} \mu_t(a) |_{t=0} = \mathbb{E} [\psi_a(Y, S_3, S_2, S_1, A, X, G) \text{SC}(Y, S_3, S_2, S_1, A, X, G)],$$

where

$$\begin{aligned} \psi_a(Y, S_3, S_2, S_1, A, X, G) &= \frac{\mathbb{I}[A = a, G = E]}{\mathbb{P}(A = a, G = E)} (h_0(S_3, S_2, A, X) - \mu(a)) \\ &\quad + \frac{\mathbb{I}[A = a, G = O]}{\mathbb{P}(A = a, G = O)} q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} \tau_t |_{t=0} &= \frac{\partial}{\partial t} \mu_t(1) |_{t=0} - \frac{\partial}{\partial t} \mu_t(0) |_{t=0} \\ &= \mathbb{E} [\psi(Y, S_3, S_2, S_1, A, X, G) \text{SC}(Y, S_3, S_2, S_1, A, X, G)], \end{aligned}$$

where

$$\begin{aligned} &\psi(Y, S_3, S_2, S_1, A, X, G) \\ &= \psi_1(Y, S_3, S_2, S_1, A, X, G) - \psi_0(Y, S_3, S_2, S_1, A, X, G) \\ &= \frac{\mathbb{I}[G = E]}{\mathbb{P}(G = E)} \frac{A - \mathbb{P}(A = 1 | G = E)}{\mathbb{P}(A = 1 | G = E)} (h_0(S_3, S_2, A, X) - \mu(A)) \\ &\quad + \frac{\mathbb{I}[G = O]}{\mathbb{P}(G = O)} \frac{A - \mathbb{P}(A = 1 | G = O)}{\mathbb{P}(A = 1 | G = O)} q_0(S_2, S_1, A, X) (Y - h_0(S_3, S_2, A, X)). \end{aligned}$$

Obviously $\psi(Y, S_3, S_2, S_1, A, X, G) \in \mathcal{S}$ for the maximal tangent space \mathcal{S} in Equation (30). Thus $\psi(Y, S_3, S_2, S_1, A, X, G)$ is the efficient influence function for τ , and its variance, which is equal to σ^2 in Theorem 6, is the semiparametric efficiency lower bound for τ relative to the maximal tangent space \mathcal{S} in Equation (30). \square

F.6 Proofs for Section 6

Proof for Theorem 7. Before proving the theorem, we note that by Bayes rule, we can easily verify that

$$\frac{\mathbb{P}(G = E | A = a) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O | A = a) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} = \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = O)} \frac{p(X | A = a, G = O)}{p(X | A = a, G = E)}.$$

Assume that condition 1 holds so we have $h = h_0$ satisfying Equation (10). In this case, for any function q , we have

$$\begin{aligned} &\mathbb{E} \left[\frac{\mathbb{P}(G = E | A = a) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O | A = a) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) | G = O \right] \\ &= \mathbb{E} \left[\frac{\mathbb{P}(G = E | A = a) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O | A = a) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} q(S_2, S_1, A, X) \right. \\ &\quad \left. \times \mathbb{E}[Y - h_0(S_3, S_2, A, X) | S_2, S_1, A, X, G = O] | G = O \right] = 0, \quad (36) \end{aligned}$$

where the last equation uses the conditional moment equation in Equation (10).

Moreover, for function $h = h_0$,

$$\begin{aligned} & \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} (h(S_3, S_2, A, X) - \bar{h}_E(A, X)) | G = E \right] \\ &= \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O) \mathbb{P}(G = E | X)} \mathbb{E} [h(S_3, S_2, a, X) - \bar{h}_E(a, X) | A = a, X, G = E] | G = E \right] = 0. \end{aligned} \quad (37)$$

Finally, we only need to prove that

$$\mu(a) = \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) | A = a, X = x, G = E] | G = O] \quad (38)$$

According to lemma 1, we already know that any function $h_0(S_3, S_2, A, X)$ that satisfies Equation (10) must be a valid bridge function in the sense of Equation (8). Thus we only need to prove Equation (38) for $h_0(S_3, S_2, A, X)$ that satisfies Equation (8). By following the proof in Theorem 1, we can show that

$$\mathbb{E} [h_0(S_3, S_2, A, X) | A = a, X, G = E] = \mathbb{E} [\mathbb{E} [Y(a) | S_2(a), U, X, G = O] | A = a, X, G = E].$$

Therefore,

$$\begin{aligned} & \mathbb{E} [\mathbb{E} [h_0(S_3, S_2, A, X) | A = a, X, G = E] | G = O] \\ &= \mathbb{E} [\mathbb{E} [\mathbb{E} [Y(a) | S_2(a), U, X, G = O] | X, G = E] | G = O] \\ &= \mathbb{E} [\mathbb{E} [\mathbb{E} [Y(a) | S_2(a), U, X, G = O] | X, G = O] | G = O] \\ &= \mathbb{E} [\mathbb{E} [Y(a) | X, G = O] | G = O] = \mathbb{E} [Y(a) | G = O]. \end{aligned}$$

Here the first equation follows from the fact that $A \perp (S(a), U) | X, G = E$ in Assumption 11, the second equation follows from Equation (28) in Lemma 6, and the third equation follows from the fact that $G \perp (S(a), U) | X$ in Assumption 10.

Combining Equations (36) to (38) proves the conclusion.

Assume that condition 2 holds so we have $q = q_0$ satisfying Equation (13) or Equation (14). We first prove that

$$\begin{aligned} \mu(a) &= \mathbb{E} \left[\frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = O)} \frac{p(X | A = a, G = O)}{p(X | A = a, G = E)} q_0(S_2, S_1, A, X) Y | G = O \right] \\ &= \mathbb{E} \left[\frac{\mathbb{P}(G = E | A = a) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O | A = a) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} q(S_2, S_1, A, X) Y | G = O \right]. \end{aligned} \quad (39)$$

To prove this, note that according to Lemma 2, any function $q_0(S_2, S_1, A, X)$ that satisfies Equation (13) or Equation (14) is a valid selection bridge function in the sense of Equation (12). Thus we only need to prove Equation (39) for any q_0 that satisfies Equation (12). We further note that

the right hand side of Equation (39) is equal to the following:

$$\begin{aligned}
& \mathbb{E} \left[\mathbb{E} \left[\frac{p(X | A = a, G = O)}{p(X | A = a, G = E)} q_0(S_2, S_1, A, X) Y | A = a, X, G = O \right] | G = O \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\mathbb{E} \left[\frac{p(X | A = a, G = O)}{p(X | A = a, G = E)} q_0(S_2, S_1, A, X) | S_2, A = a, U, X, G = O \right] \right. \right. \\
&\quad \left. \left. \times \mathbb{E}[Y | S_2, A = a, U, X, G = O] | A = a, X, G = O \right] | G = O \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\frac{p(S_2, U | A, X, G = E)}{p(S_2, U | A, X, G = O)} \mathbb{E}[Y(a) | S_2(a), U, X, G = O] | A = a, X, G = O \right] | G = O \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[Y(a) | S_2(a), U, X, G = O] | A = a, X, G = E \right] | G = O \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[Y(a) | S_2(a), U, X, G = O] | X, G = E \right] | G = O \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[Y(a) | S_2(a), U, X, G = O] | X, G = O \right] | G = O \right] \\
&= \mathbb{E}[Y(a) | G = O] = \mu(a).
\end{aligned}$$

Here the first equation uses $Y \perp S_1 | S_2, A, U, X, G = O$ which we prove in Lemma 5, the second equation uses the fact that q_0 satisfies Equation (12) and $Y(a) \perp A | S_2(a), U, X, G = O$ according to Assumption 1, the fourth equation uses that $S_2(a) \perp A | X, G = E$ according to Assumption 11, the fifth equation uses the fact that $S_2(a) \perp G | X$ according to Assumption 10.

Next, we can follow the proof above to show that for any h ,

$$\begin{aligned}
& \mathbb{E} \left[\frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = O)} \frac{p(X | A = a, G = O)}{p(X | A = a, G = E)} q_0(S_2, S_1, A, X) h(S_3, S_2, A, X) | G = O \right] \\
&= \mathbb{E}[h(S_3(a), S_2(a), a, X) | G = O]
\end{aligned} \tag{40}$$

And by change of measure, we can also verify that

$$\begin{aligned}
& \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} h(S_3, S_2, A, X) | G = E \right] \\
&= \mathbb{E}[\mathbb{E}[h(S_3, S_2, A, X) | A = a, X, G = E] | G = O] = \mathbb{E}[h(S_3(a), S_2(a), a, X) | G = O)], \tag{41}
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} \bar{h}_E(A, X) | G = E \right] \\
&= \mathbb{E}[\mathbb{E}[\bar{h}_E(A, X) | A = a, X, G = E] | G = O] = \mathbb{E}[\bar{h}_E(A, X) | G = O]
\end{aligned}$$

These show that

$$\begin{aligned}
0 &= \mathbb{E}[h_E(a, X) | G = O] \\
&+ \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = E)} (h(S_3, S_2, A, X) - \bar{h}_E(A, X)) | G = E \right] \\
&- \mathbb{E} \left[\frac{\mathbb{I}[A = a]}{\mathbb{P}(A = a | X, G = O)} \frac{p(X | A = a, G = O)}{p(X | A = a, G = E)} q_0(S_2, S_1, A, X) h(S_3, S_2, A, X) | G = O \right]. \tag{42}
\end{aligned}$$

Combining Equations (39) and (42) leads to the conclusion. \square

F.7 Proofs for Appendix

Proof for Proposition 1. First note that

$$\begin{aligned} \frac{p(S_2, U, X | A = a, G = E)}{p(S_2, U, X | A = a, G = O)} &= \frac{p(U, X | A = a, G = E)}{p(U, X | A = a, G = O)} \\ &= \frac{\mathbb{P}(A = a | U, X, G = E) \mathbb{P}(A = a | G = O)}{\mathbb{P}(A = a | U, X, G = O) \mathbb{P}(A = a | G = E)} \\ &= \frac{\mathbb{P}(A = a | G = O)}{\mathbb{P}(A = a | U, X, G = O)}, \end{aligned}$$

where the first equation follows from Lemma 6, the second equation follows from Bayes rule, and the third equation follows from the fact that $\mathbb{P}(A = a | U, X, G = E) = \mathbb{P}(A = a | G = E) = \frac{1}{2}$. Therefore, we have

$$\frac{p(S_2, U, X | A = a, G = E)}{p(S_2, U, X | A = a, G = O)} = \frac{\mathbb{E} \left[\left[1 + \exp \left((-1)^a (\kappa_1^\top U + \kappa_2^\top X) \right) \right]^{-1} \right]}{\left[1 + \exp \left((-1)^a (\kappa_1^\top U + \kappa_2^\top X) \right) \right]^{-1}}. \quad (43)$$

Second, $(S_1, S_2) | A, U, X, G = O$ follows a joint Gaussian distribution whose conditional expectation is

$$\begin{bmatrix} \tau_1 A + \beta_1 X + \gamma_1 U \\ (\tau_2 + \alpha_2 \tau_1) A + (\beta_2 + \alpha_2 \beta_1) X + (\gamma_2 + \alpha_2 \gamma_1) U \end{bmatrix}$$

and conditional covariance matrix is

$$\begin{bmatrix} \sigma_1^2 I_1 & \sigma_1^2 \alpha_2^\top \\ \sigma_1^2 \alpha_2 & \sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \end{bmatrix}.$$

It follows that $S_1 | S_2, A, U, X, G = O$ also has a Gaussian distribution function with conditional expectation

$$\lambda_1 S_2 + \lambda_2 A + \lambda_3 X + \lambda_4 U$$

and conditional variance

$$\Sigma_{1|2} = \sigma_1^2 I_1 - \sigma_1^4 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2.$$

where

$$\begin{aligned} \lambda_1 &= \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1}, \\ \lambda_2 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \tau_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \tau_2 \\ \lambda_3 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \beta_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \beta_2 \\ \lambda_4 &= \left(I_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \alpha_2 \right) \gamma_1 - \sigma_1^2 \alpha_2^\top \left(\sigma_1^2 \alpha_2 \alpha_2^\top + \sigma_2^2 I_2 \right)^{-1} \gamma_2. \end{aligned}$$

Third, for $a = 1$, we posit a selection bridge function of the following form:

$$q_0(S_2, S_1, 1, X) = c_1 \exp \left(\tilde{\theta}_2^\top S_2 + \tilde{\theta}_1^\top S_1 + \tilde{\theta}_0^\top X \right) + c_0.$$

It follows that

$$\begin{aligned}
& \mathbb{E}[q_0(S_2, S_1, 1, X) \mid S_2, A = 1, U, X, G = O] \\
&= c_1 \exp\left(\tilde{\theta}_2^\top S_2 + \tilde{\theta}_0^\top X\right) \mathbb{E}\left[\exp\left(\tilde{\theta}_1^\top S_1\right) \mid S_2, A = 1, U, X, G = O\right] + c_0 \\
&= c_1 \exp\left(\tilde{\theta}_2^\top S_2 + \tilde{\theta}_0^\top X\right) \exp\left(\tilde{\theta}_1^\top (\lambda_1 S_2 + \lambda_2 A + \lambda_3 X + \lambda_4 U) + \frac{1}{2} \tilde{\theta}_1^\top \Sigma_{1|2} \tilde{\theta}_1\right) + c_0 \\
&= c_1 \exp\left(\frac{1}{2} \tilde{\theta}_1^\top \Sigma_{1|2} \tilde{\theta}_1\right) \exp\left(\left(\tilde{\theta}_1^\top \lambda_1 + \tilde{\theta}_2^\top\right) S_2 + \tilde{\theta}_1^\top \lambda_2 A + \left(\tilde{\theta}_1^\top \lambda_3 + \tilde{\theta}_0^\top\right) X + \tilde{\theta}_1^\top \lambda_4 U\right) + c_0
\end{aligned}$$

Thus we only need the above to match Equation (43) for $a = 1$. This is possible once λ_4 has full column rank: then there exists $\tilde{\theta}_1$ such that $\tilde{\theta}_1^\top \lambda_4 = \kappa_2^\top$. Then we can choose $\tilde{\theta}_1, \tilde{\theta}_0, c_1, c_0$ accordingly. Analogously, we can also show the existence of a selection bridge function $q_0(S_2, S_1, 0, X)$ of the same form for $a = 0$. \square

Proof for Corollary 2. We first prove the conclusion for Equation (11) in Theorem 1. Following the proof for Theorem 1, we have

$$\begin{aligned}
& \mathbb{E}[h_0(S_3, S_2, A, X) \mid A = a, G = E] \\
&= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid G = E] \\
&= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = E] \mid G = E] = \mathbb{E}[Y(a) \mid G = E],
\end{aligned}$$

where the second follows from the assumption that $Y(a) \perp G \mid S(a), U, X$.

Next, we prove the conclusion for Equation (15) in Theorem 2. Following the proof for Theorem 2, we have

$$\begin{aligned}
& \mathbb{E}[q_0(S_2, S_1, A, X) Y \mid A = a, G = O] \\
&= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = O] \mid A = a, G = E] \\
&= \mathbb{E}[\mathbb{E}[Y(a) \mid S_2(a), U, X, G = E] \mid G = E] \\
&= \mathbb{E}[Y(a) \mid G = E] = \mu(a),
\end{aligned}$$

where the second equation follows from the assumption that $Y(a) \perp G \mid S(a), U, X$.

Finally, according to the proof of Theorem 3, if conditions in Theorem 1 hold and $h = h_0$ satisfies Equation (10), then

$$\begin{aligned}
& \mathbb{E}[h(S_3, S_2, A, X) \mid A = a, G = E] + \mathbb{E}[q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid A = a, G = O] \\
&= \mathbb{E}[h(S_3, S_2, A, X) \mid A = a, G = E].
\end{aligned}$$

If conditions in Theorem 2 hold and $q = q_0$ satisfies Equation (13) or Equation (14), then

$$\begin{aligned}
& \mathbb{E}[h(S_3, S_2, A, X) \mid A = a, G = E] + \mathbb{E}[q(S_2, S_1, A, X) (Y - h(S_3, S_2, A, X)) \mid A = a, G = O] \\
&= \mathbb{E}[q(S_2, S_1, A, X) Y \mid A = a, G = O].
\end{aligned}$$

Then the conclusion follows from our proof above. \square

Proof for Lemma 3. We denote the map in Equation (22) as $\Phi(\eta)$. Then we need to prove that

$$\dot{\Phi}_j(\eta^*)[\eta_j - \eta_j^*] := \frac{\partial}{\partial t} \Phi(\eta_1^*, \dots, \eta_j^* + t(\eta_j - \eta_j^*), \dots, \eta_7^*)|_{t=0} = 0, \text{ for any } \eta_j \text{ and } j \in \{1, \dots, 7\}.$$

First, we note that

$$\begin{aligned} & \dot{\Phi}_1(\eta^*)[\eta_1 - \eta_1^*] \\ &= \sum_{a \in \{0,1\}} (-1)^{1-a} \left\{ \mathbb{E} \left[\frac{\mathbb{P}(G=E) \mathbb{P}(G=O|X)}{\mathbb{P}(G=O) \mathbb{P}(G=E|X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a|X, G=E)} (h - h_0)(S_3, S_2, a, X) \mid G=E \right] \right. \\ & \left. - \mathbb{E} \left[\frac{\mathbb{P}(G=E|A=a) \mathbb{P}(G=O|X)}{\mathbb{P}(G=O|A=a) \mathbb{P}(G=E|X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a|X, G=E)} q_0(S_2, S_1, a, X) (h - h_0)(S_3, S_2, a, X) \mid G=O \right] \right\} \\ &= \sum_{a \in \{0,1\}} (-1)^{1-a} \{ \mathbb{E}[(h - h_0)(S_3(a), S_2(a), a, X) \mid G=O] - \mathbb{E}[(h - h_0)(S_3(a), S_2(a), a, X) \mid G=O] \} = 0, \end{aligned}$$

where the second equation follows from Equations (40) and (41) in the proof for Theorem 7.

Second, we have that

$$\begin{aligned} \dot{\Phi}_2(\eta^*)[\eta_2 - \eta_2^*] &= \sum_{a \in \{0,1\}} (-1)^{1-a} \left\{ \mathbb{E}[\bar{h}_E(a, X) - \bar{h}_{0,E}(a, X) \mid G=O] \right. \\ & \left. - \mathbb{E} \left[\frac{\mathbb{P}(G=E) \mathbb{P}(G=O|X)}{\mathbb{P}(G=O) \mathbb{P}(G=E|X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a|X, G=E)} (\bar{h}_E(a, X) - \bar{h}_{E,0}(a, X)) \mid G=E \right] \right\} \\ &= \sum_{a \in \{0,1\}} (-1)^{1-a} \{ \mathbb{E}[\bar{h}_E(a, X) - \bar{h}_{0,E}(a, X) \mid G=O] - \mathbb{E}[\bar{h}_E(a, X) - \bar{h}_{0,E}(a, X) \mid G=O] \} = 0, \end{aligned}$$

where the equation follows from the proof for Theorem 7.

Third, we have

$$\begin{aligned} \dot{\Phi}_3(\eta^*)[\eta_3 - \eta_3^*] &= \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\frac{\mathbb{P}(G=E|A=a) \mathbb{P}(G=O|X)}{\mathbb{P}(G=O|A=a) \mathbb{P}(G=E|X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a|X, G=E)} \right. \\ & \quad \left. \times (q - q_0)(S_2, S_1, a, X) (Y - h_0(S_3, S_2, A, X)) \mid G=O \right] \\ &= \sum_{a \in \{0,1\}} (-1)^{1-a} \mathbb{E} \left[\frac{\mathbb{P}(G=E|A=a) \mathbb{P}(G=O|X)}{\mathbb{P}(G=O|A=a) \mathbb{P}(G=E|X)} \frac{\mathbb{I}[A=a]}{\mathbb{P}(A=a|X, G=E)} \right. \\ & \quad \left. \times (q - q_0)(S_2, S_1, a, X) \mathbb{E}[Y - h_0(S_3, S_2, A, X) \mid S_2, S_1, A=a, X, G=O] \mid G=O \right] \\ &= 0. \end{aligned}$$

Fourth, we have

$$\begin{aligned}
& \dot{\Phi}_4(\eta^*)[\eta_4 - \eta_4^*] \\
= & \sum_{a \in \{0,1\}} (-1)^{1-a} \left\{ \mathbb{E} \left[\frac{\mathbb{P}(G = E) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O) \mathbb{P}(G = E | X)} \right. \right. \\
& \times \frac{\mathbb{I}[A = a]}{\mathbb{P}^2(A = a | X, G = E)} (\eta_4^* - \eta_4) \mathbb{E} [h_0(S_3, S_2, a, X) - \bar{h}_{E,0}(a, X) | A = a, X, G = O] | G = O \Big] \\
& + \mathbb{E} \left[\frac{\mathbb{P}(G = E | A = a) \mathbb{P}(G = O | X)}{\mathbb{P}(G = O | A = a) \mathbb{P}(G = E | X)} \frac{\mathbb{I}[A = a]}{\mathbb{P}^2(A = a | X, G = E)} (\eta_4^* - \eta_4) \right. \\
& \left. \left. \times q_0(S_2, S_1, a, X) \mathbb{E}[Y - h_0(S_3, S_2, A, X) | S_2, S_1, A = a, X] | G = O \right] = 0.
\end{aligned}$$

Following this proof for $\dot{\Phi}_4(\eta^*)[\eta_4 - \eta_4^*] = 0$, we can similarly show that $\dot{\Phi}_j(\eta^*)[\eta_j - \eta_j^*] = 0$ for $j = 5, 6, 7$. \square

Proof for Theorem 8. Given the asserted conditions, according to Theorem 3.1 in Chernozhukov et al. [2019], we have

$$\begin{aligned}
\hat{\tau} - \tau = & \frac{1}{n_O} \sum_{i \in \mathcal{D}_O} (\phi_1(Y_i, S_i, 1, X_i; \eta^*) - \phi_1(Y_i, S_i, 0, X_i; \eta^*) - \tau) + (\phi_3(Y_i, S_i, 1, X_i; \eta^*) - \phi_3(Y_i, S_i, 0, X_i; \eta^*)) \\
& + \frac{1}{n_E} \sum_{i \in \mathcal{D}_E} (\phi_2(Y_i, S_i, 1, X_i; \eta^*) - \phi_2(Y_i, S_i, 0, X_i; \eta^*)).
\end{aligned}$$

Then the asserted conclusion follows from central limit theorem. \square

Proof for corollary 3. We can first follow the proof for Theorem 1 to show that for any $h_0(S_3, S_2, A, X)$ that satisfies Equation (8),

$$\begin{aligned}
& \mathbb{E}[h_0(S_3, S_2, A, X) | S_1, A = a, X, G = E] \\
& = \mathbb{E}[\mathbb{E}[Y(a) | S_2(a), U, X, G = O] | S_1, A = a, X, G = E].
\end{aligned}$$

The rest of the proof is analogous to Corollary 1. \square

Proof for corollary 4. We can first follow the proof for Theorem 1 to show that for any $h_0(S_3, S_2, A, X)$ that satisfies Equation (8),

$$\begin{aligned}
& \mathbb{E}[h_0(S_3, S_2, A, X) | S_2, S_1, A = a, X, G = E] \\
& = \mathbb{E}[\mathbb{E}[Y(a) | S_2(a), U, X, G = O] | S_2, S_1, A = a, X, G = E].
\end{aligned}$$

The rest of the proof is analogous to Corollary 1. \square

Proof for Corollary 5. First, note that under Assumptions 16 and 17, we can follow the proofs for Lemmas 4 and 5 to show that $S_3 \perp G | S_2, A = a, U, X$, and $(Y, S_3) \perp S_1 | S_2, A, U, X, G = O$.

Second, following the proof for Lemma 1, we can show that for any function h_0 that satisfies Equation (10), it must also satisfy

$$\mathbb{E}[Y | S_2, A, U, X, G = O] = \mathbb{E}[h_0(S_3, S_2, A, X) | S_2, A, U, X, G = O]. \quad (44)$$

Finally, we can follow the proof for Corollary 1 to show that for any function h_0 that satisfies Equation (44), Equation (19) in Corollary 1 holds. This concludes the proof for Corollary 5. \square

Proof of Proposition 3. We already have $Z_2 \perp G \mid Z_1$. Thus, we only need to verify $G \perp Z_1$. Note that

$$\begin{aligned}
 p(z_1 \mid G = 1) &= p(z_1, A = 1 \mid G = 1) + p(z_1, A = 0 \mid G = 1) \\
 &= \frac{\mathbb{P}(G = 1 \mid A = 1, Z_1 = z_1) \mathbb{P}(A = 1) p(z_1)}{\mathbb{P}(G = 1)} \\
 &\quad + \frac{\mathbb{P}(G = 1 \mid A = 0, Z_1 = z_1) \mathbb{P}(A = 0) p(z_1)}{\mathbb{P}(G = 1)} \\
 &= p(z_1) \frac{C}{\mathbb{P}(G = 1)} \propto p(z_1),
 \end{aligned}$$

which proves the desired result. □