

Tight detection efficiency bounds of Bell tests in no-signaling theories

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No-signaling theories, which can contain nonlocal correlations stronger than quantum correlations but limited by the no-signaling condition, have deepened our understanding of the quantum theory. In practice, Bell tests are powerful tools to certify nonlocality, but their effectiveness is limited by the detector efficiency. In this work, we provide almost tight detection efficiency bounds for showing the nonlocality of no-signaling theories, by introducing a general class of Bell inequalities. In particular, we provide a tight bound for the bipartite case and an asymptotic tight bound for the multipartite case. The tightness of these bounds shows that they well characterize the structure of no-signaling theories. The bounds also imply that the detector efficiency requirement can be made arbitrarily low in both bipartite and multipartite cases by increasing the number of measurement settings. Furthermore, our work sheds light on the detector efficiency requirement for showing the nonlocality of the quantum theory.

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Quantum mechanics allows distant parties to be entangled in a way that is stronger than classical physics. Bell tests, being an elegant illustration of this phenomenon, perform projective measurements on entangled parties who cannot signal to each other, for instance, enforced by spacelike separation or a shield between the distant parties, to generate correlations of measurement outcomes that are impossible for the classical theory (i.e., local realism) [1]. By exploiting Bell tests, a more general theory beyond quantum mechanics is provided by Popescu and Rohrlich, which still keeps the no-signaling condition [2]. This immediately raises several interesting questions. One is to understand what additional restrictions are needed besides the no-signaling condition, which stimulates a line of works on quantum foundations [3]. Another is whether there exists a theory more nonlocal than the quantum theory in nature, such as the one only limited by the no-signaling condition (later referred to as *the maximally nonlocal theory*). Posing constraints in addition to the no-signaling condition, such as the uncertainty principle, induces various no-signaling theories, that are less nonlocal than the maximally nonlocal theory. Though such theories have not been experimentally found, researchers postulate that they may exist in ultra-high-density objects such as black holes [4] or when the scale is out of the quantum regime, for example, smaller than a Planck length. Independently, this research has spurred interest for device-independent quantum information processing where the adversary is only limited to the no-signaling condition [5,6].

In practice, implementing Bell tests has a major obstacle which is low detection efficiency. In fact, loophole-free Bell tests have only been very recently demonstrated [7–9]. They are all based on Bell inequalities that have two measurement settings for two parties and thus need a detection efficiency bound of at least $2/3$ [10]. It was known that $2/3$ is tight for two measurement settings [11]. However, beyond that, the efficiency bounds of no-signaling theories for more measurement settings and/or parties are largely unknown,

even for the most-studied quantum theory. The first result in this direction is due to Larsson and Semitecolos [12], who show that, for N parties and two measurement settings, the detector efficiency requirement is no larger than $N/(2N + 1)$. This bound is later shown to be tight by Massar and Pironio [10]. In the case of two parties, the best upper bound with four measurement settings is 61.8% [13] and with a large number of measurement settings the upper bound can approach zero [14]. In the tripartite case, the best upper bound for eight measurement settings is 0.501 [15]. For an infinite number of parties, a series of works [16–18] have led to the best upper bound $2/(2 + M)$ for M measurement settings.

In systems with high loss (e.g., optical systems), Bell tests with more than two measurement settings are more important than the ones with two measurement settings because they potentially have a lower efficiency requirement. Consequently they could make experimental realizations easier. The minimum detector efficiency to violate the local realism can also be viewed as a measure to characterize nonlocal correlations and thus is an important operational quantity.

II. PRELIMINARIES

Before continuing, we first formalize the problem of Bell tests. In Bell tests, there are two space-separated parties, A and B . In each turn, A (B) will be given a number $x \in [1, M_A]$ ($y \in [1, M_B]$) randomly. Then A and B will output the outcomes a and b , respectively, according to the inputs and their shared resources.

Since the two parties cannot signal to each other in a Bell test, A (B) is unaware of the input to B (A). Thus a no-signaling probability p^{NS} satisfies

$$\begin{aligned} \forall x, y \quad p^{NS}(a|x, y) &= p^{NS}(a|x), \\ p^{NS}(b|x, y) &= p^{NS}(b|y), \end{aligned} \quad (1)$$

which represents that the probability of one party A (B) outputting an outcome a (b) is independent of the input of the other party B (A) and only depends on its own input x (y).

Denote the detector efficiency as η , which is independent of the input. The probability under such inefficiency of detectors, denoted as p_η^{NS} , is related to the ideal no-signaling probability

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p^{NS} by

$$\begin{aligned} p_\eta^{NS}(a,b|x,y) &= \eta^2 p^{NS}(a,b|x,y), \\ p_\eta^{NS}(\Phi,\Phi) &= (1-\eta)^2, \\ p_\eta^{NS}(a,\Phi|x) &= \eta(1-\eta)p^{NS}(a|x), \\ p_\eta^{NS}(\Phi,b|y) &= \eta(1-\eta)p^{NS}(b|y), \end{aligned} \quad (2)$$

where Φ denotes the empty output corresponding to a failed detection. The derivation of these relations is straightforward. For example, for the first relation, the outcomes a and b would be obtained only when both detectors respond. Hence the probability shrinks by a factor η^2 because the detector of each party responds with a probability η and the detectors of different parties are independent.

A local hidden variable (LHV) model assumes that A and B share a random variable λ but cannot communicate. The strategy of A (B) can then be characterized by the probability $p_A(a|x,\lambda)$ ($p_B(b|y,\lambda)$), which uses λ and x (y) to determine the probability of outputting a (b). If p_η^{NS} can be simulated by a LHV model, there exists a local strategy such that for any outcomes a and b (including the empty output Φ)

$$p_\eta^{NS}(a,b|x,y) = \int_\lambda p(\lambda)p(a|x,\lambda)p(b|y,\lambda)d\lambda. \quad (3)$$

Since the efficiency η and the simulation capability of a LHV model are monotonic (as shown in the Appendix), one can define the minimum value of detector efficiency η_{NS}^* for showing the nonlocality of the maximally nonlocal theory.

III. RESULTS

We are now ready to state our main results.

Theorem 1. In Bell tests with two parties,

$$\eta_{NS}^* \geq \frac{M_A + M_B - 2}{M_A M_B - 1}, \quad (4)$$

where M_A and M_B are the numbers of inputs of A and B , respectively.

Proof. Theorem 1 in Ref. [10] designs a LHV model to simulate any quantum strategy if the efficiency is lower than or equals $(M_A + M_B - 2)/(M_A M_B - 1)$. Since this design only leverages the no-signaling condition, the efficiency bound also holds in the maximally nonlocal theory which completes the proof. ■

When $M_A = M_B = 2$, Theorem 1 gives a well-known detector efficiency bound of $2/3$ which is tight for the quantum theory [12]. We now show that, for arbitrary M_A and M_B , the efficiency bound $(M_A + M_B - 2)/(M_A M_B - 1)$ is tight for the maximally nonlocal theory. The same statement is open for the quantum theory.

Theorem 2. In two-party Bell tests with M_A and M_B inputs, respectively,

$$\eta_{NS}^* \leq \frac{M_A + M_B - 2}{M_A M_B - 1}. \quad (5)$$

Proof. Denote P as the minimal prime number such that $P \geq \max(M_A, M_B)$. We construct a Bell inequality with P outputs excluding Φ :

$$\begin{aligned} I_{M_A M_B (P+1)(P+1)} &= \sum_{a,b,x,y} f(a,b,x,y)p(a,b|x,y) + \sum_{a,x} g(a,x)p(a|x) + \sum_{b,y} h(b,y)p(b|y), \\ f(a,b,x,y) &= \begin{cases} 1 & ((x > 1) \vee (y > 1)) \wedge (a + b \equiv xy) \\ 0 & (x = 1) \wedge (y = 1) \wedge (a + b \equiv xy) \\ -P^4 & \text{otherwise} \end{cases}, \\ g(a,x) &= \begin{cases} -1 & x > 1 \\ 0 & x = 1 \end{cases}, \quad h(b,y) = \begin{cases} -1 & y > 1 \\ 0 & y = 1 \end{cases}, \end{aligned} \quad (6)$$

where $1 \leq x \leq M_A, 1 \leq y \leq M_B, 0 \leq a, b < P$ and the modulo is over P . An illustration of this Bell inequality is shown in Fig. 1(a).

First, we prove $I_{M_A M_B (P+1)(P+1)} \leq 0$ with LHV strategies. Since LHV strategies can be regarded as the linear combination of deterministic strategies, the LHV strategy can be assumed deterministic:

$$F(x) = a, G(y) = b. \quad (7)$$

In other words, the input will determine output by the functions F and G .

Case 1. If $F(x) = \Phi \forall x$, then $I_{M_A M_B (P+1)(P+1)} \leq 0$. Similarly, if $G(y) = \Phi \forall y$, then $I_{M_A M_B (P+1)(P+1)} \leq 0$.

Case 2. If $x_1 \neq x_2, y_1 \neq y_2, a_1 = F(x_1), a_2 = F(x_2), b_1 = G(y_1), b_2 = G(y_2)$ and none of a_1, a_2, b_1, b_2 equals Φ ,

then one of $f(a_1, b_1, x_1, y_1), f(a_1, b_2, x_1, y_2), f(a_2, b_1, x_2, y_1), f(a_2, b_2, x_2, y_2)$ must equal $-P^4$. Otherwise

$$a_1 + b_1 \equiv x_1 y_1 \pmod{P}, \quad (8)$$

$$a_1 + b_2 \equiv x_1 y_2 \pmod{P}, \quad (9)$$

$$a_2 + b_1 \equiv x_2 y_1 \pmod{P}, \quad (10)$$

$$a_2 + b_2 \equiv x_2 y_2 \pmod{P}. \quad (11)$$

By applying Eq. (8)+Eq. (11)–Eq. (9)–Eq. (10), we get $(x_1 - x_2)(y_1 - y_2) \equiv 0 \pmod{P}$. Since P is a prime number, and $1 \leq x_1 \neq x_2 \leq P, 1 \leq y_1 \neq y_2 \leq P$, there is a contradiction. Since P^4 is bigger than the sum of positive coefficients in the Bell inequality, $I_{M_A M_B (P+1)(P+1)} \leq 0$ follows.

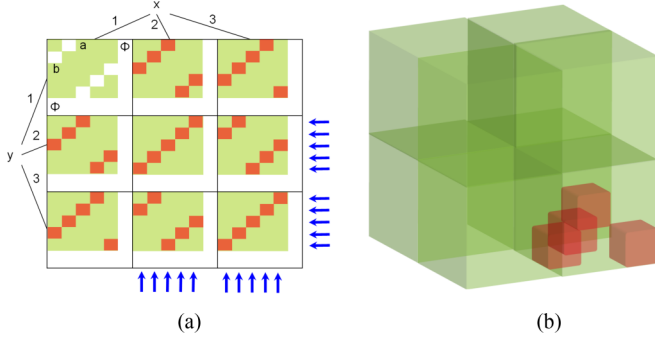


FIG. 1. Geometric interpretation of the Bell inequalities $I_{M_A M_B (P+1)(P+1)}$ and I_{P+1}^{multi} . (a) In $I_{M_A M_B (P+1)(P+1)}$, which is a linear combination of $p(a, b|x, y)$, the coefficients of which are first divided into blocks according to x and y and then divided into entries according to a and b . The red, green, and white entries are one, $-P^4$, and zero, respectively. Each blue arrow pointing to a column or a row is a term $-p(a|x)$ or $-p(b|y)$ in $I_{M_A M_B (P+1)(P+1)}$. (b) I_{P+1}^{multi} is similarly divided into higher-dimensional blocks and entries. The red entries still maintain the property that there is one and only one red entry on every line within a block according to the construction.

Case 3. If there is only one x such that $F(x) = a \neq \Phi$, the Bell inequality is simplified to

$$\begin{aligned} & \sum_y f(a, G(y), x, y) + \sum_y h(G(y), y) + g(a, x) \\ &= \sum_{y \neq 1} (f(a, G(y), x, y) - 1) + f(a, G(1), x, 1) + g(a, x) \\ &\leq f(a, G(1), x, 1) + g(a, x) \leq 0. \end{aligned} \quad (12)$$

If there is only one y such that $G(y) = b \neq \Phi$, similarly $I_{M_A M_B (P+1)(P+1)} \leq 0$.

Next, we construct a no-signaling strategy:

$$p^{NS}(a, b|x, y) = \begin{cases} \frac{1}{P} & a + b \equiv xy \\ 0 & \text{otherwise} \end{cases}. \quad (13)$$

This probability distribution satisfies the no-signaling condition because when x, y, a are determined there is a unique b such that $p^{NS}(a, b|x, y) = \frac{1}{P}$ and therefore

$$\forall y, \quad p^{NS}(a|x, y) = \sum_b p^{NS}(a, b|x, y) = \frac{1}{P}; \quad (14)$$

similarly $\forall x, \quad p^{NS}(b|x, y) = 1/P$. Its Bell value is

$$\begin{aligned} & I_{M_A M_B (P+1)(P+1)} \\ &= \sum_{a, b, x, y} f(a, b, x, y) p^{NS}(a, b|x, y) + \sum_{a, x} g(a, x) p^{NS}(a|x) \\ &+ \sum_{b, y} h(b, y) p^{NS}(b|y) \\ &= \sum_{x > 1, y > 1} \eta^2 - \sum_{x > 1} \eta - \sum_{y > 1} \eta \\ &= \eta^2 (M_A M_B - 1) - \eta (M_A + M_B - 2). \end{aligned} \quad (15)$$

Recall that $\eta > \eta_{NS}^*$ is necessary to violate the Bell inequality, i.e., $I_{M_A M_B (P+1)(P+1)} > 0$. Hence Eq. (5) holds. ■

Through proving Theorem 2, we design a useful Bell inequality $I_{M_A M_B (P+1)(P+1)}$. It has a similar condition with the Clauser-Horne-Shimony-Holt (CHSH) inequality: $a \oplus b \equiv xy$ [19], but generalizes module 2 to module P . Compared to another generalization of the CHSH inequality [20], our Bell inequality is more advantageous at persisting nonlocality when detector inefficiency occurs. There are relatively few quantum efficiency upper bound results. The quantum efficiency upper bound is 61.8% when $M_A = M_B = 4$, showing the $2/3$ bound can be beaten with relatively few inputs [13]. With more inputs $M_A = M_B = 2^d$, the quantum efficiency bound can be as low as $\eta = d^{1/2} 2^{-0.0035d}$ [14], which is, however, hard to be met in practice, requiring 10^{285} inputs when the efficiency is $1/10$. Our Bell inequality suggests much fewer inputs might suffice.

We next generalize the efficiency bound η_{NS}^* to N parties and have the following lower bound.

Theorem 3. In Bell tests with $N \leq 500$ space-separated parties, the efficiency bound satisfies

$$\eta_{NS}^* \geq \frac{N}{M(N-1)+1}, \quad (16)$$

where all N parties have M inputs.

Proof. From Ref. [10] Conjecture 2, a multipartite LHV model is designed to simulate any quantum strategy when the efficiency is no larger than $M/[M(N-1)+1]$ and $N \leq 500$. Since the only condition used in that proof is the no-signaling condition, this finishes the proof. Reference [10] also conjectures that this bound holds for any value of N . ■

When $N \rightarrow \infty$, this theorem suggests $1/M$ is the detector efficiency lower bound for showing nonlocality. We also have the following asymptotically matched upper bound.

Theorem 4. Consider N -party Bell tests with M_1, M_2, \dots, M_N inputs, respectively:

$$\eta_{NS}^* \leq \left(\frac{M_1 + M_2 + \dots + M_N - N}{M_1 M_2 \dots M_N - 1} \right)^{\frac{1}{N-1}}. \quad (17)$$

Proof. Denote P as the minimal prime number such that $P > \max(M_1, M_2, \dots, M_N)$. We construct a general Bell inequality with P outputs excluding Φ :

$$\begin{aligned} & I_{P+1}^{\text{multi}} = \sum_{\mathbf{a}, \mathbf{x}} f(\mathbf{a}, \mathbf{x}) p(\mathbf{a}|\mathbf{x}) + \sum_i \sum_{a_i, x_i} g_i(a_i, x_i) p(a_i|x_i), \\ & f(\mathbf{a}, \mathbf{x}) = \begin{cases} 1 & (\exists t, x_t \neq 1) \wedge (\sum_j a_j \equiv \prod_i x_i) \\ 0 & (\forall t, x_t = 1) \wedge (\sum_i a_i \equiv \prod_i x_i) \\ -P^{2N} & \text{otherwise} \end{cases}, \\ & g_i(a_i, x_i) = \begin{cases} -1 & x_i > 1 \\ 0 & x_i = 1 \end{cases}, \end{aligned} \quad (18)$$

where $\mathbf{a} = \{a_1, \dots, a_N\}$, $\mathbf{x} = \{x_1, \dots, x_N\}$, $1 \leq x_i \leq M_i, 0 \leq a_i < P$, and the modulo is over P . An illustration of this Bell inequality is shown in Fig. 1(b).

First, we prove $I_{P+1}^{\text{multi}} \leq 0$ for any LHV strategy. Since a LHV strategy can always be regarded as a linear combination of deterministic strategies, we consider a deterministic strategy $F_i(x_i) = a_i \forall i$.

Case 1. If there exists some i such that $F_i(x_i) = \Phi \forall 1 \leq x_i \leq M_i$, then $I_{P+1}^{\text{multi}} \leq 0$.

Case 2. There exist $i, j, x'_i, x''_i, x'_j, x''_j$, where x'_i, x''_i are different inputs of the i th party; x'_j, x''_j are different inputs of the j th party, such that none of the corresponding outputs $a'_i = F_i(x'_i), a''_i = F_i(x''_i), a'_j = F_j(x'_j), a''_j = F_j(x''_j)$ equals Φ . Also for any k , there exists x_k such that $F_k(x_k) = a_k \neq \Phi$ because otherwise Case 1 arises. Denote $\mathbf{a}_1 = \{a_k\}$, $\mathbf{x}_1 = \{x_k\}$ and $\mathbf{a}_{1a'_i, a'_j}$ as the vector which replaces the i th and the j th items in \mathbf{a}_1 with a'_i, a'_j , respectively, and similarly for $\mathbf{x}_{1x'_i, x'_j}$. Consider the following four Bell coefficients:

$$\begin{aligned} f(\mathbf{a}_{1a'_i, a'_j}, \mathbf{x}_{1x'_i, x'_j}), & \quad f(\mathbf{a}_{1a''_i, a''_j}, \mathbf{x}_{1x'_i, x'_j}), \\ f(\mathbf{a}_{1a'_i, a''_j}, \mathbf{x}_{1x'_i, x''_j}), & \quad f(\mathbf{a}_{1a''_i, a''_j}, \mathbf{x}_{1x''_i, x''_j}). \end{aligned}$$

Denote $c = \sum_{k \neq i, k \neq j} a_k, z = \prod_{k \neq i, k \neq j} x_k$ and consider

$$a'_i + a'_j + c \equiv x'_i x'_j z \pmod{P}, \quad (19)$$

$$a'_i + a''_j + c \equiv x'_i x''_j z \pmod{P}, \quad (20)$$

$$a''_i + a'_j + c \equiv x''_i x'_j z \pmod{P}, \quad (21)$$

$$a''_i + a''_j + c \equiv x''_i x''_j z \pmod{P}. \quad (22)$$

If one of the above equations does not hold, then $-P^{2N}$ will contribute to the Bell inequality which induces $I_{P+1}^{\text{multi}} \leq 0$, because the sum of the positive coefficients is smaller than P^{2N} . If all equations hold, Eq. (19)+Eq. (22)–Eq. (20)–Eq. (21) yields $(x'_i - x''_i)(x'_j - x''_j)z \equiv 0$, which contradicts with the fact that P is a prime number and $1 \leq x'_i, x''_i, x'_j, x''_j, x_k < P$.

Case 3. Without loss of generality, we assume for $1 \leq i \leq N-1$ that there is only one x'_i such that $F_i(x'_i) = a'_i \neq \Phi$. Denote $\mathbf{a}' = \{a'_i\}_{1 \leq i \leq N-1}$, $\mathbf{x}' = \{x'_i\}_{1 \leq i \leq N-1}$. Then $I_{P+1}^{\text{multi}} = \sum_{x_n} f(\mathbf{a}', F_n(x_n), \mathbf{x}', x_n) + \sum_{i=1}^{N-1} g_i(a'_i, x'_i) + \sum_{x_n} g_n(F_n(x_n), x_n) = \sum_{x_n \neq 1} (f(\mathbf{a}', F_n(x_n), \mathbf{x}', x_n) - 1) + \sum_{i=1}^{N-1} g_i(a'_i, x'_i) + f(\mathbf{a}', F_n(1), \mathbf{x}', 1) \leq \sum_{i=1}^{N-1} g_i(a'_i, x'_i) + f(\mathbf{a}', F_n(1), \mathbf{x}', 1) \leq 0$. The last inequality holds because if $f(\mathbf{a}', F_n(1), \mathbf{x}', 1) = 1$, then there exists $1 \leq j < n$ such that $x'_j > 1$ and consequently $g_j(a'_j, x'_j) = -1$.

Next, we construct a no-signaling strategy:

$$p^{NS}(\mathbf{a}|\mathbf{x}) = \begin{cases} \frac{1}{P^{N-1}} & \sum_{i=1}^N a_i \equiv \prod_{i=1}^N x_i, \\ 0 & \text{otherwise} \end{cases}, \quad (23)$$

which satisfies the no-signaling condition

$$\forall \mathbf{a}, \mathbf{x}, 1 \leq k \leq N \quad p^{NS}(a_k|\mathbf{x}) = p^{NS}(a_k|x_k), \quad (24)$$

because for any value of a_k

$$p^{NS}(a_k|\mathbf{x}) = \sum_{\mathbf{a}' : a'_k = a_k, \sum_i a'_i \equiv \prod_i x_i} \frac{1}{P^{N-1}} = \frac{1}{P}. \quad (25)$$

The last equality holds because there are only $N-2$ free variables.

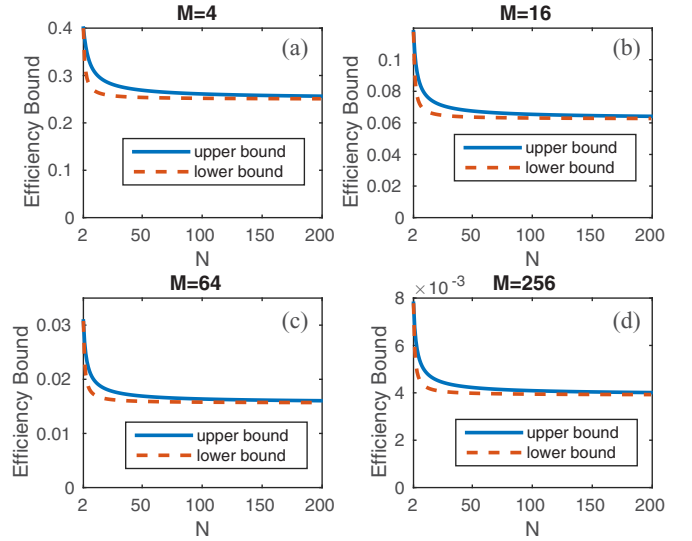


FIG. 2. (a)–(d) vary M from 4 to 256 and for each M the efficiency upper bound and lower bound are shown for $N = 2, 3, \dots, 200$. When M increases, the ratio between the efficiency bound of $N = 2$ and 200 gradually increases but is always smaller than 2.

The Bell value of the no-signaling strategy p_{η}^{NS} is

$$\begin{aligned} I_{P+1}^{\text{multi}} &= \sum_{\mathbf{a}, \mathbf{x}} f(\mathbf{a}, \mathbf{x}) p(\mathbf{a}|\mathbf{x}) + \sum_i \sum_{a_i, x_i} g_i(a_i, x_i) \\ &= \sum_{\mathbf{x}: \exists i, x_i \neq 1} \eta^N - \sum_i \sum_{x_i > 1} \eta \\ &= \eta^N \left(\prod_{i=1}^n M_i - 1 \right) - \eta \left(\sum_{i=1}^n M_i - N \right). \quad (26) \end{aligned}$$

Combined with the definition of η_{NS}^* that $\eta > \eta_{NS}^*$ is necessary to violate the Bell inequality, i.e., $I_{P+1}^{\text{multi}} > 0$, this leads to Eq. (17). ■

We compare the efficiency lower bound and upper bound in Theorems 3 and 4 for different numbers of parties N in Fig. 2. In the comparison, the number of inputs for each party is taken to be the same value M . It can be seen that when $N = 2$ the two bounds coincide. Also when N becomes large, the two bounds converge to the same value. This is consistent with the analytic upper and lower bound formulas, which both converge to $1/M$ when N goes to infinity.

It is also instructive to fix M and compare the efficiency bound for different N . It can be seen that with increasing N the efficiency bound is lowered. However, even when N goes to infinity, the efficiency bound is lowered by at most half compared to $N = 2$. On the other hand, increasing the number of input settings M greatly reduces the efficiency requirement. For example, when $M = 256$ and $N = 2$, an efficiency of 8% is enough to demonstrate nonlocality. Thus in this respect, increasing the number of input settings is more efficient than increasing the number of parties.

Compared to quantum efficiency bounds, there exist significant gaps. When $N = 3$ and $M = 8$, the best bound of the quantum theory is 0.501 [15] while the best bound of the maximally nonlocal theory is 0.20. For $N \rightarrow \infty$, the best

bound of the quantum theory $2/(2 + M)$ [18] is approximately twice of the best bound of the maximally nonlocal theory $1/M$. Therefore the advantage brought by our no-signaling strategies may provide inspiration to design more robust quantum strategies.

IV. DISCUSSION

In summary, we have investigated the efficiency requirement for the violation of Bell inequalities in the maximally nonlocal theory. Our result implies that, for showing the maximally nonlocal theory, the detector efficiency requirement can be arbitrarily low with enough input settings. Our work opens a few interesting avenues for future investigations. First, though our work essentially closes the gap between the upper bound and the lower bound of the detector efficiency for the maximally nonlocal theory, the corresponding question in the quantum theory is still wide open. It would be interesting to apply our techniques to solve the analog question in the quantum theory. Second, there is a small gap between the detector efficiency bounds of the maximally nonlocal theory when $2 < N < \infty$. Closing this gap may require some more delicate estimates on both the upper bound and the lower bound.

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APPENDIX: MONOTONICITY BETWEEN THE EFFICIENCY AND THE SIMULATION CAPABILITY OF THE LHV MODEL

The following lemma shows monotonicity between the efficiency η and the simulation capability of LHV models.

Lemma 1. For $\eta_1 < \eta_2$, if $p_{\eta_2}^{NS}$ can be simulated by a LHV model, then $p_{\eta_1}^{NS}$ can also be simulated by a LHV model.

Proof. Assume the LHV model p_2 which can simulate $p_{\eta_2}^{NS}$ is

$$\begin{aligned} p_{\eta_2}^{NS}(a, b|x, y) &= \int_{\lambda} p_2(\lambda) p_2(a|x, \lambda) p_2(b|y, \lambda) d\lambda, \\ p_{\eta_2}^{NS}(a, \Phi|x, y) &= \int_{\lambda} p_2(\lambda) p_2(a|x, \lambda) p_2(\Phi|y, \lambda) d\lambda, \\ p_{\eta_2}^{NS}(\Phi, b|x, y) &= \int_{\lambda} p_2(\lambda) p_2(\Phi|x, \lambda) p_2(b|y, \lambda) d\lambda, \\ p_{\eta_2}^{NS}(\Phi, \Phi|x, y) &= \int_{\lambda} p_2(\lambda) p_2(\Phi|x, \lambda) p_2(\Phi|y, \lambda) d\lambda. \end{aligned} \quad (A1)$$

Then, we define a LHV model p_1 such that

$$\begin{aligned} p_1(a|x, \lambda) &= p_2(a|x, \lambda) \frac{\eta_1}{\eta_2}, \\ p_1(\Phi|x, \lambda) &= 1 - \frac{\eta_1}{\eta_2} + p_2(\Phi|x, \lambda) \frac{\eta_1}{\eta_2}, \\ p_1(b|y, \lambda) &= p_2(b|y, \lambda) \frac{\eta_1}{\eta_2}, \\ p_1(\Phi|y, \lambda) &= 1 - \frac{\eta_1}{\eta_2} + p_2(\Phi|y, \lambda) \frac{\eta_1}{\eta_2}. \end{aligned} \quad (A2)$$

It is easy to verify that p_1 simulates $p_{\eta_1}^{NS}$. Intuitively, we modify the strategy p_2 to p_1 by assuming the detector efficiency of p_2 to be η_1/η_2 : outputting Φ (empty) with a probability $1 - \eta_1/\eta_2$. ■

By Lemma 1, we denote the function $e : p^{NS} \rightarrow [0, 1]$, such that $e(p^{NS})$ is the maximal efficiency which satisfies that $p_{e(p^{NS})}^{NS}$ can be simulated by a LHV model.

Thus, the efficiency bound η_{NS}^* of the no-signaling strategy can be defined as follows (with the number of inputs fixed):

$$\eta_{NS}^* = \inf_{p^{NS}} e(p^{NS}). \quad (A3)$$

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