Bell nonlocality in conventional and topological quantum phase transitions

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We investigate the critical behaviors of the Bell nonlocality, which are measured by the Bell-function values, in conventional and topological quantum-phase transitions. First, by using the quantum-renormalization-group method, we show that the Bell function value of a three-qubit Bell inequality in the anisotropic spin-1/2 XY model exhibits singular behaviors at the critical point. Moreover, for topological quantum-phase transitions, we find that the first-order derivative of the Bell-function value of the well-known Clauser-Horne-Shimony-Holt inequality in the Kitaev-Castelnovo-Chamon model behaves singularly at the topological-phase-transition point. Our results have established a link between quantum nonlocality and phase transitions.

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I. INTRODUCTION

Bell nonlocality, a kind of stronger nonclassical correlation which is unexplainable by any local-hidden-variable (LHV) theory or shared randomness only [1], differentiates quantum mechanics from classical physics in a very profound way. It refutes the concept of local realism and instead implies, to some extent, notable evidence of "spooky action" [2]. In fact, it is now understood that quantum correlation has a hierarchy in which Bell nonlocality is the strongest, followed by Einstein-Podolsky-Rosen (EPR) steering [3], then entanglement, and finally discord [4]. Bell nonlocality plays a key role in many quantum-information and -computation processes, such as quantum-key distribution (QKD) [5], nonlocal quantum computation [6], etc. However, despite extensive studies in the field of quantum information, it rarely appears in condensed-matter physics. Here, we investigate Bell nonlocality in many-body systems and try to explore its significance in quantum-phase transitions.

Generally speaking, quantum-phase transitions (QPTs), which are induced by the change of an external parameter or coupling constant [7], happen at a temperature of absolute zero where all the thermal fluctuations are frozen and quantum fluctuations become dominant. Conventional QPTs can be characterized by the Landau-Ginzburg-Wilson (LGW) spontaneous-symmetry-breaking theory in which the correlation function of local-order parameters plays a crucial role [7]. Nevertheless, examples beyond the LGW paradigm do exist. For instance, topological ordered phases of some strongly correlated quantum many-body systems depend on the system topology [8] and do not have local-order parameters, leading to the absence of LGW-symmetry-breaking mechanisms. An archetypal physical realization of such a phase is in the quantum Hall system [9], which possesses many unconventional characteristics, including fractional statistical behaviors and ground-state topological degeneracy [10]. A particular interest in topological ordered states is their robustness against local perturbations which can lead to several consequences, such as topological insulators [11] and topological quantum computations [12].

Not surprisingly, the exotic properties of the topological phase demand new ways to analyze topological quantum-phase transitions (TQPTs). For example, the phase transition between an Abelian and a non-Abelian topological phase in a chiral-spin liquid might be characterized by global flux and generalized topological-entanglement entropy [13]. More remarkably, for time-reversal invariant anyonic quantum systems, Gils *et al.* have recently showed that the topological phases could be uniformly described in terms of fluctuations of the two-dimensional surfaces and their topological changes [14]. However, despite a vast number of pioneering works, a universal characterization and detection of the topological phase and its transitions are still missing.

During the past few years, several important concepts in the quantum-information field have been borrowed to characterize QPTs and TQPTs, including entanglement [15], fidelity [16], fidelity susceptibility [17], discord [18], Bell inequality [19], etc. A brief review of the progress related to this issue is given in Ref. [20] and the references therein. Notwithstanding the great successes in marking QPTs and TQPTs in some physical systems, each approach above has its own disadvantages [20]. Take the fidelity approach, for example; to witness the QPTs, one has to find out the exact ground state. However, for most of the physical systems, finding out the exact ground state is very difficult. In addition, it is also a challenge to measure the fidelity in experiments on large systems. In this paper, we investigate the critical behaviors of Bell nonlocalities in QPTs and TQPTs. To begin with, we show, by using the quantum-renormalization-group (QRG) method, that the Bell nonlocality in the anisotropic spin-1/2 XY model exhibits singular behaviors at the critical point. On the other hand, for TQPTs, our discussion is mainly based on the Kitaev-Castelnovo-Chamon (KCC) model [21], which exhibits a second-order TQPT at the critical point.

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We show that the first-order derivative of the Bell-function value (BFV) of the well-known Clauser-Horne-Shimony-Holt (CHSH) inequality in the KCC model behaves singularly at the topological-phase-transition point. More interestingly, through this approach, one can analytically obtain the critical value of the transition point.

The paper is organized as follows. In Sec. II, we discuss the critical behaviors of Bell nonlocality in conventional quantum-phase transitions. We take the one-dimensional (1D) anisotropic spin-1/2 XY model as an example, and the Bell nonlocality is characterized by a three-qubit inequality. In Sec. III, we consider the case of topological quantum-phase transitions. The discussion here is mainly based on the CHSH inequality and the KCC model. Finally, we summarize the paper and provide some discussions of future work in Sec. IV.

II. BELL NONLOCALITY IN CONVENTIONAL QUANTUM-PHASE TRANSITIONS

The model we focus on in this section is the one-dimensional anisotropic spin-1/2 XY model. The Hamiltonian reads

$$H = J\sum_{i}^{N}\left[(1+h)\sigma_i^x\sigma_{i+1}^x + (1-h)\sigma_i^y\sigma_{i+1}^y\right], \quad (1)$$

where J describes the exchange coupling, h is the anisotropy parameter, and $\sigma^{x,y}$ are Pauli matrices. The system undergoes a QPT at the critical point $h=0$, differing the spin-fluid phase $h=0$ from the Ising-like phase $0<h\leqslant 1$. By resorting to the quantum-renormalization-group (QRG) method, the renormalized model is described by [22]

$$H^{\mathrm{Re}} = J'\sum_{L}^{N/3}\left[(1+h')\sigma_L^x\sigma_{L+1}^x + (1-h')\sigma_L^y\sigma_{L+1}^y\right], \quad (2)$$

where $J' = J\frac{3h^2+1}{2(1+h^2)}$, $h' = \frac{h^3+3h}{3h^2+1}$, and $\sigma_L^{x,y}$ are block Pauli matrices (see Ref. [22] for details). The renormalized ground state is of the three-qubit type:

$$|\mathrm{G}(g)\rangle = \frac{1}{2\sqrt{1+h'^2}}\big(-\sqrt{1+h'^2}|\mathscr{U}\mathscr{U}\mathscr{D}\rangle + \sqrt{2}|\mathscr{U}\mathscr{D}\mathscr{U}\rangle - \sqrt{1+h'^2}|\mathscr{D}\mathscr{U}\mathscr{U}\rangle + \sqrt{2}h'|\mathscr{D}\mathscr{D}\mathscr{D}\rangle\big), \quad (3)$$

where $|\mathscr{D}\rangle$ and $|\mathscr{U}\rangle$ are two orthogonal renormalized states and $g=(1+h')/(1-h')$ is a function of h' which we introduce for convenience. In the known literature [23–26], various types of three-qubit Bell inequalities are presented. Here, we use the one given in Ref. [25] to investigate the relation between Bell nonlocality and the QPT in the XY model. The three-qubit Bell inequality is of the form [25]

$$\mathcal{I}_3 = \tfrac{1}{4}(Q_{122} - Q_{123} + Q_{132} - Q_{133} + Q_{212} + Q_{213} + Q_{221} + Q_{222} - Q_{231} + Q_{233} - Q_{312} - Q_{313} + Q_{321} + Q_{323} - Q_{331} + Q_{332}) \leqslant 1, \quad (4)$$

where $Q_{ijk} = \int_\Gamma \mu(\lambda)X_1(\mathbf{n}_i^{X_1},\lambda)X_2(\mathbf{n}_j^{X_2},\lambda)X_3(\mathbf{n}_k^{X_3},\lambda)d\lambda$ is the correlation function with $X_l(\mathbf{n}_m^{X_l},\lambda)$ denoting the mth observable on the lth particle $(i,j,k,l,m=1,2,3)$, Γ is the total space of the hidden variable λ, and $\mu(\lambda)$ is a statistical distribution of λ, satisfying $\int_\Gamma \mu(\lambda)d\lambda = 1$. The inequality (4) is tight, and all LHV models should obey it [25]. However, quantum mechanically, the above inequality is violated by some entangled states, either pure or mixed. In fact, the expression of the correlation function for any three-qubit state ρ reads $Q_{ijk}^Q = \mathrm{Tr}[(\mathbf{n}_i^{X_1}\cdot\vec{\sigma})\otimes(\mathbf{n}_j^{X_2}\cdot\vec{\sigma})\otimes(\mathbf{n}_k^{X_3}\cdot\vec{\sigma})\rho]$. Here, $\mathbf{n}_m^{X_l}$ $(m,l=1,2,3)$ are the unit vectors in three-dimensional Hilbert space, and $\vec{\sigma}$ is the Pauli matrix vector. For a specific three-qubit state ρ, we define the *Bell-function values* (BFVs) of this inequality as

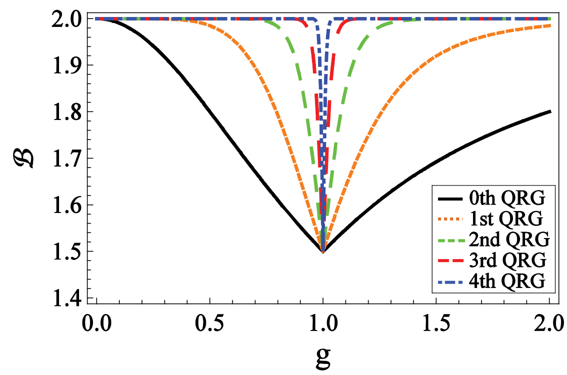

FIG. 1. (Color online) Numerical results of the BFV $\mathscr{B}(|\mathrm{G}(g)\rangle)$ versus g for the three-qubit-type ground state $|\mathrm{G}(g)\rangle$.

$$\mathscr{B}(\rho) \equiv \max \mathcal{I}_3^Q, \quad (5)$$

where $\mathcal{I}_3^Q$ is the quantum expression of $\mathcal{I}_3$ and the maximization is performed over all possible unit vectors $\mathbf{n}_m^{X_l}$. If $\mathscr{B}(\rho)$ is greater than 1, then the inequality (4) is violated. The state ρ exhibits Bell nonlocality and cannot be described by any LHV theories. Generally speaking, for every specific three-qubit quantum state ρ, we need to carry out the procedure of the maximization to obtain its BFV $\mathscr{B}(\rho)$. An alternative approach is to rewrite $\mathcal{I}_3^Q$ as a scalar product of two real vectors and then maximize this scalar product [25]. We numerically calculated the BFV for the renormalized ground state $|\mathrm{G}(g)\rangle$ of the 1D XY model. Figure 1 shows the variation of $\mathscr{B}(|\mathrm{G}(g)\rangle)$ versus g with different QRG steps. From the figure, the BFV $\mathscr{B}(|\mathrm{G}(g)\rangle)$ is always greater than 1 and hence reflects the Bell nonlocality of the system. It is clear that the BFV exhibits singular behavior at the critical point $h'_c = 0$ or $g_c = 1$. The more steps of QRG, the sharper $\mathscr{B}(|\mathrm{G}(g)\rangle)$ is at the critical point. In other words, the QPT is captured by $\mathscr{B}(|\mathrm{G}(g)\rangle)$ due to the nonanalytic behavior of the Bell nonlocality. This explicitly manifests the usefulness of the BFV in marking conventional QPTs.

III. BELL NONLOCALITY IN TOPOLOGICAL QUANTUM-PHASE TRANSITIONS

A. The Bell-CHSH inequality

As we have shown the usefulness of Bell nonlocality in conventional QPTs, now let us move to the TQPT case. To this end, we first make a brief introduction of the famous Bell-CHSH inequality. Basically, the Bell-CHSH inequality is a two-qubit inequality which provides the smallest testbed for experimental verifications of quantum mechanics against the predictions of local realistic models. It can be written as [1]

$$\mathcal{I} = Q_{11} + Q_{12} + Q_{21} - Q_{22} \leqslant 2, \quad (6)$$

where Q_{ij} is defined analogously as in Eq. (4). Quantum mechanically, the above inequality is violated by all pure entangled states of two qubits [27], and the expression of the correlation function for any two-qubit state ρ reads $Q_{ij}^{Q} = \mathrm{Tr}[(\mathbf{n}_i^{X_1}\cdot\vec{\sigma})\otimes(\mathbf{n}_j^{X_2}\cdot\vec{\sigma})\rho]$. Similarly, we can define the BFV of the Bell-CHSH inequality as

$$\mathscr{B}(\rho) \equiv \max \mathcal{I}^{Q}, \tag{7}$$

where $\mathcal{I}^{Q} = Q_{11}^{Q} + Q_{12}^{Q} + Q_{21}^{Q} - Q_{22}^{Q}$ and the maximization is performed over all possible vectors $\mathbf{n}_m^{X_k}$. Generally speaking, for every specific two-qubit quantum state ρ, we need to carry out the procedure of the maximization to obtain its BFV $\mathscr{B}(\rho)$. Fortunately, in Ref. [28], the authors introduced another method to calculate $\mathscr{B}(\rho)$, which can circumvent the tedious maximization. It was proved there that

$$\mathscr{B}(\rho) = 2\sqrt{\upsilon_1 + \upsilon_2}, \tag{8}$$

where υ_1 and υ_2 are the two greater eigenvalues of the 3×3 symmetric matrix $\mathscr{L}_\rho^{\mathrm{T}}\mathscr{L}_\rho$, $\mathscr{L}_\rho$ is a 3×3 matrix with elements defined by $(\mathscr{L}_\rho)_{\varsigma\tau} = \mathrm{Tr}[\rho\sigma^{\varsigma}\otimes\sigma^{\tau}]$ $(\varsigma,\tau = x,y,z)$, and $\mathscr{L}_\rho^{\mathrm{T}}$ is the transpose of $\mathscr{L}_\rho$. In the experimental situation, in order to obtain the BFV, the observers of the first (second) qubit should carry out two measurements $\mathbf{n}_1^{X_1}\cdot\vec{\sigma}$ and $\mathbf{n}_2^{X_1}\cdot\vec{\sigma}$ ($\mathbf{n}_1^{X_2}\cdot\vec{\sigma}$ and $\mathbf{n}_2^{X_2}\cdot\vec{\sigma}$), just the same as in many Bell-CHSH-inequality testing experiments [29].

B. The Kitaev-Castelnovo-Chamon model

The physical model we consider in this part was introduced by Castelnovo and Chamon [21]; it is a deformation of the Kitaev-toric-code model [30]. The Hamiltonian of the KCC model with periodic-boundary conditions reads

$$\mathscr{H} = -\mathcal{J}_m \sum_{\mathcal{F}\in\mathrm{T}^2} B_{\mathcal{F}} - \mathcal{J}_e \sum_{\mathcal{V}\in\mathrm{T}^2} A_{\mathcal{V}} + \mathcal{J}_e \sum_{\mathcal{V}\in\mathrm{T}^2} e^{-\beta\sum_{j\in\mathcal{V}}\sigma_j^z},$$

where $\mathcal{J}_m,\mathcal{J}_e > 0$, β is a coupling constant, and $A_{\mathcal{V}} = \prod_{j\in\mathcal{V}}\sigma_j^x$ and $B_{\mathcal{F}} = \prod_{j\in\mathcal{F}}\sigma_j^z$ are the vertex and face operators in the original Kitaev-toric-code model, respectively [30]. A brief sketch of this model is shown in Fig. 2. The ground state in the topological sector containing the fully magnetized state $|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$ can be analytically obtained [21]:

$$|G(\beta)\rangle = \mathcal{Z}(\beta)^{-\frac{1}{2}} \sum_{g\in\mathcal{G}} e^{\beta\sum_j\sigma_j^z(g)/2} g|0\rangle, \tag{9}$$

where $\mathcal{Z}(\beta) = \sum_{g\in\mathcal{G}} e^{\beta\sum_j\sigma_j^z(g)}$, $\mathcal{G}$ is the Abelian group generated by the vertex operators $\{A_{\mathcal{V}}\}$, and $\sigma_j^z(g)$ is the value of spin at site j in state $g|0\rangle$. Obviously, when $\beta = 0$, $|G(\beta)\rangle$ reduces to the topologically ordered ground state of the toric-code model [30] while when $\beta\to\infty$, $|G(\beta)\rangle$ becomes the fully magnetized state $|0\rangle$. At the point $\beta_c = \frac{1}{2}\ln(\sqrt{2}+1)$, there exists a second-order TQPT at which the topological entanglement entropy $\mathcal{S}_{\mathrm{topo}} = 1$ for $\beta < \beta_c$ changes to $\mathcal{S}_{\mathrm{topo}} = 0$ for $\beta > \beta_c$ [21].

As shown by Castelnovo and Chamon, there exists a one-to-one mapping between the configurations $\{g\} = \mathcal{G}$ and the configurations $\{\theta\}$ of the classical 2D Ising model [21]. In the mapping, the Hamiltonian of the Ising model has the form $\mathscr{H}_{\mathrm{Ising}} = -\mathscr{C}\sum_{\langle r,r'\rangle}\theta_r\theta_{r'}$, where $\mathscr{C}$ is a coupling constant and $\theta_r,\theta_{r'} = +1$ or -1 depending on whether or not the corresponding vertex operator $A_{\mathcal{V}}$ is acting on the site r. Thus, $\sigma_i^z = \theta_r\theta_{r'}$ with i being the edge between the nearest-neighbor vertexes. An illustration of this mapping is shown in Fig. 2.

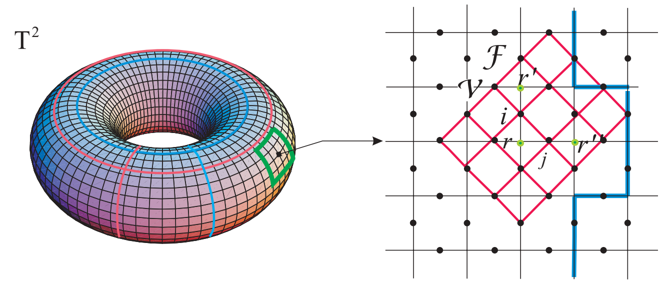

FIG. 2. (Color online) An illustration of the Kitaev-Castelnovo-Chamon spin-lattice model and its map to the 2D Ising model. In the KCC model, each black dot on the edge of the lattice represents a qubit, and $\mathcal{V}$ and $\mathcal{F}$ denote the vertex and the face, respectively. The blue line across the lattices stands for a string operator along the nontrivial loop on the torus T^2. The system is invariant under transformation along the blue line [32]. In its corresponding 2D Ising model, the qubits exist on the vertices (green dots). Mapping: $\sigma_i^z = \theta_r\theta_{r'}$, where i is the bond between the neighboring vertices $\langle r,r'\rangle$. Thus, for the i and j nearest neighbors, the mapping gives $\langle\sigma_i^z\sigma_j^z\rangle = \langle\theta_r\theta_{r'}\theta_{r''}\theta_r\rangle = \langle\theta_{r'}\theta_{r''}\rangle$, namely, the nearest qubits in the KCC model become next-nearest in the corresponding 2D Ising model.

C. Signaling TQPTs by the BFV

Since the BFV introduced in expression (7) only accounts for two-qubit states, we need to calculate the reduced density matrix of two-qubit ρ_{ij} based on the ground state $|G(\beta)\rangle$ and the symmetry of the Hamiltonian $\mathscr{H}$. It was shown in Ref. [31] that ρ_{ij} has the following form (details are given in Refs. [21,32], and references therein):

$$\rho_{ij} = \tfrac{1}{4}\big[\mathbf{I} + \langle\sigma_i^z\rangle(\sigma_i^z + \sigma_j^z) + \langle\sigma_i^z\sigma_j^z\rangle\sigma_i^z\sigma_j^z\big], \tag{10}$$

where $\mathbf{I}$ is the 4×4 identity matrix. Based on Eq. (10), the BFV can be calculated by using the simplified formula for $\mathscr{B}(\rho_{ij})$ in Eq. (8). For convenience and simplicity, we only examine two cases in which i and j are nearest and next-to-nearest neighbors, respectively. In the thermodynamic limit, the mapping to the 2D Ising model gives $\langle\sigma_i^z\rangle = \langle\theta_r\theta_{r'}\rangle = -\coth(2\beta)\{\pi + [4\tanh^2(2\beta) - 2]\mathscr{K}(\chi)\}/(2\pi)$, where $\mathscr{K}(\chi) = \int_0^{\pi/2} d\varphi(1-\chi^2\sin^2\varphi)^{-1/2}$ and $\chi = 2\sinh(2\beta)/\cosh^2(2\beta)$. For the calculation of $\langle\sigma_i^z\sigma_j^z\rangle$, the equivalence between the 2D Ising model and the quantum 1D XY model yields the following two alternatives.

First, for i and j of the nearest case, $\langle\sigma_i^z\sigma_j^z\rangle = \langle\theta_r\theta_{r'}\rangle = \frac{1}{\pi}\int_0^{\pi} d\phi\{[\sinh^{-2}(2\beta) - \cos\phi]\cos\phi - \sin^2\phi\}/\{\sin^2\phi + [\sinh^{-2}(2\beta) - \cos\phi]^2\}^{1/2}$. Summarizing all the relations above enables us to obtain the BFV $\mathscr{B}(\rho_{ij})$. For i and j of the nearest case, the numerical results for the first-order derivative of the BFV $\frac{d\mathscr{B}(\rho_{ij})}{d\beta}$ with regard to the variance of β are displayed in Fig. 3(a) from which we see a distinct rapid increase of $\frac{d\mathscr{B}(\rho_{ij})}{d\beta}$ around the critical point $\beta_c \approx 0.44$. Note that we only focus on a small neighboring region of β around the TQPT point β_c, namely, $0.4 \leqslant \beta \leqslant 0.5$. The smaller the $\delta\beta$, the greater the rapid increase is. When $\delta\beta\to 0$,

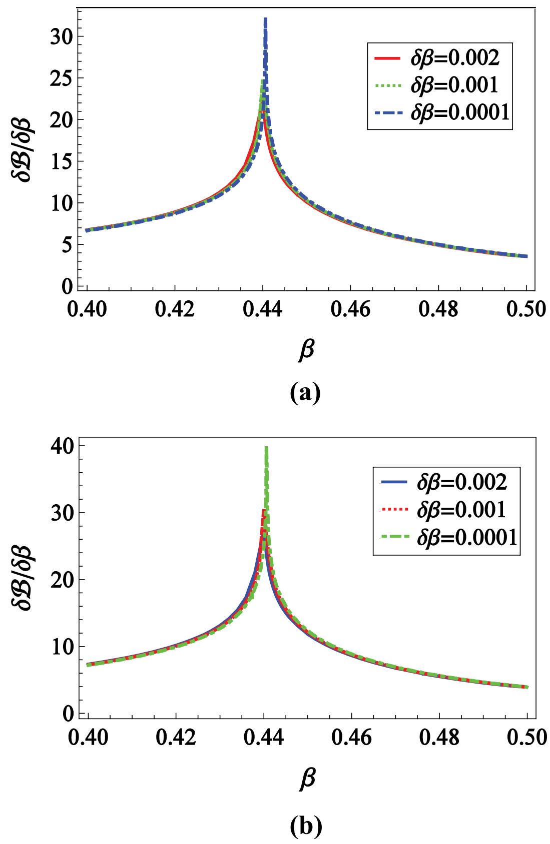

FIG. 3. (Color online) Numerical results of the first-order derivative of the BFV $\mathscr{B}(\rho_{ij})$ versus β for different $\delta\beta$. (a) i and j are the nearest case; (b) i and j are the next-to-nearest case.

$\frac{d\mathscr{B}(\rho_{ij})}{d\beta} \to +\infty$, indicating its singularity at the TQPT point β_c.

More interestingly, after long, tedious, but straightforward calculations, we arrive at an analytical formula for $\frac{d\mathscr{B}(\rho_{ij})}{d\beta}$, potentially enabling us to obtain the analytical value of β_c:

$$\frac{d\mathscr{B}}{d\beta} = \int_0^{\pi} d\phi \operatorname{csch}^2(2\beta)\sin^2\phi\, \Upsilon(\phi,\beta)/\pi, \tag{11}$$

where $\Upsilon(\phi,\beta)$=8 coth $(2\beta)\operatorname{csch}^2(2\beta)/[1-2\cos\phi \operatorname{csch}^2(2\beta) + \operatorname{csch}^4(2\beta)]^{3/2}$. What is interesting is that we can analytically obtain the critical point from Eq. (11). To this end, one can rewrite $\Upsilon(\phi,\beta)$ as $\Upsilon(\phi,\beta) = 2\sqrt{2}\coth(2\beta)/(\{\frac{1}{2}[\operatorname{csch}^2(2\beta) + 1/\operatorname{csch}^2(2\beta)] - \cos\phi\}^{3/2}\operatorname{csch}(2\beta))$. Obviously, Eq. (11) has only one singular point because $\operatorname{csch}^2(2\beta) + 1/\operatorname{csch}^2(2\beta) \geqslant 2$, and so the singularity happens at $\operatorname{csch}^2(2\beta) + 1/\operatorname{csch}^2(2\beta) = 2$, namely, $\beta_c = \frac{1}{2}\ln(\sqrt{2}+1)$. This also explicitly exhibits one of the advantages of the BFV approach to TQPTs.

Second, for i and j of the next-to-nearest case, direct calculations show that $\langle\sigma_i^z\sigma_j^z\rangle = \cosh^2(\beta^*)(\mathcal{T}_{-1}^2 - \mathcal{T}_{-2}\mathcal{T}_0) - \sinh^2(\beta^*)(\mathcal{T}_1^2 - \mathcal{T}_2\mathcal{T}_0)$, where $\mathcal{T}_\kappa = \frac{1}{\pi}\int_0^{\pi} d\phi[(\xi - \cos\phi)\cos\kappa\phi + \gamma\sin\phi\sin\kappa\phi]/[(\gamma\sin\phi)^2 + (\xi-\cos\phi)^2]^{1/2}$. Here, $\tanh(\beta^*) = e^{-2\beta}$, $\gamma = [\cosh(2\beta^*)]^{-1}$, and $\xi = (1-\gamma^2)^{1/2}/\tanh(2\beta)$. We also plot the numerical results of $\frac{d\mathscr{B}(\rho_{ij})}{d\beta}$ versus β in Fig. 3(b). From this figure, one can observe that $\frac{d\mathscr{B}(\rho_{ij})}{d\beta}$ has a singularity at the TQPT point β_c. It is also obvious that $\frac{d\mathscr{B}(\rho_{ij})}{d\beta}$ behaves quite similarly between the nearest and next-to-nearest cases. This result accords with the results in Ref. [31] wherein the reduced fidelity and reduced-fidelity susceptibilities are only slightly different between the two cases, respectively. Since the BFV indicates the qubit correlations in the system, it seems that the correlation of nearest qubits is similar to that of next-to-nearest qubits in topological ordered states. This is, to some extent, counterintuitive because in many physical systems the interaction between nearest particles is usually greater than that of next-to-nearest particles.

It is worthwhile to note that the reduced density matrix ρ_{ij} is diagonal, indicating that the correlations between any two local spins in the ground state of the KCC model are always classical. Consequently, the BFV of ρ_{ij} cannot be greater than 2, the classical bound. Another interesting consideration here is similar to that discussed in Ref. [32]: we can calculate the BFV between a local qubit denoted by i and the rest of the whole lattice by rewriting the ground state as $|G(\beta)\rangle = \mathscr{Y}_+|\mathscr{P}\rangle|0\rangle_i + \mathscr{Y}_-|\mathscr{Q}\rangle|1\rangle_i$, where $\mathscr{Y}_\pm^2 = (1 \pm \langle\theta_{0,0}\theta_{0,1}\rangle)/2$ and $|\mathscr{P}\rangle$ and $|\mathscr{Q}\rangle$ are two orthogonal normalized vectors. Consequently, we can regard $|G(\beta)\rangle$ as a simple, pure two-qubit entangled state. In this case, the BFV has a one-to-one monotonous relation with entanglement [33] and, thus, also with quantum discord since for a pure two-qubit state, the quantum discord is the same as the entanglement of entropy [34]. As a result, the BFV should behave similarly as the quantum discord does at the critical point β_c (for details, see Ref. [32]). However, it is worth clarifying that there is a distinctive difference between the BFV approach and the quantum-discord approach. For the quantum discord, its value becomes trivially zero for the reduced two-qubit state of Eq. (10) and, thus, cannot signal the TQPT at the critical point. Nevertheless, as shown above, the first-order derivative of the BFV is an excellent marker of the transitions. In addition, the physical meanings of the BFV and quantum discord are different. Generally speaking, quantum discord is a measurement of the *quantumness* of a system while the BFV measures the nonlocality of the system when it is greater than the classical bound. In this case, $\mathscr{B}(|G(\beta)\rangle) = 2\sqrt{1 + 4\mathscr{Y}_+^2\mathscr{Y}_-^2} > 2$. Thus, the BFV $\mathscr{B}(|G(\beta)\rangle)$ can measure the nonlocality of the ground state, establishing a link between quantum nonlocality and TQPTs.

IV. SUMMARY AND DISCUSSION

To summarize, based on the 1D anisotropic spin-1/2 XY model and the KCC model, we have introduced the BFV approach to both conventional QPTs and topological QPTs. Our results show that the BFV serves as an accurate marker for both types of phase transitions. Since the BFV also serves as a measure of nonlocality, which is a pure quantum phenomenon and cannot be described by any local realism theory, our work has established a link between quantum nonlocality and phase transitions.

Although we have only focused on two specific models, we believe this approach is applicable to other models as well. For instance, for the model recently introduced by

Son *et al.*, which is described by a cluster Hamiltonian $\mathscr{H}(\mathscr{F}) = -\sum_{i=1}^{N}(\sigma_{i-1}^x\sigma_i^z\sigma_{i+1}^x + \mathscr{F}\sigma_i^y\sigma_{i+1}^y)$ and has an exotic phase transition at the critical point $\mathscr{F} = 1$ [35], our numerical results show that the first-order derivative of the BFV can explicitly capture the transition. The cluster Hamiltonian mentioned above can be simulated in a triangular configuration of an optical lattice of two atomic species [35], thus also leading to the possibility of testing the BFV approach experimentally. To further investigate the BFV in QPTs, we also have considered the one-dimensional Ising model. Without surprise, the numerical results confirm the singular behaviors of the BFV at the critical point once again.

It would be interesting and significant to apply this approach to QPTs and TQPTs in various physical systems, such as the quantum-spin Hall system, both theoretically and experimentally. It would also be interesting to use the BFV based on other Bell inequalities to investigate QPTs and TQPTs. More specifically, studying the BFV of the pure ground states based on multipartite Bell inequalities, such as the famous Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality [26], might shed light on the behavior of the quantum nonlocality of the whole system in QPTs and TQPTs.

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