

Unification of quantum resources in distributed scenarios

Hongyi Zhou,^{1,*} Xiao Yuan,^{2,†} and Xiongfeng Ma^{1,‡}

¹*Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing, 100084 China*

²*Department of Materials, University of Oxford, Parks Road, Oxford OX1 3PH, United Kingdom*

Quantum resources, such as coherence, discord, and entanglement, play as a key role for demonstrating advantage in many computation and communication tasks. In order to find the nature behind these resources, tremendous efforts have been made to explore the connections between them. In this work, we extend the single party coherence resource framework to the distributed scenario and relate it to basis-dependent discord. We show the operational meaning of basis-dependent discord in quantum key distribution. By formulating a framework of basis-dependent discord, we connect these quantum resources, coherence, discord, and entanglement, quantitatively, which leads to a unification of measures of different quantum resources.

I. INTRODUCTION

Coherence, discord, and entanglement are fundamental resources in many tasks that cannot be achieved by classical physics. Coherence characterizes the superpositions [1, 2], serving as a resource of quantum randomness generation [3–5], quantum metrology [6–9], quantum computation [10–13], and quantum thermodynamics [14–19]. As one of the most widely used quantum resources, entanglement [20–27] plays a key role in quantum teleportation [28], quantum key distribution [29, 30], and dense coding [31], and also interprets the violation of Bell inequalities. Discord characterizes quantum correlations beyond entanglement [32–38]. It is the resource for remote state preparation [37], and might explain the acceleration in discrete quantum computation with one qubit and other quantum computation circuits [39].

Although these quantum resources play different roles in different tasks, the nature behind the resources might be the same. To find out such a non-classical nature, a natural idea is to build a unification framework of these quantum resources. Recently some researches have made progress for this goal [40–43]. Early researches in this field focus on the transformation between distillable entanglement and discord [44, 45]. Since the framework of coherence is proposed [2], there have been substantial attempts for unifying coherence and entanglement resource theory by designing protocols where these two resources can be converted into each other [46–50]. One example is that a single partite state with non-zero coherence is shown to be able to generate entanglement with bipartite incoherent operations [46]. Similar results are extended to discord and generalized to multipartite systems in [12], where it is shown that the quantum discord created by multipartite incoherent operations is bounded by the quantum coherence consumed in its subsystems. Another connection between coherence and entanglement

lies in quantum state merging [51]. A standard quantum state merging can lead to a gain of entanglement, while the incoherent quantum state merging [52] where one of the parties is restricted with local incoherent operations only, shows that entanglement and coherence cannot be gained at the same time.

All the works above are trying to connect part of these resources. Recently a unification of all the three resources based on an interferometric scenario is proposed in [53]. Considering a phase encoding process of an input state, the interferometry power, i.e., how much phase information can be obtained is determined by the quantum resource contained in the input state. In such an interferometric framework, different quantum resources corresponds to the interferometry power in different scenarios. Although coherence, discord, and entanglement are qualitatively unified in the interferometric framework, a quantitative unification is still an open problem.

In this work, we construct such a quantitative unification of the three resources. We first review the general definitions of resource frameworks and summarize the corresponding definitions for coherence, discord, and entanglement. Then, we extend the single party coherence resource framework to the bipartite distributed scenario in several different ways. It turns out that one of the definitions is identical to basis-dependent (BD) discord [12, 43]. We construct the resource framework of BD-discord, where we propose its operational meaning in quantum key distribution (QKD) and give some examples of BD-discord measures. With the help of BD-discord, measures of coherence, discord, and entanglement can be naturally defined and unified. We believe our unified framework of quantum resources can make a substantial progress in understanding the quantum nature.

II. PRELIMINARIES

In this section, we first review the definitions of a general resource framework. Then, we briefly summarize the coherence framework and refer the reader to Appendix A

* zhouhy14@mails.tsinghua.edu.cn

† xiao.yuan.ph@gmail.com

‡ xma@tsinghua.edu.cn

for a detailed review of discord and entanglement frameworks.

A. Resource framework

A general resource framework [2, 13, 54–62] consists of the definition of free state, free operation, and resource quantifiers.

Free state is a set of states \mathcal{F} that contain no resource while a state $\rho \notin \mathcal{F}$ contains resource.

Free operations are physical realizable operations characterized by completely positive and trace preserving (CPTP) maps. They should at least transform free states only into free states, i.e. $\Lambda_{\text{CPTP}}(\rho) \in \mathcal{F}, \forall \rho \in \mathcal{F}$ which can be rewritten as $\sum_n K_n \rho K_n^\dagger \in \mathcal{F}, \forall \rho \in \mathcal{F}$ in Kraus presentation. Here $\{K_n\}$ is the set of Kraus operators satisfying $\sum_n K_n^\dagger K_n = I$. Different other free operations can be defined based on different extra requirements.

Quantifiers are real-valued functions f mapping states to non-negative real numbers. The free states should be mapped to zero, i.e., $f(\rho) = 0, \forall \rho \in \mathcal{F}$. And for an arbitrary state, the function value should not increase under free operations, i.e., $f(\rho) \geq f(\Lambda_{\text{CPTP}}(\rho))$. Other principles are required for different resources and different tasks.

B. Framework of coherence

The general resource framework reduces to a specific one when we consider coherence, discord, and entanglement as the resource. We briefly review the coherence framework introduced in [2, 61], focusing on quantum states in a d -dimensional Hilbert space.

Incoherent and maximally coherent states. Given a classical computational basis $J = \{|j\rangle\}$, ($j = 1, 2, \dots, d$), an incoherent state refers to a state without superposition on the basis, which can be described by

$$\sigma = \sum_{j=1}^d p_j |j\rangle\langle j|, \quad (1)$$

where $p_{j_A} \in [0, 1], \forall j$ and $\sum_j p_{j_A} = 1$. At the meantime, maximally coherent states can be expressed as:

$$|\Psi_d\rangle = \frac{1}{d} \sum_{j=1}^d e^{i\phi_j} |j\rangle, \quad (2)$$

where $\phi_j \in [0, 2\pi)$.

Incoherent operations. Incoherent operations map an incoherent state only to an incoherent state. That is, $\sum_n \hat{K}_n \rho \hat{K}_n^\dagger \subset \mathcal{C}, \forall \rho \in \mathcal{C}$, where \mathcal{C} is the set of incoherent states, $\{\hat{K}_n\}$ is a series of Kraus operators satisfying $\sum_n \hat{K}_n^\dagger \hat{K}_n = I$.

Coherence measures. A coherence measure $C(\rho)$ is defined by a function that maps a quantum states ρ to a

real non-negative number, which satisfies the following conditions in Table. 1:

Table 1: Properties of a coherence quantifier.

- (C1) $C(\sigma) = 0$ when σ is an incoherent state. A stronger condition is (C1') $C(\sigma) = 0$ if and only if σ is an incoherent state;
- (C2) *Monotonicity:* Coherence should not increase under incoherent operations, that is, (C2a) $C(\rho) \geq C[\Phi_{\text{ICPTP}}(\rho)]$, (C2b) $C(\rho) \geq \sum_n p_n C(\rho_n)$, where $\rho_n = K_n \rho K_n^\dagger / \text{tr}(K_n \rho K_n^\dagger)$;
- (C3) *Convexity:* Coherence cannot increase under mixing, that is, $\sum_e p_e C(\rho_e) \geq C(\sum_e p_e \rho_e)$.

We leave the framework of the other two quantum resources, discord and entanglement in Appendix A.

III. EXTENDING COHERENCE TO THE DISTRIBUTED SCENARIO

Quantum coherence is defined in the single party scenario while discord and entanglement are defined for at least two parties. Therefore, to unify the three measures, we should generalize coherence to multiple parties. In this section, we consider three approaches to generalize coherence to the bipartite distributed scenario, where we begin with three possible generalized definitions of the incoherent state.

A. Incoherent-incoherent bipartite coherence

A natural extension is the bipartite coherence proposed in [46], which considers the joint basis $J_A J_B = \{|j_A\rangle|j_B\rangle\}$ ($j_A = 1, 2, \dots, d_A, j_B = 1, 2, \dots, d_B$) with d_A and d_B being dimensions of the local Hilbert spaces of system A and B , respectively. The bipartite incoherent state in can be rewritten as

$$\sigma_{AB}^I = \sum_{j_A, j_B} p_{j_A j_B} |j_A\rangle\langle j_A| \otimes |j_B\rangle\langle j_B|. \quad (3)$$

It is not hard to see that the bipartite incoherent state defined above is a specific type of classical-classical state $\sigma_{AB}^{CC} = \sum_{m,n} p_{mn} |m\rangle\langle m| \otimes |n\rangle\langle n|$ with certain local bases. Here we call Eq. (3) as *incoherent-incoherent (II) state*. A bipartite state contains bipartite coherence if it is not an incoherent-incoherent state.

B. Incoherent-classical bipartite coherence

When focusing the coherence in a local basis of system A (say J_A) and ignore the local basis of system B , we define the *incoherent-classical (IC)* state as

$$\sigma_{AB}^{IC} = \sum_{j_A, n} p_{j_A n} |j_A\rangle\langle j_A| \otimes |n\rangle\langle n| \quad (4)$$

which is still a classical-classical state. Although $|j_A\rangle$ is still from the J_A basis, $|n\rangle$ can be from an arbitrary basis of the system B . Note that any incoherent-classical state can be obtained by applying a local unitary operation on system B to a incoherent-incoherent state, i.e., $\sigma_{AB}^{IC} = U_B \sigma_{AB}^{II} U_B^\dagger$. Therefore, the set of incoherent-classical states is larger than the set of incoherent-incoherent states. A bipartite state contains incoherent-classical bipartite coherence if it is not an incoherent-classical state.

C. Incoherent-quantum bipartite coherence

In the above generalization, we still consider the incoherent state as a classical-classical state. If we only focus on the coherence in a local basis of system A (say J_A), and totally ignore the other party (B), we can generalize coherence to be incoherent-quantum (IQ) coherence [41, 63],

$$\sigma_{AB}^{IQ} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_B^{j_A}. \quad (5)$$

Equivalently, it can be written as

$$\sigma_{AB}^{IQ} = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \left(\sum_{l_{j_A}} l_{j_A} |l_{j_A}\rangle\langle l_{j_A}| \right), \quad (6)$$

with a spectral decomposition in party B . It is not hard to see that incoherent-quantum state can be obtained by mixing incoherent-classical states. Or we can regard the set of incoherent-quantum state as the convex hull of the set of the incoherent-classical states. A bipartite state contains incoherent-quantum bipartite coherence if it is not an incoherent-quantum state.

We generalize the bipartite coherence in distributed scenarios with the track of $\sigma_{AB}^{II} \rightarrow \sigma_{AB}^{IC} \rightarrow \sigma_{AB}^{IQ}$. The incoherent-incoherent state is a subset of incoherent-classical state which is further a subset of incoherent-quantum state. We illustrate the relationship of these states in Fig. 1. The incoherent-quantum bipartite coherence is actually identical to the basis-dependent (BD) discord [12, 43], which is the key resource for our unification framework.

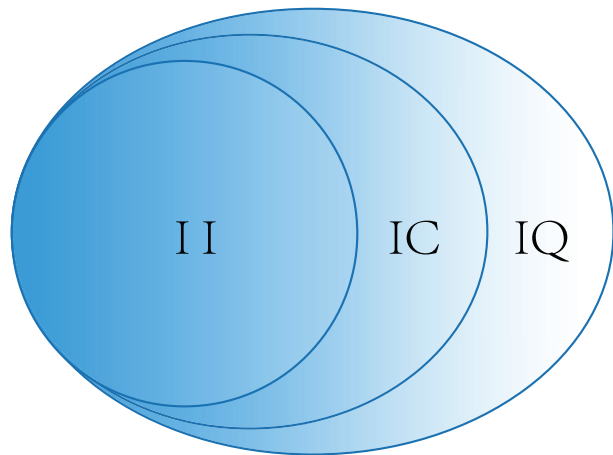


FIG. 1. Illustration of free states for different types of bipartite coherence. *II*: incoherent-incoherent states; *IC*: incoherent-classical states; *IQ*: incoherent-quantum states.

IV. BASIS-DEPENDENT DISCORD

A. Framework of basis-dependent discord

The concept of BD-discord has been proposed in [12, 43] when studying discord. Here we formulate its resource framework, beginning with definitions of free states for BD-discord given a local computational basis $J_A = \{|j_A\rangle\} (j = 1, 2, \dots, d_A)$ on system A .

Definition 1. A zero basis-dependent discord state in $J_A = \{|j_A\rangle\}$ is an incoherent-quantum state in Eq. (5)

Second we define free operations for BD-discord, which map incoherent-quantum states to incoherent-quantum states.

Definition 2. The free operations for BD-discord are separable-quantum-incoherent (SQI) operations [64]

$$\Lambda_{SQI}(\sigma_{AB}^{IQ}) = \sum_n \hat{A}_n \otimes \hat{B}_n \sigma_{AB} \hat{A}_n^\dagger \otimes \hat{B}_n^\dagger \subset \delta_{IQ}, \quad (7)$$

where δ_{IQ} is the set of incoherent-quantum states, $\hat{A}_n \otimes \hat{B}_n$ is a series of Kraus operators satisfying the completeness condition $\sum_n \hat{A}_n^\dagger \hat{A}_n \otimes \hat{B}_n^\dagger \hat{B}_n = I$, and $\{\hat{A}_n\}$ is a set of incoherent operations on A .

Finally we define the measures of BD-discord, $BD_{J_A}(\rho_{AB})$, which map a bipartite quantum states ρ_{AB} to a real non-negative number, satisfying the conditions in Table 2.

B. Examples of basis-dependent discord measures

Here we give two categories of BD-discord measures that fulfill the conditions in Table. 2. One is the distance-based measure. The BD-discord equals to the distance

Table 2: Properties of a basis-dependent discord quantifier.

- (BD1) Basis-dependent discord vanishes for incoherent-quantum state $\sigma_{AB}^{j_A} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_B^{j_A}$
- (BD2) *Monotonicity*: Basis-dependent discord should not increase under SQI operations, i.e., $BD_{J_A}(\Lambda_{SQI}(\rho_{AB})) \leq BD_{J_A}(\rho_{AB})$
- (BD3) Basis-dependent discord is invariant under a local incoherent unitary operation on A and a unitary operation on B

from IQ states, which is expressed as

$$BD_{J_A}(\rho_{AB}) = \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB} || \sigma_{AB}^{IQ}). \quad (8)$$

Specifically, the distance can be various of measures given in Table 3, where the superscript in ρ_{AB}^{Adiag} means a local dephasing operation on A . Actually these measures are widely used in entanglement, discord and coherence.

The other is the convex roof of local randomness,

$$BD_{J_A}(\rho_{AB}) = \min_{p_e, |\Psi_{AB}\rangle_e} \sum_e p_e R(|\Psi_{AB}\rangle_e) \quad (9)$$

where the minimization is over all possible pure state decompositions of ρ_{AB} , and $R(|\Psi_{AB}\rangle_e)$ is the local randomness given by von Neumann entropy of party A

$$R(|\Psi_{AB}\rangle_e) = S\left(\sum_{j_A} \langle j_A | \text{tr}_B(\rho_{AB}) | j_A \rangle | j_A \rangle \langle j_A | \right) \quad (10)$$

We prove that Eq. (8) and Eq. (9) satisfy all conditions of a BD-discord measure in Appendix B.

Table 3: Some possible measures of basis-dependent discord.

- (1) relative entropy, $S(\rho_{AB}^{Adiag}) - S(\rho_{AB})$
- (2) l_1 norm, $\min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} \|\rho_{AB} - \sigma_{AB}^{IQ}\|_{l_1}$
- (3) geometric measure, $1 - \max_{\sigma_{AB}^{IQ} \in \delta_{IQ}} F(\rho_{AB}, \sigma_{AB}^{IQ})$
- (4) fidelity measure, $1 - \max_{\sigma_{AB}^{IQ} \in \delta_{IQ}} \sqrt{F(\rho_{AB}, \sigma_{AB}^{IQ})}$

C. Operational meaning of the basis-dependent discord

In this section, we consider the operational meaning of BD-discord, which is the local randomness of the raw key in QKD. In the QKD security analysis, the communication partners, Alice and Bob, share a bipartite state ρ_{AB} , while the adversary Eve, is assumed to hold a purification $|\Psi_{ABE}\rangle$ of Alice's and Bob's system AB , which enables her to obtain the most information. The Devetak-Winter formula [65] gives an asymptotic key rate with one-way direct reconciliation. When ρ_{AB} is known to Alice and Bob, the formula is expressed as

$$K = S(Z_A|E) - S(Z_A|Z_B) \quad (11)$$

where $S(\cdot)$ is the von Neumann entropy function and $Z_{A(B)}$ is a local key generation measurement expressed as $\{|j_{A(B)}\rangle\langle j_{A(B)}|\}$, $j_{A(B)} = 1, 2, \dots, d_{A(B)}$. We will show that the first term in Eq. (11) is actually a basis-dependent discord measure.

Proposition 1. *The local randomness in QKD, i.e., the conditional entropy $S(Z_A|E)$ in the Devetak-Winter formula, is a BD-discord measure.*

Proof. The conditional entropy $S(Z_A|E)$ can be expressed as

$$S(Z_A|E) = S(\rho_{AE}^{Adiag}) - S(\rho_E). \quad (12)$$

Suppose the tripartite state after Alice's local measurement is $\rho_{ABE}^{Adiag} = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_{BE}^{j_A}$, where $\rho_{BE}^{j_A}$ is a pure state since $|\Psi_{ABE}\rangle$ is a pure state, then

$$\begin{aligned} \rho_{AB}^{Adiag} &= \text{tr}_E(\rho_{ABE}^{Adiag}) \\ &= \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \text{tr}_E(\rho_{BE}^{j_A}) \\ &= \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_B^{j_A} \end{aligned} \quad (13)$$

and ρ_{AE}^{Adiag} has a similar expression of $\rho_{AE}^{Adiag} = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_E^{j_A}$. Consider the von Neumann entropy of a classical-quantum state,

$$\begin{aligned} S(\rho_{AE}^{Adiag}) &= H(\{p_{j_A}\}) - \sum_{j_A} p_{j_A} S(\rho_E^{j_A}) \\ &= H(\{p_{j_A}\}) - \sum_{j_A} p_{j_A} S(\rho_B^{j_A}) \\ &= S(\rho_{AB}^{Adiag}), \end{aligned} \quad (14)$$

where $H(\cdot)$ is the Shannon entropy function and the second equality uses the fact that $S(\rho_B^{j_A}) = S(\rho_E^{j_A})$ when $\rho_{BE}^{j_A}$ is a pure state, then Eq. (12) becomes

$$\begin{aligned} S(Z_A|E) &= S(\rho_{AB}^{Adiag}) - S(\rho_E) \\ &= S(\rho_{AB}^{Adiag}) - S(\rho_{AB}) \\ &= BD_{J_A}(\rho_{AB}) \end{aligned} \quad (15)$$

where the last equation is the relative entropy measure of basis-dependent discord given in Table. 3. \square

V. UNIFYING MEASURES OF QUANTUM RESOURCES

With the help of the framework of BD-discord, now we are ready to unify the measures of different quantum resources.

A. BD-discord to coherence

In previous section, BD-discord is extended from bipartite coherence. And now we redefine the original single partite coherence [2] with BD-discord.

Theorem 1. *The BD-discord measure of a tensor product state $\rho_A \otimes \rho_B$ is a coherence monotone of ρ_A , i.e.,*

$$C_{J_A}(\rho_A) = BD_{J_A}(\rho_A \otimes \rho_B) \quad (16)$$

For simplicity, we can calculate the coherence of ρ_A by $BD_{J_A}(\rho_A \otimes I_B)$, where I_B is an identity matrix of B . If $BD_{J_A}(\rho_A \otimes \rho_B)$ is further convex over $\rho_A \otimes \rho_B$, $C_{J_A}(\rho_A)$ becomes a coherence measure.

Proof. First, for an incoherent state $\sigma_A = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A|$, $\sigma_A \otimes \rho_B = \sum_{j_A} p_{j_A} |j_A\rangle\langle j_A| \otimes \rho_B$ is an IQ state. Then the rhs of Eq. (16) equals to zero, which means $C_{J_A}(\sigma_A) = 0$ for an incoherent state σ_A .

Second, according to the contractivity of a BD-discord measure under Λ_{SQI} , $BD_{J_A}(\Lambda_{SQI}(\rho_A \otimes \rho_B)) = BD_{J_A}(\Lambda_{IO}(\rho_A) \otimes \Phi(\rho_B)) \leq BD_{J_A}(\rho_A \otimes \rho_B)$, where Λ_{IO} is an incoherent operation and Φ is an arbitrary operation. Then $C_{J_A}(\Lambda_{IO}(\rho_A)) \leq C_{J_A}(\rho_A)$, which means $C_{J_A}(\rho_A)$ is contractive under Λ_{IO} .

Finally, if $BD_{J_A}(\rho_A \otimes \rho_B)$ is convex over $\rho_A \otimes \rho_B$, i.e., $BD_{J_A}(\rho_A \otimes \rho_B) \leq \sum_n p_n BD_{J_A}(\rho_A^n \otimes \rho_B^n)$, where $\rho_A \otimes \rho_B = \sum_n p_n \rho_A^n \otimes \rho_B^n$. Then $C_{J_A}(\rho_A) \leq \sum_n p_n C_{J_A}(\rho_A^n)$, which shows the convexity of $C_{J_A}(\rho_A)$. A coherence monotone with convexity is a coherence measure. \square

B. BD-discord to discord

Furthermore, we can define a discord measure from any BD-discord measure. The free state for discord we consider here is the classical-quantum state, i.e.,

$$\sigma_{AB}^{CQ} = \sum_n p_n |n\rangle\langle n| \otimes \rho_B^n, \quad (17)$$

where $\{|n\rangle\}$ is orthogonal for different n , $p_n \in [0, 1], \forall n$ and $\sum_n p_n = 1$. As the set of classical-quantum state contains all the incoherent-quantum state in different local bases, one can regard discord as a basis-independent version of BD-discord. Based on such an intuition, we can define a discord measure by Theorem 2.

Theorem 2. *A discord measure is a minimization of BD-discord measure over local bases, i.e.,*

$$D(\rho_{AB}) = \min_{U_A} BD_{J_A}(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I) \quad (18)$$

We leave the proof in Appendix C.

C. BD-discord to entanglement

To define entanglement measures from BD-discord measures, we consider the strong adversary scenario in [53]. For a given input state ρ_{AB} , some phase information is encoded in the local basis J_A , i.e., by a local operation of $U_A = \sum_{j_A=1}^{d_A} e^{i\phi_{j_A}} |j_A\rangle\langle j_A|$. After the phase encoding, a joint measurement is performed on both A and B to extract the phase information. It turns out that the interferometry power, i.e., how much phase information can be extracted corresponds to the BD-discord of the input state ρ_{AB} . A strong adversary holds a purification of ρ_{AB} with $\rho_{AB} = \text{tr}_E(|\Psi\rangle\langle\Psi|_{ABE})$. In order to let the extracted phase information as little as possible, the adversary will choose an optimal measurement on her local quantum system E and rotate the phase-encoding basis according to the measurement results. In this case the interferometry power corresponds to entanglement. Since the the local measurement on E will effectively make the remaining system be with a certain decomposition $\rho_{AB} = \sum_e p_e |\psi_{AB}\rangle\langle\psi_{AB}|_e$ and the basis rotation operation depends on e , the interferometry power will be minimized over all kinds of decompositions and the local unitary operations on A . Therefore we have the following theorem.

Theorem 3. *An entanglement measure is a convex roof of a discord measure, i.e.,*

$$E(\rho_{AB}) = \min_{p_e, |\psi_{AB}\rangle_e} \sum_e p_e \min_{U_A^e} BD_{J_A}(U_A^e \otimes I |\psi_{AB}\rangle\langle\psi_{AB}|_e U_A^{\dagger e} \otimes I) \quad (19)$$

where the minimization is over all possible decompositions of $\rho_{AB} = \sum_e p_e |\psi_{AB}\rangle\langle\psi_{AB}|_e$ and $|\psi_{AB}\rangle_e$ is a pure state.

We leave the proof in Appendix D.

D. Example with distance-based measures

In this section we show an example of the measure unification of different quantum resources, the distance-based measures. Given distance-based BD-discord in Eq. (8), the distance-based coherence, discord and entanglement measures are given by Theorem 1, Theorem 2 and Theorem 3

$$\begin{aligned}
C_{J_A}(\rho_A) &= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_A \otimes \rho_B || \sigma_{AB}^{IQ}) \\
D(\rho_{AB}) &= \min_{U_A} \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I || \sigma_{AB}^{IQ}) \\
E(\rho_{AB}) &= \min_{p_e, |\psi_{AB}\rangle_e} \sum_e p_e \min_{U_A^e} \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^e \otimes I |\psi_{AB}\rangle_e \langle \psi_{AB}|_e U_A^{\dagger e} \otimes I || \sigma_{AB}^{IQ}).
\end{aligned} \tag{20}$$

We note that the unification results will also be applied for other measures. And the operational meanings of each resource will be consistent in our unification framework. For example, the relative entropy measure of BD-discord will be transformed into distillable coherence, discord and entanglement by (20) which are also quantified by relative entropy.

VI. DISCUSSION AND CONCLUSION

In this work, we propose a unification framework on coherence, basis-dependent discord, discord and entanglement. We begin with constructing a resource framework of basis-dependent discord. As a bridge, basis-dependent discord connects coherence for their basis-dependence nature. On the other hand, it relates discord and entanglement since they all characterize bipartite quantum correlations. A unification framework of these quantum resources is established with the help of BD-discord. Moreover, we give the operational meanings of basis-dependent discord in QKD, which correspond to the local randomness of keys.

For future work, it is interesting to generalize these results to continuous variable cases, especially for Gaussian states. Discord and entanglement for Gaussian states have been well defined based on covariance matrix presentations [38, 66], however, the quantum coherence or a coherence-like basis-dependent quantity is still missing. This work can provide an inspiration to complete the unifications of quantum resources for continuous variables. And this will also help us understand the quantum resource behind the secure keys in continuous variable QKD.

Acknowledgement We acknowledge Y. Zhou and X. Zhang for the insightful discussions. This work was supported by the National Natural Science Foundation of China Grants No. 11674193, the National Key R&D Program of China (2017YFA0303900, 2017YFA0304004), the National Research Foundation (NRF), NRF-Fellowship (Reference No: NRF-NRFF2016-02), BP plc and the EPSRC National Quantum Technology Hub in Networked Quantum Information Technology (EP/M013243/1).

H.Z. and X.Y. contributed equally to this work.

Appendix A: Framework of discord and entanglement

1. Discord

In this part, we briefly review the framework for quantum discord [38, 67, 68] in a bipartite system AB .

Definition of classical state. A state is classical for discord when it is a classical-quantum state, i.e.,

$$\sigma_{AB}^{CQ} = \sum_n p_n |n\rangle\langle n| \otimes \rho_B^n, \tag{A1}$$

where $\{|n\rangle\}$ is orthogonal for different n , $p_n \in [0, 1], \forall n$ and $\sum_n p_n = 1$.

Definition of classical operation. The classical operation for discord is defined by local operations on B , i.e., $I_A \otimes \Phi_B$.

Discord measure. A discord measure $D(\rho_{AB})$ is defined by a function that maps a quantum states ρ to a real non-negative number, which satisfies the following conditions in Table 4:

Table 4: Discord properties.

- (D1) $D(\sigma_{AB})$ vanishes for classical-quantum states, $\sigma_{AB} = \sum_n p_n |n\rangle\langle n| \otimes \rho_B^n$.
- (D2) *Monotonicity:* $D(\rho_{AB})$ cannot increase under local operations, $D(I_A \otimes \Phi_B(\rho_{AB})) \leq D(\rho_{AB})$.
- (D3) $D(\rho_{AB})$ is invariant under all local unitary operations, $D(\rho_{AB}) = D(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger)$.

2. Entanglement

In this part, we summarize the framework for entanglement [24, 69, 70] in a bipartite system AB .

Definition of classical state. A state is classical for entanglement when it is separable, i.e.,

$$\sigma_{AB}^{sep.} = \sum_n p_n \rho_A^n \otimes \rho_B^n, \tag{A2}$$

where $p_n \in [0, 1], \forall n$ and $\sum_n p_n = 1$.

Definition of classical operation. The classical operation for entanglement is defined by local operation and classical communication (LOCC). In the following, we denote LOCC operations by Λ_{LOCC} .

Entanglement measure. An entanglement measure $D(\rho_{AB})$ is defined by a function that maps a quantum states ρ to a real non-negative number, which satisfies the following conditions in Table 5:

Table 5: Entanglement properties.

- (E1) $E(\rho_{AB})$ vanishes when ρ_{AB} is separable.
- (E2) *Monotonicity:* $E(\rho_{AB})$ cannot increase under LOCC operation, that is, (E2a) $E[\Lambda_{LOCC}(\rho_{AB})] \leq E(\rho_{AB})$. This condition is often replaced by another stronger one. (E2b) $E(\rho_{AB})$ should not increase on average under LOCC operations which map ρ_{AB} to ρ_{AB}^k with probability p_k , then $\sum_k p_k E(\rho_{AB}^k) \leq E(\rho_{AB})$.
- (E3) *Convexity:* $E(\rho_{AB})$ decreases under mixing, $E(\sum_k p_k \rho_{AB}^k) \leq \sum_k p_k E(\rho_{AB}^k)$.
- (E4) $E(\rho_{AB})$ is invariant under all local unitary operations, that is, $E(\rho_{AB}) = E(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger)$.

Appendix B: Proofs for BD-discord measures

In order to formulate the conditions of BD-discord measures, we investigate the properties of incoherent unitary operations.

Lemma 1. *The Kraus operator of a unitary operation is unique.*

Proof. Consider a unitary operation U , one possible Kraus operator representation can be written as UU^\dagger which is rank 1. All of its other Kraus operator representations are $E_i = \sum_j u_{ij} U_j$, where u_{ij} is a unitary matrix. Since U is rank 1, the matrix u_{ij} reduces to 1 and $E_i = U$. \square

Lemma 2. *If a unitary operation is an incoherent operation, its inverse operation is also an incoherent operation*

Proof. Consider a unitary operation U , it has unique Kraus operator representation UU^\dagger according to Lemma 1. If it is an incoherent operation, we have

$$C(U\rho U^\dagger) \leq C(\rho) \quad (\text{B1})$$

for an arbitrary state ρ . Assume that its reverse operation, $U^{-1} = U^\dagger$, is not an incoherent operation, then $C(\rho) = C(U^\dagger U \rho U^\dagger U) > C(U\rho U^\dagger)$, which leads to a contradiction with Eq. (B1). \square

Lemma 3. *The coherence of an arbitrary state is invariant under incoherent unitary operations.*

Proof. Consider an incoherent unitary operation U . Its reverse operation U^\dagger is also a unitary operation U according to Lemma 2. Then $C(\rho) = C(U^\dagger U \rho U^\dagger U) \leq C(U\rho U^\dagger)$. On the other hand, $C(U\rho U^\dagger) \leq C(\rho)$ since U is an incoherent operation, which leads to $C(\rho) = C(U\rho U^\dagger)$. \square

With the lemmas above, we can first prove that the distance-based measure in Eq. (8) satisfy all the conditions of a BD-discord measure.

Proof. Proof of (BD1). It is straightforward that $BD_{J_A}(\sigma_{AB}^{QI}) = 0$ according to the definition.

Proof of (BD2). We have such relations

$$\begin{aligned} & BD_{J_A}[\Lambda_{SQI}(\rho_{AB})] \\ &= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\Lambda_{SQI}(\rho_{AB}) || \sigma_{AB}^{IQ}) \\ &= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\Lambda_{SQI}(\rho_{AB}) || \Lambda_{SQI}(\sigma_{AB}^{IQ})) \quad (\text{B2}) \\ &\leq \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB} || \sigma_{AB}^{IQ}) \\ &= BD_{J_A}(\rho_{AB}), \end{aligned}$$

where the second equality is because $\Lambda_{SQI}(\sigma_{AB}^{IQ}) \in \delta_{IQ}$ and the inequality is due to the contractive nature of a distance measure, i.e, the distance will not increase under a completely positive and trace preserving (CPTP) map.

Proof of (BD3). Note the local incoherent unitary operation on A and a unitary operation on B as $U_A^I \otimes U_B$, which is a SQI operation, then

$$\begin{aligned} & \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^I \otimes U_B \rho_{AB} U_A^{I\dagger} \otimes U_B^\dagger || \sigma_{AB}^{IQ}) \\ & \leq \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB} || \sigma_{AB}^{IQ}) \quad (\text{B3}) \end{aligned}$$

On the other hand, the reverse operation $U_A^{I\dagger} \otimes U_B^\dagger$ is also a SQI operation since $U_A^{I\dagger}$ is an incoherent operation according to Lemma 2, then

$$\begin{aligned} & \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB} || \sigma_{AB}^{IQ}) \\ &= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^I U_A^{I\dagger} \otimes U_B U_B^\dagger \rho_{AB} U_A^{I\dagger} U_A^I \otimes U_B^\dagger U_B || \sigma_{AB}^{IQ}) \\ &\leq \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^I \otimes U_B \rho_{AB} U_A^{I\dagger} \otimes U_B^\dagger || \sigma_{AB}^{IQ}) \quad (\text{B4}) \end{aligned}$$

Thus we conclude that

$$\begin{aligned} & \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(\rho_{AB} || \sigma_{AB}^{IQ}) \\ &= \min_{\sigma_{AB}^{IQ} \in \delta_{IQ}} d(U_A^I \otimes U_B \rho_{AB} U_A^{I\dagger} \otimes U_B^\dagger || \sigma_{AB}^{IQ}) \quad (\text{B5}) \end{aligned}$$

\square

Next we prove that the convex roof measure Eq. (9) also satisfies all conditions of a BD-discord measure.

Proof. Proof of (BD1). Consider the spectral decomposition of $\rho_B^{j_A}$ in Eq. (6), an IQ state can be rewritten as

$$\sigma_{AB}^{IQ} = \sum_{j_A, l_{j_A}} p_{j_A} l_{j_A} |j_A l_{j_A}\rangle \langle j_A l_{j_A}|. \quad (\text{B6})$$

□

For each pure state component $|j_A l_{j_A}\rangle$, the local randomness is zero according to Eq. (10). And such a decomposition is an optimal decomposition due to the non-negativity of a BD-discord measure.

Proof of (BD2). Suppose the optimal decomposition is $\rho_{AB} = \sum_e p_e |\psi_{AB}\rangle \langle \psi_{AB}|_e$. For an arbitrary component $|\psi_{AB}\rangle_e$, the local randomness is

$$R(|\psi_{AB}\rangle_e) = S\left(\sum_{j_A} |\langle j_A | \text{tr}_B |\psi_{AB}\rangle|^2 |j_A\rangle \langle j_A|\right) \quad (\text{B7})$$

We notice that Eq. (B7) is equal to the relative entropy of BD-discord measure of $|\psi_{AB}\rangle$, which is a distance-based measure and contractive under Λ_{SQI} . The convex roof is a mixture of the local randomness for each pure state component, and the mixture is also contractive under Λ_{SQI} .

Proof of (BD3). Same as the proof for distance-based measure.

Appendix C: Proof of Theorem 2

Proof. Proof of (D1). For a classical-quantum state in Eq. (A1), we set $U_A|n\rangle = |j_A\rangle$ for $n = 1, 2, \dots, d_A$, then

$$\begin{aligned} & BD_{J_A} \left[U_A \otimes I \left(\sum_n^{d_A} p_n |n\rangle \langle n| \otimes \rho_B^n \right) U_A^\dagger \otimes I \right] \\ &= BD_{J_A} \left[\sum_n^{d_A} p_n (U_A |n\rangle \langle n| U_A^\dagger) \otimes \rho_B^n \right] \\ &= BD_{J_A} \left(\sum_{j_A}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \rho_B^n \right) \\ &= 0. \end{aligned} \quad (\text{C1})$$

We can see that such a U_A is optimal, which realizes a minimization of $D(\rho_{AB})$ due to the non-negativity of a basis-dependent discord measure.

Proof of (D2). Since $I \otimes \Phi_B \subset \Lambda_{SQI}$, from (BD2) we have

$$BD_{J_A}[I \otimes \Phi_B(\rho_{AB})] \leq BD_{J_A}(\rho_{AB}) \quad (\text{C2})$$

and their minimization on the local basis also satisfies

$$\begin{aligned} & \min_{U_A} BD_{J_A}[I \otimes \Phi_B(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I)] \\ & \leq \min_{U_A} BD_{J_A}(U_A \otimes I \rho_{AB} U_A^\dagger \otimes I) \end{aligned} \quad (\text{C3})$$

Proof of (D3). Our target is to prove

$$\begin{aligned} & \min_{U_A} BD_{J_A}[(U_A \otimes U_B)(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)(U_A^\dagger \otimes U_B^\dagger)] \\ &= \min_{U_A} BD_{J_A}[(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \end{aligned} \quad (\text{C4})$$

Note that, in our definition of discord, the minimization is over all local basis, it is equal to prove that

$$\begin{aligned} & \min_{U_A} BD_{J_A}[(I \otimes U_B)(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)(I \otimes U_B^\dagger)] \\ &= \min_{U_A} BD_{J_A}[(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \end{aligned} \quad (\text{C5})$$

Since $I \otimes U_B \subset I \otimes \Phi_B$, according to (D2) we have

$$\begin{aligned} & \min_{U_A} BD_{J_A}[(I \otimes U_B)(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)(I \otimes U_B^\dagger)] \\ & \leq \min_{U_A} BD_{J_A}[(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \end{aligned} \quad (\text{C6})$$

Apply local operation $I \otimes U_B^\dagger$ on both sides,

$$\begin{aligned} & \min_{U_A} BD_{J_A}[(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \\ & \leq \min_{U_A} BD_{J_A}[(I \otimes U_B^\dagger)(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)(I \otimes U_B)] \end{aligned} \quad (\text{C7})$$

As the local operation $I \otimes U_B^\dagger \subset I \otimes \Phi_B$, we also have

$$\begin{aligned} & \min_{U_A} BD_{J_A}[(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)] \\ & \leq \min_{U_A} BD_{J_A}[(I \otimes U_B^\dagger)(U_A \otimes I) \rho_{AB} (U_A^\dagger \otimes I)(I \otimes U_B)] \end{aligned} \quad (\text{C8})$$

Thus we prove Eq. (C4). □

Appendix D: Proof of Theorem 3

Proof. Since condition (E2a) can be derived with (E2b) and (E3),

$$\begin{aligned} E(\Lambda_{LOCC}(\rho_{AB})) &= E\left(\sum p_n \rho_{AB}^n\right) \\ &\stackrel{C3}{\leq} \sum p_n E(\rho_{AB}^n) \\ &\stackrel{C2b}{\leq} E(\rho_{AB}), \end{aligned} \quad (\text{D1})$$

where $\rho_{AB}^n = \hat{K}_n \rho_{AB} \hat{K}_n^\dagger / p_n$ and $p_n = \text{Tr}(\hat{K}_n \rho_{AB} \hat{K}_n^\dagger)$, we only need to prove (E1), (E2b), (E3) and (E4).

Proof of (E1). Since the set of separable states is convex and closed, a separable state $\sigma_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j$ can always be expressed as a mixture of pure separable states, i.e., product states.

$$\sigma_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j = \sum_e p_e |\psi_A\rangle_e \langle \psi_A|_e |\psi_B\rangle_e \langle \psi_B|_e \quad (\text{D2})$$

Substitute Eq. (D2) into Eq. (19), for each pure state component $|\psi_A\rangle_e \langle \psi_B|_e$, we set a certain U_A^e such that $U_A^e |\psi_A\rangle = |j_A\rangle$, then

$$\begin{aligned}
& \min_{p_e, |\psi_{AB}\rangle_e} \sum_e p_e \min_{U_A^e} BD_{J_A}(U_A^e \otimes I |\psi_A\rangle_e |\psi_B\rangle_e \langle \psi_A|_e \langle \psi_B|_e U_A^{\dagger e} \otimes I) \\
&= \min_{p_e, |\psi_{AB}\rangle_e} \sum_e p_e BD_{J_A}(|j_A\rangle \langle j_A| \otimes |\psi_B\rangle_e \langle \psi_B|) \\
&= 0
\end{aligned} \tag{D3}$$

We can see that such a st of U_A^e and decomposition are optimal, which realizes a minimization of $E(\rho_{AB})$ due to the non-negativity of a basis-dependent discord measure.

Proof of (E3). Suppose an arbitrary decomposition of $\rho_{AB} = \sum_l p_l \rho_{AB}^l$, and

$$\begin{aligned}
& \sum_l p_l E(\rho_{AB}^l) = \\
& \sum_l p_l \min_{p_e} \sum_e p_e \min_{U_A^e} BD_{J_A}(U_A^e \otimes I |\psi_{AB}\rangle_e \langle \psi_{AB}|_e U_A^{\dagger e} \otimes I)
\end{aligned} \tag{D4}$$

where we simplify the subscript of minimizing decomposition $p_e, |\psi_{AB}\rangle_e$ to p_e . Suppose for each component ρ_{AB}^l the optimal decomposition is $\rho_{AB}^l = \sum_{e_l} p_{e_l}^l |\psi_{AB}\rangle_{e_l}^l \langle \psi_{AB}|_{e_l}^l$, and we can further rewrite Eq. (D4) as

$$\begin{aligned}
& \sum_l p_l E(\rho_{AB}^l) \\
&= \sum_l p_l \sum_{e_l} p_{e_l}^l \min_{U_A^e} BD_{J_A}(U_A^e \otimes I |\psi_{AB}\rangle_{e_l}^l \langle \psi_{AB}|_{e_l}^l U_A^{\dagger e} \otimes I) \\
&= \sum_l \sum_{e_l} p_l p_{e_l}^l \min_{U_A^e} BD_{J_A}(U_A^e \otimes I |\psi_{AB}\rangle_{e_l}^l \langle \psi_{AB}|_{e_l}^l U_A^{\dagger e} \otimes I)
\end{aligned} \tag{D5}$$

Similarly we assume the optimal decomposition for ρ_{AB} is $\rho_{AB} = \sum_e p_e |\psi_{AB}\rangle_e \langle \psi_{AB}|_e$

$$\begin{aligned}
& E(\rho_{AB}) = \\
& \sum_e p_e \min_{U_A^e} BD_{J_A}(U_A^e \otimes I |\psi_{AB}\rangle_e \langle \psi_{AB}|_e U_A^{\dagger e} \otimes I)
\end{aligned} \tag{D6}$$

Compare Eq. (D5) and Eq. (D6), we can see that they are all probabilistic mixture of bipartite pure state discord. However, the ways of decomposition in Eq. (D6) is more than those in Eq. (D5) since the latter is constrained by the decomposition $\rho_{AB} = \sum_l p_l |\psi_{AB}\rangle_e \langle \psi_{AB}|_e$. Then we conclude that

$$E(\rho_{AB}) \geq \sum_l p_l E(\rho_{AB}^l) \tag{D7}$$

Proof of (E2b). Suppose the decomposition of ρ_{AB} that achieves minimum of $E(\rho_{AB})$ is $\rho_{AB} = \sum_e p_e \rho_{AB}^e$, where ρ_{AB}^e is a pure state. After the CPTP channel of

LOCC,

$$\begin{aligned}
\rho_{AB}^n &= \frac{\hat{K}_n \rho_{AB} \hat{K}_n^\dagger}{p_n} \\
&= \sum_e \frac{p_e}{p_n} \hat{K}_n \rho_{AB}^e \hat{K}_n^\dagger \\
&= \sum_e \frac{p_e}{p_n} p_{en} \rho_{AB}^{en}
\end{aligned} \tag{D8}$$

where $p_{en} = \text{Tr}(\hat{K}_n \rho_{AB}^e \hat{K}_n^\dagger)$ and $\rho_{AB}^{en} = \hat{K}_n \rho_{AB}^e \hat{K}_n^\dagger / p_{en}$. Then we have

$$\begin{aligned}
E(\rho_{AB}) &= \sum_e p_e \min_{U_A} BD_{J_A}(U_A \otimes I \rho_{AB}^e U_A^\dagger \otimes I) \\
&\geq \sum_e p_e \min_{U_A} \sum_n p_{en} BD_{J_A}(U_A \otimes I \rho_{AB}^{en} U_A^\dagger \otimes I) \\
&\geq \sum_e p_e \sum_n p_{en} \min_{U_{An}} BD_{J_A}(U_{An} \otimes I \rho_{AB}^{en} U_{An}^\dagger \otimes I) \\
&= \sum_n \sum_e p_e p_{en} \min_{U_{An}} BD_{J_A}(U_{An} \otimes I \rho_{AB}^{en} U_{An}^\dagger \otimes I) \\
&= \sum_n p_n \sum_e \frac{p_e p_{en}}{p_n} \min_{U_{An}} BD_{J_A}(U_{An} \otimes I \rho_{AB}^{en} U_{An}^\dagger \otimes I) \\
&\geq \sum_n p_n E(\rho_{AB}^n),
\end{aligned} \tag{D9}$$

where the first inequality is due to the selective monotonicity of distance-based BD-discord measure,

$$BD_{J_A}(\rho_{AB}) \geq \sum_n p_n BD_{J_A}(\rho_{AB}^n) \tag{D10}$$

the second inequality is because the minimization over local basis according to each component after channel U_{An} is more powerful than an entire minimization U_A .

Proof of (E4). Local unitary operations $U_A \otimes U_B$ belong to LOCC. Then according to (E2a),

$$E(\rho_{AB}) \geq E(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) \tag{D11}$$

Apply $U_A^\dagger \otimes U_B^\dagger$ to the last equation,

$$E(U_A^\dagger \otimes U_B^\dagger \rho_{AB} U_A \otimes U_B) \geq E(\rho_{AB}) \tag{D12}$$

On the other hand, operations $U_A^\dagger \otimes U_B^\dagger$ also belong to LOCC.

$$E(U_A^\dagger \otimes U_B^\dagger \rho_{AB} U_A \otimes U_B) \leq E(\rho_{AB}) \tag{D13}$$

Therefore we have

$$E(\rho_{AB}) = E(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) \tag{D14}$$

□

-
- [1] J. Aberg, eprint arXiv:quant-ph/0612146 (2006), quant-ph/0612146.
- [2] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [3] X. Yuan, H. Zhou, Z. Cao, and X. Ma, *Phys. Rev. A* **92**, 022124 (2015).
- [4] X. Yuan, Q. Zhao, D. Girolami, and X. Ma, arXiv preprint arXiv:1605.07818 (2016).
- [5] M. Hayashi and H. Zhu, *Phys. Rev. A* **97**, 012302 (2018).
- [6] V. Giovannetti, S. Lloyd, and L. Maccone, *Science* **306**, 1330 (2004).
- [7] I. Marvian and R. W. Spekkens, *Phys. Rev. A* **94**, 052324 (2016).
- [8] B. Escher, R. de Matos Filho, and L. Davidovich, *Nature Physics* **7**, 406 (2011).
- [9] P. Giorda and M. Allegra, *Journal of Physics A: Mathematical and Theoretical* **51**, 025302 (2017).
- [10] M. Hillery, *Phys. Rev. A* **93**, 012111 (2016).
- [11] N. Anand and A. K. Pati, arXiv preprint arXiv:1611.04542 (2016).
- [12] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, *Phys. Rev. Lett.* **116**, 160407 (2016).
- [13] J. Matera, D. Egloff, N. Killoran, and M. Plenio, *Quantum Science and Technology* **1**, 01LT01 (2016).
- [14] J. Åberg, *Phys. Rev. Lett.* **113**, 150402 (2014).
- [15] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, *Phys. Rev. X* **5**, 021001 (2015).
- [16] P. Źwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, *Phys. Rev. Lett.* **115**, 210403 (2015).
- [17] M. T. Mitchison, M. P. Woods, J. Prior, and M. Huber, *New Journal of Physics* **17**, 115013 (2015).
- [18] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, *Journal of Physics A: Mathematical and Theoretical* **49**, 143001 (2016).
- [19] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyán, M. Huber, and A. Acín, *Phys. Rev. Lett.* **118**, 070601 (2017).
- [20] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [21] S. Hill and W. K. Wootters, *Phys. Rev. Lett.* **78**, 5022 (1997).
- [22] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).
- [23] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
- [24] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [25] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
- [26] M. B. Plenio and S. Virmani, *Quantum Information & Computation* **7**, 1 (2007).
- [27] M. Horodecki and J. Oppenheim, *International Journal of Modern Physics B* **27**, 1345019 (2013).
- [28] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
- [29] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing* (IEEE Press, New York, 1984) pp. 175–179.
- [30] A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
- [31] C. H. Bennett and S. J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).
- [32] H. Ollivier and W. H. Zurek, *Phys. Rev. Lett.* **88**, 017901 (2001).
- [33] J. Maziero, L. C. Céleri, R. M. Serra, and V. Vedral, *Phys. Rev. A* **80**, 044102 (2009).
- [34] B. Dakić, V. Vedral, and i. c. v. Brukner, *Phys. Rev. Lett.* **105**, 190502 (2010).
- [35] D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, and A. Winter, *Phys. Rev. A* **83**, 032324 (2011).
- [36] M. Gu, H. M. Chrzanowski, S. M. Assad, T. Symul, K. Modi, T. C. Ralph, V. Vedral, and P. K. Lam, *Nature Physics* **8**, 671 (2012).
- [37] B. Dakić, Y. O. Lipp, X. Ma, M. Ringbauer, S. Kropatschek, S. Barz, T. Paterek, V. Vedral, A. Zeilinger, Č. Brukner, *et al.*, *Nature Physics* **8**, 666 (2012).
- [38] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, *Rev. Mod. Phys.* **84**, 1655 (2012).
- [39] A. Datta, A. Shaji, and C. M. Caves, *Phys. Rev. Lett.* **100**, 050502 (2008).
- [40] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, *Phys. Rev. A* **92**, 022112 (2015).
- [41] K. Bu, L. Li, S.-M. Fei, and J. Wu, arXiv preprint arXiv:1712.09167 (2017).
- [42] M.-L. Hu and H. Fan, *Phys. Rev. A* **95**, 052106 (2017).
- [43] B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral, *Phys. Rev. X* **6**, 041028 (2016).
- [44] M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, *Phys. Rev. Lett.* **106**, 220403 (2011).
- [45] A. Streltsov, H. Kampermann, and D. Bruß, *Phys. Rev. Lett.* **106**, 160401 (2011).
- [46] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, *Phys. Rev. Lett.* **115**, 020403 (2015).
- [47] X. Qi, T. Gao, and F. Yan, *Journal of Physics A: Mathematical and Theoretical* **50**, 285301 (2017).
- [48] S. Chin, *Journal of Physics A: Mathematical and Theoretical* **50**, 475302 (2017).
- [49] H. Zhu, Z. Ma, Z. Cao, S.-M. Fei, and V. Vedral, *Phys. Rev. A* **96**, 032316 (2017).
- [50] K. C. Tan, H. Kwon, C.-Y. Park, and H. Jeong, *Phys. Rev. A* **94**, 022329 (2016).
- [51] M. Horodecki, J. Oppenheim, and A. Winter, *Nature* **436**, 673 (2005).
- [52] A. Streltsov, E. Chitambar, S. Rana, M. N. Bera, A. Winter, and M. Lewenstein, *Phys. Rev. Lett.* **116**, 240405 (2016).
- [53] X. Yuan, H. Zhou, M. Gu, and X. Ma, *Phys. Rev. A* **97**, 012331 (2018).
- [54] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern, *Physics Reports* **583**, 1 (2015).
- [55] R. Gallego and L. Aolita, *Phys. Rev. X* **5**, 041008 (2015).
- [56] S. Du, Z. Bai, and X. Qi, *Quantum Information & Computation* **15**, 1307 (2015).
- [57] F. G. S. L. Brandão and G. Gour, *Phys. Rev. Lett.* **115**, 070503 (2015).
- [58] A. Winter and D. Yang, *Phys. Rev. Lett.* **116**, 120404 (2016).

- [59] E. Chitambar and G. Gour, Phys. Rev. Lett. **117**, 030401 (2016).
- [60] E. Chitambar and M.-H. Hsieh, Phys. Rev. Lett. **117**, 020402 (2016).
- [61] A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. **89**, 041003 (2017).
- [62] T. Theurer, N. Killoran, D. Egloff, and M. B. Plenio, Phys. Rev. Lett. **119**, 230401 (2017).
- [63] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein, Phys. Rev. Lett. **116**, 070402 (2016).
- [64] A. Streltsov, S. Rana, M. N. Bera, and M. Lewenstein, Phys. Rev. X **7**, 011024 (2017).
- [65] I. Devetak and A. Winter, in *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Vol. 461 (The Royal Society, 2005) pp. 207–235.
- [66] M. Horodecki, J. Oppenheim, and A. Winter, Communications in Mathematical Physics **269**, 107 (2007).
- [67] L. Henderson and V. Vedral, Journal of Physics A: Mathematical and General **34**, 6899 (2001).
- [68] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
- [69] C. Bennett, D. DiVincenzo, J. Smolin, and W. Wootters, Phys. Rev. A **54**, 3824 (1996).
- [70] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).