# Bounded Budget Betweenness Centrality Game for Strategic Network Formations

Xiaohui Bei[1], Wei Chen[2], Shang-Hua Teng[3,⋆], Jialin Zhang[1], and Jiajie Zhu[4]

[1] Institute for Theoretical Computer Science, Tsinghua University
{bxh08,zhanggl02}@mails.tsinghua.edu.cn
[2] Microsoft Research Asia
weic@microsoft.com
[3] University of Southern California
shanghua@usc.edu
[4] University of California at Los Angeles
jiajie@cs.ucla.edu

**Abstract.** In this paper, we introduce the *bounded budget betweenness centrality* game, a strategic network formation game in which nodes build connections subject to a budget constraint in order to maximize their betweenness centrality, a metric introduced in the social network analysis to measure the information flow through a node. To reflect real world scenarios where short paths are more important in information exchange, we generalize the betweenness definition to only consider shortest paths of length at most $\ell$. We present both complexity and constructive existence results about Nash equilibria of the game. For the nonuniform version of the game where node budgets, link costs, and pairwise communication weights may vary, we show that Nash equilibria may not exist and it is NP-hard to decide whether Nash equilibria exist in a game instance. For the uniform version of the game where link costs and pairwise communication weights are one and each node can build $k$ links, we construct two families of Nash equilibria based on shift graphs, and study the properties of Nash equilibria. Moreover, we study the complexity of computing best responses and show that the task is polynomial for uniform 2-B$^3$C games and NP-hard for other games.

**Keywords:** algorithmic game theory, network formation game, Nash equilibrium, betweenness centrality.

## 1 Introduction

Many network structures in real life are not designed by central authorities. Instead, they are formed by autonomous agents who often have selfish motives [17]. Typical examples of such networks include the Internet where autonomous systems linked together to achieve global connection, peer-to-peer networks where peers connect to one another for online file sharing (e.g. [7,19]), and social networks where individuals connect to one another for information exchange and other social functions [18]. Since these autonomous agents have their selfish motives and are not under any centralized control,

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they often act strategically in deciding whom to connect to in order to improve their own benefits. This gives rise to the field of *network formation games*, which studies the game-theoretic properties of the networks formed by these selfish agents as well as the process in which all agents dynamically adjust their strategies [1,9,13,14,15].

A key measure of importance of a node is its *betweenness centrality* (or betweenness for short), which is introduced originally in social network analysis [10,16]. If we view a network as a graph $G = (V, E)$ (directed or undirected), the betweenness of a node (or vertex) $i$ in $G$ is

$$btw_i(G) = \sum_{u \neq v \neq i \in V,\, m(u,v)>0} w(u,v) \frac{m_i(u,v)}{m(u,v)} \tag{1}$$

where $m(u, v)$ is the number of shortest paths from $u$ to $v$ in $G$, $m_i(u, v)$ is the number of shortest paths from $u$ to $v$ that pass $i$ in $G$, and $w(u, v)$ is the weight on pair $(u, v)$. Intuitively, if the amount of information from $u$ to $v$ is $w(u, v)$, and the information is passed along all shortest paths from $u$ to $v$ equally, then the betweenness of node $i$ measures the amount of information passing through $i$ among all pair-wise exchanges.

In this paper, we generalize the betweenness definition with a parameter $\ell$ such that only shortest paths with length at most $\ell$ are considered in betweenness calculation. Formally, we define

$$btw_i(G, \ell) = \sum_{u \neq v \neq i \in V,\, m(u,v,\ell)>0} w(u,v) \frac{m_i(u,v,\ell)}{m(u,v,\ell)}, \tag{2}$$

where $m(u, v, \ell)$ is the number of shortest paths from $u$ to $v$ in $G$ with length at most $\ell$, and $m_i(u, v, \ell)$ is the number of shortest paths from $u$ to $v$ that passes $i$ in $G$ with length at most $\ell$. It is easy to see that $btw_i(G) = btw_i(G, n-1)$, where $n$ is the number of vertices in $G$.

Betweenness with path length constraint is reasonable in real-world scenarios. In peer-to-peer networks such as Gnetella [19], query requests are searched only on nodes with a short graph distance away from the query initiator. In social networks, researches (e.g. [4,5]) show that short connections are much more important than long-range connections.

In a decentralized network with autonomous agents, each agent may have incentive to maximize its betweenness in the network. For example, in computer networks and peer-to-peer networks, a node in the network may be able to charge the traffic that it helps relaying, in which case the revenue of the node is proportional to its betweenness in the network. So the maximization of revenue is consistent with the maximization of the betweenness. In a social network, an individual may want to gain or control the most amount of information travelling in the network by maximizing her betweenness.

In this paper, we introduce a network formation game in which every node in a network is a selfish agent who decides which other nodes to connected to in order to maximize its own betweenness. Building connections with other nodes incur costs. Each node has a budget such that the cost of building its connections cannot exceed its budget. We call this game the *bounded budget betweenness centrality* game or the $B^3C$ game. When distinction is necessary, we use $\ell$-$B^3C$ to denote the games using generalized betweenness definition $btw_i(G, \ell)$.

Bounded budget assumption, first incorporated into a network formation game in [14], reflects real world scenarios where there are physical limits to the number of connections one can make. In computer and peer-to-peer networks, each node usually has a connection limit. In social networks, each individual only has a limited time and energy to create and maintain relationships with other individuals. An alternative treatment to connection costs appearing in more studies [1,9,13,15] is to subtract connection costs from the main objectives to be maximized. This treatment, however, restricts the variety of Nash equilibria exhibited by the game, e.g. only allowing dense graphs to be Nash equilibria [13]. Therefore, in this paper we choose to incorporate the bounded budget assumption, even though it makes the game model more complicated.

In this paper, we consider the directed graph variant of the game, in which nodes can only establish outgoing links to other nodes. Since incoming links help increasing nodes' betweenness, nodes should be happy to accept incoming links created by other nodes. This mitigates the concern on many network formation games in which connection creation is one-sided decision. Since the game allows some trivial Nash equilibria (such as a network with no links at all), we study a stronger form called *maximal* Nash equilibria, in which no node can add more outgoing links without exceeding its budget constraint. Adding outgoing links of a node can only help its betweenness, so it is reasonable to study maximal Nash equilibria in the B$^3$C games.

We present both complexity and existence results about B$^3$C games. First, we study the existence of maximal Nash equilibria in *nonuniform* $\ell$-B$^3$C game, which is specified by several parameters concerning the node budgets, link costs, and pairwise communication weights (Section 3). We show that a nonuniform $\ell$-B$^3$C game may not have any maximal Nash equilibria for any $\ell \geq 2$. Moreover, given these parameters as input, it is NP-hard to determine whether the game has a maximal Nash equilibrium. The result indicates that finding Nash equilibria in general $\ell$-B$^3$C games is a difficult task.

Second, we address the complexity of computing best responses in $\ell$-B$^3$C games (Section 4). For uniform $\ell$-B$^3$C games where all pair weights are one, all link costs are one, and all node budgets are given as an integer $k$, we show that with $\ell = 2$, computing a best response takes $O(n^3)$ time. For all other cases (uniform games with $\ell \geq 3$ or nonuniform games with $\ell \geq 2$), the task is NP-hard.

Finally, we turn our attention to the construction and the properties of Nash equilibria in the *uniform* $\ell$-B$^3$C game with $n$ nodes and $k$ outgoing edges from each node (Section 5). We introduce a type of multi-partite graphs that we call *shift graphs*, which are variants of better known De Bruijn graphs and Kautz graphs. Based on these shift graphs, we construct two different families of Nash equilibria for *uniform* $\ell$-B$^3$C games. One family gives a stronger form of Nash equilibria called strict Nash equilibria, while the other family belongs to what we call *$\ell$-path-unique graphs* ($\ell$-PUGs), which we show are always Nash equilibria for uniform $\ell$-B$^3$C games. We then use $\ell$-PUGs to study several properties of Nash equilibria. In particular, we show that (a) for any $\ell$, $k$ and large enough $n$ ($n \geq (k+\ell)!/k!$), a maximal Nash equilibrium exists; (b) Nash equilibria may exhibit rich structures, e.g. they may be disconnected or have unbalanced in-degrees and betweenness among nodes; and (c) for 2-B$^3$C games, all maximal Nash equilibria must be 2-PUGs if the maximum in-degree is $o(n)$ (with $k$ being a constant). The proofs for all results in this paper are available in our full technical report [2].

**Related work.** There are a number of studies on network formation games with Nash equilibrium as the solution concept [1,3,9,13,14,15]. Most of the above work belong to a class of games in which nodes try to minimize their average shortest distances to other nodes in the network [1,9,14,15], which is called *closeness centrality* in social network analysis [10].

Our research is partly motivated by the work of [13], in which Kleinberg et al. study a different type of network formation games related to the concept of structural holes in organizational social network research. In this game, each node tries to bridge other pairs of nodes that are not directly connected. In a sense, this is a restricted type of betweenness where only length-2 shortest paths are considered. Besides some difference in the game setup, there are two major differences between our work and theirs. First, we consider betweenness with a general path length constraint of $\ell$ as well as no path length constraints, while they only consider the bridging effect between two immediate neighbors of a node. Second, we incorporate budget constraints to restrict the number of links one node can build, while their work subtracts link costs in the payoff function of each node. As the result of their treatment to link costs, they show that all Nash equilibria are limited to dense graphs with $\Omega(n^2)$ edges where $n$ is the number of vertices. This is what we want to avoid in our study. A couple of other studies [6,11] also address strategic network formations with structural holes, but they do not address the computation issue, and their game formats have their own limitations (e.g. star networks as the only type of equilibria [11] or limited to length-2 paths [6]).

Our game is also inspired by the Bounded Budget Connection game of Laoutaris et al. [14]. This game considers directed links and bounded budgets on nodes, using minimization of average shortest distances to others as the objective for each node. It shows hardness results in determining the existence of Nash equilibria in general games, and provides tree-like structures as Nash equilibria for the uniform version of the game. It also shows that Abelian Cayley graphs cannot be Nash equilibria in large networks.

Solution concepts other than Nash equilibrium are also used in the study of network formation games. Authors in [8,12] consider games in which two end points of a link have to jointly agree on adding the link, and they use pairwise stability as an alternative to Nash equilibrium.

## 2 Problem Definition

A *(nonuniform) bounded-budget betweenness centrality* ($\mathrm{B}^3\mathrm{C}$) game with parameters $(n, b, c, w)$ is a network formation game defined as follows. We consider a set of $n$ players $V = \{1, 2, \ldots, n\}$, which are also nodes in a network. Function $b : V \to \mathbb{N}$ specifies the budget $b(i)$ for each node $i \in V$ ($\mathbb{N}$ is the set of natural numbers). Function $c : V \times V \to \mathbb{N}$ specifies the cost $c(i, j)$ for the node $i$ to establish a link to node $j$, for $i, j \in V$. Function $w : V \times V \to \mathbb{N}$ specifies the weight $w(i, j)$ from node $i$ to node $j$ for $i, j \in V$, which can be interpreted as the amount of traffic $i$ sends to $j$, or the importance of the communication from $i$ to $j$.

The strategy space of player $i$ in $\mathrm{B}^3\mathrm{C}$ game is $S_i = \{s_i \subseteq V \setminus \{i\} \mid \sum_{j \in s_i} c(i, j) \leq b(i)\}$, i.e., all possible subsets of outgoing links of node $i$ within $i$'s budget. A strategy profile $s = (s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n$ is referred to as a *configuration*

in this paper. The graph induced by configuration $s$ is denoted as $G_s = (V, E)$, where $E = \{(i, j) \mid i \in V, j \in s_i\}$. For convenience, we will also refer $G_s$ as a configuration.

In the game without path length constraint, the utility of a node $i$ in configuration $G$ is defined by the *betweenness centrality* of $i$ as given in equation (1). When we generalize betweenness centrality and consider only shortest paths of length at most $\ell$, the utility of node $i$ is given as in equation (2). We use $\ell$-B$^3$C to denote the generalized version of game with parameter $\ell$.

In a configuration $s$, if no node can increase its own utility by changing its own strategy unilaterally, we say that $s$ is a *(pure) Nash equilibrium*, and we also say that $s$ is *stable*. Moreover, if in configuration $s$ any strategy change of any node strictly decreases the utility of the node, we say that $s$ is a *strict* Nash equilibrium.

The following Lemma shows the monotonicity of node betweenness when adding new edges to a node, which motivates our definition of maximal Nash equilibrium.

**Lemma 1.** *For any graph $G = (V, E)$, let $G' = (V, E \cup \{(i, j)\})$ where $i, j \in V$ and $(i, j) \notin E$. Then $btw_i(G) \leq btw_i(G')$, and $btw_i(G, \ell) \leq btw_i(G', \ell)$ for all $\ell \geq 2$.*

Given a nonuniform B$^3$C game with parameters $(n, b, c, w)$, a *maximal strategy* of a node $v$ is a strategy with which $v$ cannot add any outgoing edges without exceeding its budget. We say that a graph (configuration) is *maximal* if all nodes use maximal strategies in the configuration. A configuration is a *maximal Nash equilibrium* if it is a maximal graph and it is a Nash equilibrium. By Lemma 1, it makes sense to study maximal Nash equilibria where no node can add more edges within its budget limit. Moreover, trivial non-maximal Nash equilibria exist (e.g. graphs with no edges), making it less interesting to study all Nash equilibria. Therefore, for the rest of the paper, we focus on maximal Nash equilibria in B$^3$C games. The following lemma states the relationship between maximal Nash equilibria and strict Nash equilibria, a direct consequence of the monotonicity of betweenness centrality.

**Lemma 2.** *Given a B$^3$C game with parameters $(n, b, c, w)$, any strict Nash equilibrium in the game is a maximal Nash equilibrium.*

Based on the above lemma, for positive existence of Nash equilibria, we sometimes study the existence of strict Nash equilibria to make our results stronger.

A special case of B$^3$C game is the *uniform* game, which has parameters $n, k \in \mathbb{N}$ such that $b(i) = k$ for all $i \in V$, and $c(i, j) = w(i, j) = 1$ for all $i, j \in V$.

## 3 Determining Nash Equilibria in Nonuniform Games

In this section we show that a nonuniform B$^3$C game may not have any maximal (or strict) Nash equilibrium, and determining whether a game has a maximal (or strict) Nash equilibrium is NP-hard. For simplicity, we address the B$^3$C game without path length constraint first, and then present the results on the $\ell$-B$^3$C game.

### 3.1 Nonexistence of Maximal Nash Equilibria

We fist show that some B$^3$C game with nonuniform edge cost has no maximal (or strict) Nash equilibrium. We construct a family of graphs, which we refer to as the gadget, and

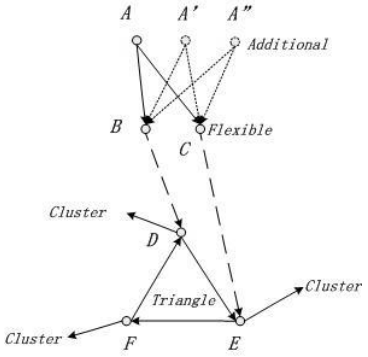



**Fig. 1.** Main structure of the gadget that has no maximal (or strict) Nash Equilibrium

show that B$^3$C games based on the gadget do not have any maximal Nash equilibrium. The gadget is shown in Figure 1. There are $5 + 3t + r$ nodes in the gadget, where $t \in \mathbb{N}$ and $r = 1, 2, 3$. The values of $t$ and $r$ allow us to construct a graph of any size great than 5. There are $r$ nodes, denoted as $A, A', A''$ in the figure, which establish edges to $B$ and $C$. Both $B$ and $C$ can establish at most one edge to a node in $\{D, E, F\}$ respectively. Each node in $\{D, E, F\}$ connects to a cluster of size $t$ each (not shown in the figure). The only requirement for these three clusters is that they are identical to each other and are all strongly connected, so $D, E, F$ can reach all nodes in their corresponding clusters. Nodes in the three clusters do not establish edges to the other clusters or to $A, A', A'', B, C, D, E, F$.

We classify nodes and edges as follows. Nodes $B$ and $C$ are *flexible* nodes since they can choose to connect one node in $\{D, E, F\}$. Nodes $D, E, F$ are *triangle* nodes, nodes in the clusters are *cluster* nodes, and nodes $A, A', A''$, are *additional* nodes. Edges $(i, j)$ with $i \in \{B, C\}$ and $j \in \{D, E, F\}$ are *flexible* edges. Other edges shown in the figure plus the edges in the clusters are *fixed* edges. The remaining pairs with no edge connected (e.g. $(A, D)$, $(A, E)$, etc.) are referred to as *forbidden* edges.

We use the parameters $(n, b, c, w)$ of a B$^3$C game to realize the gadget. In particular, (a) $n = 5 + 3t + r$; (b) $b(i) = 1$ for all $i \in V$; (c) $c(i, j) = 0$ if $(i, j)$ is a fixed edge, $c(i, j) = 1$ if $(i, j)$ is a flexible edge, $c(i, j) = M > 1$ if $(i, j)$ is a forbidden edge; and (d) $w(i, j) = 1$ for all $i, j \in V$. Note that in the game only the edge costs are nonuniform. With the above construction, we can show the following theorem.

**Theorem 1.** *The B$^3$C game based on the gadget of Figure 1 does not have any maximal (or strict) Nash equilibrium. This implies that for any $n \geq 6$, there is an instance of B$^3$C game with $n$ players that does not have any maximal (or strict) Nash equilibrium.*

*proof (sketch).* In any maximal graph, each of the flexible nodes $B$ and $C$ must have exactly one flexible edges pointing to one of the triangle nodes $\{D, E, F\}$. It is mechanical to verify that if one flexible node points to a traingle node $X \in \{D, E, F\}$, the best response of the other flexible node is to point to $Y \in \{D, E, F\}$ that is "downstream" from $X$, i.e. $(X, Y)$ is a fixed edge. Then the best responses of $B$ and $C$ will

cycle through the triangle nodes forever. Therefore, there is no Nash equilibrium for this game. By Lemma 2, there is no strict Nash equilibrium either. □

It is easy to verify that in the proof of Theorem 1 the critical paths that matter for the argument are from $A$, $A'$ and $A''$ to nodes $D, E, F$, and the lengths of these critical paths are at most three. Therefore, with the same argument, we directly know that for all $\ell \geq 3$, the $\ell$-B$^3$C game based on Figure 1 does not have maximal Nash equilibrium either. We develop a different gadget in [2] and show that for $\ell = 2, 3$ that $\ell$-B$^3$C game based on that gadget has no maximal (or strict) Nash equilibrium. Therefore, we have

**Theorem 2.** *For any $\ell \geq 2$ and $n \geq 6$, there is an instance of $\ell$-B$^3$C game with $n$ players that does not have any maximal (or strict) Nash equilibrium.*

### 3.2 Hardness of Determining the Existence of Maximal Nash Equilibria

In this section we use the gadget given in Figure 1 as a building block to show that determining the existence of maximal Nash equilibria given a nonuniform B$^3$C game is NP-hard. In fact, we use strict Nash equilibria to obtain a stronger result.

We define a problem TWOEXTREME as follows. The input of the problem is $(n, b, c, w)$ as the parameter of a B$^3$C game. The output of the problem is Yes or No, such that (a) if the game has a strict Nash equilibrium, the output is Yes; (b) if the game has no maximal Nash equilibrium, the output is No; and (c) for other cases, the output could be either Yes or No. Both deciding the existence of maximal Nash equilibria and deciding the existence of strict Nash equilibria are stronger problems than TWOEXTREME, because their outputs are valid for the TWOEXTREME problem by Lemma 2. We show the following result by a reduction from the 3-SAT problem.

**Theorem 3.** *The problem of* TWOEXTREME *is NP-hard.*

**Corollary 1.** *Both deciding the existence of maximal Nash equilibria and deciding the existence of strict Nash equilibria of a B$^3$C game are NP-hard.*[1]

We obtain the same result for the $\ell$-B$^3$C game.

**Theorem 4.** *For any $\ell \geq 2$, both deciding the existence of maximal Nash equilibria and deciding the existence of strict Nash equilibria in an $\ell$-B$^3$C game are NP-hard.*

## 4 Complexity of Computing Best Responses

The *best response* of a node in a configuration is its strategy that gives the node the best utility (i.e. best betweenness). In this section, we show the complexity of computing best responses for uniform games first, and then extend it for nonuniform games.

[1] In fact, the decision problem for any intermediate concept between maximal Nash equilibrium and strict Nash equilibrium is also NP-hard. For example, deciding the existence of nontransient Nash equilibria [9] is also NP-hard because any strict Nash equilibrium is a nontransient Nash equilibrium while the existence of a nontransient Nash equilibrium implies the existence of a maximal Nash equilibrium in B$^3$C games.

In a uniform game with parameters $(n, k)$, one can exhaustively search all $\binom{n-1}{k}$ strategies and find the one with the largest betweenness. Computing the betweenness of nodes given a fixed graph can be done by all-pair shortest paths algorithms in polynomial time. Therefore, the entire brute-force computation takes polynomial time if $k$ is a constant. However, if $k$ is not a constant, the result depends on $\ell$, the parameter bounding the shortest path length in the $\ell$-B$^3$C game.

Consider first the case of a uniform 2-B$^3$C game. Let $G = (V, E)$ be a directed graph. For a node $v$ in $G$, let $G_{v,S}$ be the graph where $v$ has outgoing edges to nodes in $S \subseteq V \setminus \{v\}$ and all other nodes have the same outgoing edges as in $G$. Then we have

**Lemma 3.** *For all $S \subseteq V \setminus \{v\}$, $btw_v(G_{v,S}, 2) = \sum_{u \in S} btw_v(G_{v,\{u\}}, 2)$.*

The lemma shows that for a uniform 2-B$^3$C game, the betweenness of a node can be computed by the sum of its betweenness when adding each of its outgoing edges alone.

**Theorem 5.** *Computing the best response in a uniform 2-B$^3$C game with parameters $(n, k)$ can be done in $O(n^3)$ time.*

*proof (sketch).* For each $u \in V \setminus \{v\}$, node $v$ computes $btw_v(G_{v,\{u\}}, 2)$ in $O(n^2)$ time. Then $v$ selects the top $k$ nodes $u$ with the largest $btw_v(G_{v,\{u\}}, 2)$ as its strategy, which is guaranteed to be $v$'s best response by Lemma 3. □

For cases other than the uniform 2-B$^3$C game, we show that best response computation is NP-hard, via a reduction from either the knapsack problem (for nonuniform the 2-B$^3$C game) or the set cover problem (the other cases).

**Theorem 6.** *It is NP-hard to compute the best response in either a nonuniform 2-B$^3$C game, or an $\ell$-B$^3$C game with $\ell \geq 3$ (uniform or not), or a B$^3$C game without path length constraint (uniform or not).*

## 5 Nash Equilibria in Uniform Games

In this section we focus on uniform $\ell$-B$^3$C games. we first define a family of graph structures called *shift graphs* and show that they are able to produce Nash equilibria for B$^3$C games. We then study some properties of Nash equilibria in uniform games.

### 5.1 Construction of Nash Equilibria via Shift Graphs

We first define *shift graphs* and *non-rotational shift graphs*. Then we show that for any $\ell, k$ and any $\ell' \geq \ell$, the *non-rotational shift graphs* with $n = (\ell' + k)!/k!$ nodes are all Nash equilibria in the uniform $\ell$-B$^3$C game with parameter $n$ and $k$. Moreover, we use shift graphs to construct *strict* Nash equilibria for both $\ell$-B$^3$C games and B$^3$C games without path length constraint, for certain combinations of $n$ and $k$ where $k = \Theta(\sqrt{n})$.

**Definition 1.** *A* shift graph $G = (V, E)$ *with parameters $m, t \in \mathbb{N}_+$ and $t \geq m$, denoted as $SG(m, t)$, is defined as follows. Each vertex of $G$ is labeled by an $m$-dimensional vector such that each dimension has $t$ symbols and no two dimensions have the same symbol appeared in the label. That is, $V = \{(x_1, x_2, \ldots, x_m) \mid x_i \in [t]$*

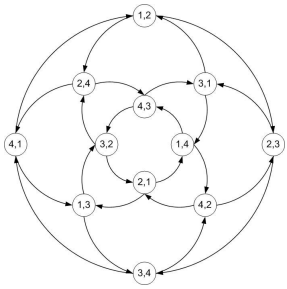



**Fig. 2.** Non-rotational shift graph $SG_{nr}(2,4)$

*for all $i \in [m]$, and $x_i \neq x_j$ for all $i, j \in [m], i \neq j$}. A vertex $u$ has a directed edge pointing to a vertex $v$ if we can obtain $v$'s label by shifting $u$'s label to the left by one digit and appending the last digit on the right. That is, $E = \{(u,v) \mid u, v \in V, u[2:m] = v[1:(m-1)]\}$, where $u[i:j]$ denote the sub-vector $(x_i, x_{i+1}, \ldots, x_j)$ with $u = (x_1, x_2, \ldots, x_m)$.*

In the shift graph $SG(m,t)$, we know that the number of vertices is $n = t \cdot (t-1) \cdots (t-m+1) = t!/(t-m)!$, and each vertex has out-degree $t-m+1$. Notice that the definition requires that $m$ dimensions have all different symbols. If they are allowed to be the same, then the graphs are the well-known De Bruijn graphs, whereas if we require only that the two adjacent dimensions have different symbols, the graphs are Kautz graphs, which are iterative line graphs of complete graphs.

**Definition 2.** *A* non-rotational shift graph *with parameter $m, t \in \mathbb{N}+$ and $t \geq m+1$, denoted as $SG_{nr}(m,t)$, is a shift graph with the further constraint that if $(u,v)$ is an edge, then $v$'s label is not a rotation of $u$'s label to the left by one digit. That is, $E = \{(u,v) \mid u, v \in V, u[2:m] = v[1:(m-1)]$ and $u[1] \neq v[m]\}$, where $u[i]$ denotes the $i$-th element of $u$.*

Graph $SG_{nr}(m,t)$ also has $t!/(t-m)!$ vertices but the out-degree of every vertex is $t-m$. A simple non-rotational shift graph $SG_{nr}(2,4)$ is given in Figure 2 as an example. Non-rotational shift graphs is a class of vertex-transitive graphs that are Eulerian and strongly connected. More importantly, they have one property that makes them Nash equilibria of $\ell$-B$^3$C games, as we now explain.

We say that a vertex $v$ in a graph $G$ is *$\ell$-path-unique* if any path that passes through $v$ (neither starting nor ending at $v$) with length no more than $\ell$ is the unique shortest path from its starting vertex to its ending vertex. A graph is *$k$-out-regular* if every vertex in the graph has out-degree $k$. A $k$-out-regular graph is an *$\ell$-path-unique graph* (or *$\ell$-PUG* for short) if every vertex in the graph is $\ell$-path-unique.

**Lemma 4.** *Non-rotational shift graph $SG_{nr}(\ell, k+\ell)$ is an $\ell$-PUG.*

**Lemma 5.** *If a directed graph $G$ has $n$ nodes and is $k$-out-regular and $\ell$-path-unique, then $G$ is a maximal Nash equilibrium for the uniform $\ell$-B$^3$C game with parameter $n$ and $k$.*

With the above results, we immediately have

**Theorem 7.** *For any $\ell \geq 2$, $\ell' \geq \ell$, $k \in \mathbb{N}_+$, graph $SG_{nr}(\ell', k+\ell')$ is a maximal Nash equilibrium of the uniform $\ell$-B$^3$C game with parameters $n = (k+\ell')!/k!$ and $k$.*

The above construction of maximal Nash equilibria is based on path-unique graphs. Next we show that shift graphs also lead to another family of Nash equilibria not based on path uniqueness. In fact, we show that they are strict Nash equilibria for uniform $\ell$-B$^3$C games for every $\ell \geq 2$ as well as B$^3$C games without path length constraint.

**Definition 3.** *Given a graph $G = (V, E)$, a* vertex-duplicated graph *$G' = (V', E')$ of $G$ with parameter $d \in \mathbb{N}_+$, denoted as $D(G, d)$, is a new graph such that each vertex of $G$ is duplicated to $d$ copies, and each duplicate inherits all edges incident to the original vertex. That is, $V' = \{(v, i) \mid v \in V, i \in [d]\}$, and $E' = \{((u, i), (v, j)) \mid u, v \in V, (u, v) \in E, i, j \in [d]\}$.*

**Theorem 8.** *For any $t \geq 2$, $d \geq 2$, graph $D(SG(2, t), d)$ is a strict Nash equilibrium of the uniform $\ell$-B$^3$C game with parameters $n = dt(t-1)$ and $k = d(t-1)$. It is also a strict Nash equilibrium of the uniform B$^3$C game without the path length constraint.*

In the simple case of $t = 2$, graph $D(SG(2, 2), d)$ is the complete bipartite graph with $d$ nodes on each side. For larger $t$, $D(SG(2, t), d)$ is a $t$-partite graph with more complicated structure. When $d = 2$, we have $n = 2t(t-1)$ and $k = 2(t-1)$. Thus, we have found a family of strict Nash equilibria with $k = \Theta(\sqrt{n})$.

An important remark is that when $d \geq 2$, each node is split into at least two nodes inheriting all incoming and outgoing edges, and thus graphs $D(SG(2, t), d)$ for all $t \geq 2$ and $d \geq 2$ are not $\ell$-PUGs for any $\ell \geq 2$. Therefore, the construction by splitting nodes in shift graphs $SG(2, t)$ are a new family of construction not based on path-unique graphs.

### 5.2 Properties of Nash Equilibria

From Lemma 5, we learn that $\ell$-PUGs are good sources for maximal Nash equilibria for uniform $\ell$-B$^3$C games. Thus we start by looking into the properties of $\ell$-PUGs to obtain more ways of constructing Nash equilibria. The following lemma provides a few ways to construct new $\ell$-PUGs given one or more existing $\ell$-PUGs.

**Lemma 6.** *Suppose that $G$ is a $k$-out-regular $\ell$-PUG. The following statements are all true:*

*(1) If $G'$ is a $k'$-out-regular subgraph of $G$ for some $k' \leq k$, then $G'$ is an $\ell$-PUG.*

*(2) Let $v$ be a node of $G$ and $\{v_1, v_2, \ldots, v_k\}$ be $v$'s $k$ outgoing neighbors. We add a new node $u$ to $G$ to obtain a new graph $G'$. All edges in $G$ remains in $G'$, and $u$ has $k$ edges connecting to $v_1, v_2, \ldots, v_k$. Then $G'$ is also an $\ell$-PUG.*

*(3) If $G'$ is another $k$-out-regular $\ell$-PUG and $G'$ does not shared any node with $G$, then the new graph $G''$ simply by putting $G$ together with $G'$ is also an $\ell$-PUG.*

Lemma 6 has several important implications. First, by repeatedly applying Lemma 6 (2) on an existing $\ell$-PUG, we can obtain an $\ell$-PUG with an arbitrary size. Combining it with Theorem 7, it immediately implies the following theorem.

**Theorem 9.** *For any $\ell \geq 2$, $k \in \mathbb{N}_+$, and $n \geq (k+\ell)!/k!$, there is a maximal Nash equilibrium in the uniform $\ell$-B$^3$C game with parameters $n$ and $k$.*

Next, Lemma 6 implies that Nash equilibria of uniform $\ell$-B$^3$C games could be disconnected, or weakly connected, or have very unbalanced in-degrees or betweenness among nodes, which implies that there exist rich structures among Nash equilibria.

Finally, we investigate non-PUG maximal Nash equilibria in the uniform 2-B$^3$C game with parameters $(n, k)$, which by Theorem 5 is the most interesting case since its best response computation is polynomial. We want to see that when we fix $k$, whether we can find non-PUG maximal Nash equilibria for arbitarily large $n$. Let $maxInd(G)$ denotes the maximum in-degree in graph $G$. The following result provides the condition under which all maximal Nash equilibria are PUGs.

**Theorem 10.** *Let $G$ be a $k$-out-regular graph with $n$ nodes. If $maxInd(G) \leq \frac{n-k}{k^2+k+1}$, then $G$ is a maximal Nash equilibrium for the uniform 2-B$^3$C game with parameter $n$ and $k$ if and only if $G$ is a 2-PUG.*

The above theorem implies that non-PUG equilibria is only possible if $maxInd(G) = \Theta(n)$ when $k$ is a constant, which means that non-PUG equilibria must have very unbalanced in-degrees when $n$ is large. In [2], we show an example of how to construct such a non-PUG equilibria for arbitrarily large $n$ when $k = 2$.

Theorem 10 can also be used to eliminate some families of graphs with balanced in-degrees as maximal Nash equilibria. For example, in [2], we show that when $n \geq k^3 + k^2 + 2k$, a family of symmetrical graphs called Abelian Cayley graphs cannot be maximal Nash equilibria for uniform 2-B$^3$C games.

## 6 Future Work

There are a number of directions to continue the study of B$^3$C games. First, besides the Nash equilibria we found in the paper, there are other Nash equilibria in the uniform games, some of which have been found by our experiments. We plan to further search for other Nash equilibrium structures and more properties of Nash equilibria. Second, we may also look into other variants of the game and solution concept, such as undirected connections or approximate Nash equilibria. Another direction is to study beyond betweenness definitions based on shortest paths, e.g. betweenness definitions based on network flows or random walks. This can be coupled with enriching the strategy set of the nodes to include fractional weighted edges.

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