

Mutual information and spontaneous symmetry breaking

A. Hamma,^{1,*} S. M. Giampaolo,² and F. Illuminati²

¹*Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, People's Republic of China*

²*Dipartimento di Ingegneria Industriale, Università degli Studi di Salerno, I-84084 Fisciano (SA), Italy*

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We show that the metastable, symmetry-breaking ground states of quantum many-body Hamiltonians have vanishing quantum mutual information between macroscopically separated regions and are thus the most classical ones among all possible quantum ground states. This statement is obvious only when the symmetry-breaking ground states are simple product states, e.g., at the factorization point. On the other hand, symmetry-breaking states are in general entangled along the entire ordered phase, and to show that they actually feature the least macroscopic correlations compared to their symmetric superpositions is highly nontrivial. We prove this result in general, by considering the quantum mutual information based on the two-Rényi entanglement entropy and using a locality result stemming from quasiadiabatic continuation. Moreover, in the paradigmatic case of the exactly solvable one-dimensional quantum XY model, we further verify the general result by considering also the quantum mutual information based on the von Neumann entanglement entropy.

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I. INTRODUCTION

The emergence of a macroscopic classical behavior from a microscopic quantum world can be explained in terms of decoherence to the environment that quickly destroys the coherent superpositions of macroscopic objects (Schrödinger cats) [1]. The selected pointer states must then be factorized states with respect to a tensor product structure that is local in real space [2,3]. Similarly, superselection induced by decoherence due to weak interactions with the environment plays a key role also in the phenomenon of spontaneous symmetry breaking, where different ordered sectors with broken symmetry are dynamically disconnected and are thus the only states that are metastable [4], from which comes the notion of spontaneous symmetry breaking [5].

In the paradigmatic case of the quantum Ising model, the ground space of the ferromagnetic phase at zero transverse field h is spanned by two orthogonal product states $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$ which are in the same class of pointer states of the typical decoherence argument, while the symmetric states $\Psi_{\pm} = 1/\sqrt{2}(|0\rangle^{\otimes N} \pm |1\rangle^{\otimes N})$ realize macroscopic coherent superpositions that are not stable under decoherence [1,4]. Therefore, at zero transverse field h , the situation is very clear: the only stable states are those that maximally break the symmetry of the Hamiltonian and, at the same time, those that feature vanishing macroscopic total correlations, including entanglement, between spatially separated regions.

As we turn on the external field h , we have a whole range of values where, before a critical value $h = h_c$ is reached, there is a magnetic order associated to spontaneous symmetry breaking [6], and the decoherence argument applies within the entire ordered phase. This means that, again, the only stable states are those that maximally break the Hamiltonian symmetry [7,8]. However, now the symmetry-breaking states are entangled, and their mixed-state reductions on arbitrary subsystems possess in general nonvanishing entanglement

[9–11], as well as quantum [12–14] and classical correlations [6]. Indeed, the symmetry-breaking ground states can be, “locally,” more entangled than some nearby symmetric states [15]. On the other hand, it is always implicitly assumed that such states are not *macroscopically* correlated, while their symmetric superpositions are, in complete analogy with the case $h = 0$. Although this is a very plausible picture, a rigorous proof has never been provided, due to the mathematical difficulties in dealing with measures of entanglement and correlations based on the von Neumann entropy; see, e.g., the difficulties in proving the boundary (*area*) law in generic gapped systems [16,17], or in proving the stability of topological entanglement entropy in topologically ordered states [18]. The symmetry-breaking states obey the boundary law for entanglement [19–22], while the macroscopic correlations featured by the superposition of two different symmetry-broken sectors are of order 1.

The question is then about which quantity one should look at in order to distinguish the presence of macroscopic correlations, among all possible sources of entanglement and correlations. Historically, the key concepts that have been considered are the off-diagonal long range order (ODLRO) [23] and, more recently, the two-site concurrence (entanglement of formation) at large distance [24,25]. If there is either ODLRO or nonvanishing concurrence between two sites in different clusters A and B , then also the two clusters must be entangled, since the total state of the global system is pure. This is an important point, because the reduced subsystems being in a separable state does not imply that there must be no entanglement between the two clusters in a pure state. Even if all the remaining correlations are classical, they are due to the fact that the overall state is a pure entangled state.

On the other hand, the reverse argument need not apply: it is possible that macroscopic clusters are entangled even if measures of two-point correlations, like the concurrence, are vanishing. For example, this can happen in the two-dimensional toric code [26] or in the one-dimensional cluster models [27–29] where all two-site concurrences are identically zero and yet the macroscopic block entanglement entropy is

*Corresponding author: hama.alioscia@gmail.com

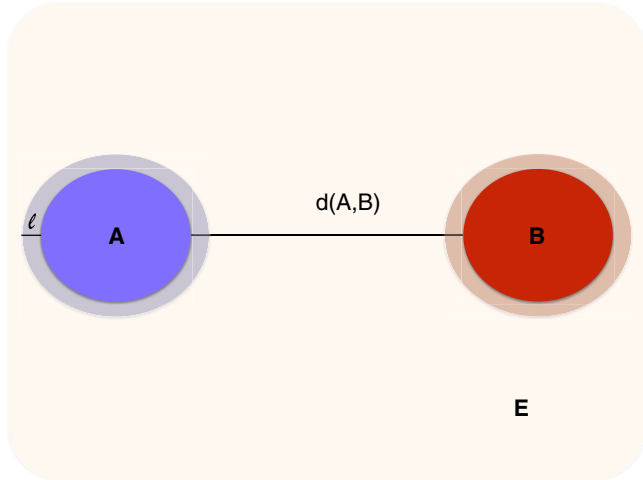


FIG. 1. A many-body quantum system is partitioned in three distinguishable subsystems, A, B , and the remainder E , so that the total Hilbert space acquires the tensor product structure $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$. The quantity $d(A, B)$ is the distance between the two regions A and B , and l is the distance defining the new effective support after an adiabatic deformation of operators with initial support on \mathcal{H}_A and \mathcal{H}_B .

finite. Also, states that possess volume law for the bipartite entanglement typically have no two-site entanglement, because of monogamy. Moreover, concurrence and ODLRO have to be computed for every specific model and cannot provide universal classifications.

In the present work, we will show that macroscopic correlations are generally nonvanishing in generic symmetric states and vanishing in symmetry-breaking states. To this end, we consider the total correlations (classical plus quantum) between two generic subsystems A and B of the total system, as measured by the mutual information [30]:

$$\mathcal{I}(A|B) = S(A) + S(B) - S(AB), \quad (1)$$

where $S(X)$ is the von Neumann entropy of the density matrix pertaining to subsystem X , as shown pictorially in Fig. 1. The mutual information is indeed a bona fide measure of total correlations (classical plus quantum) [31]. If for two arbitrary subsystems A, B spatially separated by arbitrarily large distances $d(A, B)$ the mutual information $\mathcal{I}(A|B)$ is vanishing, we are assured that there are no macroscopic correlations and, in particular, no macroscopic entanglement and no macroscopic quantum correlations (all correlation and entanglement measures are non-negative defined). Otherwise, taking into account that the total system is in a global pure state, the two subsystems must be macroscopically entangled and quantum correlated. It is immediate to verify that $\mathcal{I}(A|B)$ vanishes on symmetry-breaking states that are product states of the form $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$, while for symmetric superpositions Ψ_{\pm} it is always of order 1, irrespective of the actual value of the distance $d(A, B)$.

We will show that the above result is in fact valid in general. In order to prove such a statement, our strategy is the following. Starting from a factorization point, i.e., a point in which the system admits a fully separable pointer state

as global ground state [32–35], we will consider adiabatic deformations of the ground state and we will study the behavior of the macroscopic mutual information. The deformation corresponds to the adiabatic continuation of the ground state obtained by switching on the transverse magnetic field h . We will prove that, in the entire symmetry-breaking phase, the total macroscopic correlations and, *a fortiori*, the macroscopic entanglement, as measured by the mutual information, vanish in the maximally symmetry-breaking ground states at large spatial separations between arbitrarily selected macroscopic subsystems. On the other hand, we will also prove that, as long as the deformation is sufficiently small, the macroscopic mutual information remains finite in symmetric states. Later on, we will apply the results of the general analysis to specific spin models, showing that the result holding for slightly deformed symmetric states is in fact valid for all symmetric states in the entire symmetry-breaking (ordered) phase, until a quantum phase transition point is reached.

Proving the general result analytically would be quite a daunting task if one were to consider the von Neumann entropy S . Rather, we will resort to a related quantity, the two-Rényi entropy, S_2 , and the corresponding two-Rényi mutual information. For specific, computable, examples, we will then show that the general conclusions reached using the two-Rényi entropy hold as well using the von Neumann entropy.

The Rényi entropies of index α are defined as

$$S_{\alpha}(A) = (1 - \alpha)^{-1} \log_2 \text{Tr} \rho_A^{\alpha}, \quad (2)$$

and they are all bona fide entanglement monotones [36]. In particular, the two-Rényi entropy S_2 is an experimentally accessible quantity, since it is the expectation value of the swap operator on two copies of a quantum state [37], while the von Neumann entropy S is not an experimentally friendly observable, because it requires complete state tomography, which is essentially impossible on a many-body state. Concrete proposals for measuring S_2 resort to quantum switches [38], or to multiparticle interferometry [39].

We thus consider the two-Rényi mutual information defined as

$$\mathcal{I}_2(A|B) := S_2(A) + S_2(B) - S_2(AB). \quad (3)$$

Unlike the quantum mutual information defined in Eq. (1), the two-Rényi mutual information may not be positive defined for every state in Hilbert space. In order to ensure positivity, one can regularize this definition in terms of two-Rényi relative entropies, as shown in Refs. [40,41]. However, on the class of states of interest for the present investigation, nonregularized two-Rényi mutual information is always positive definite. Moreover, as already mentioned, in the paradigmatic case of the exactly solvable one-dimensional quantum XY model, we will also make direct comparison with the von Neumann-based mutual information, finding complete agreement. Finally, the clustering property and other general properties of the two-Rényi mutual information that are at the core of our investigation and that we will prove below can be proven without too much effort to hold valid also for the regularized versions. Therefore, in the following, also in order to avoid unnecessary formal mathematical complications, we will

consider only the nonregularized version of the two-Rényi mutual information.

By adopting such a strategy, we will obtain the following main result: in the ordered phase corresponding to the spontaneous breaking of some symmetry of a many-body Hamiltonian with nonvanishing local order parameter, the mutual information between two arbitrarily selected subsystems A and B separated by a distance $d(A, B)$ reaches its maximum in the symmetry-preserving ground states and is upper bounded by $\exp\{-O[d(A, B)]\}$ in the maximally symmetry-breaking ground states, i.e., the ground states that maximize the order parameter.

Therefore, we establish rigorously that—at least for some finite range of values within the phase—spontaneous symmetry breaking corresponds to the suppression of macroscopic coherent superpositions, and symmetry-breaking ground states are the ones selected in the real world by environmental decoherence, in complete analogy with pointer states. In the following, we focus specifically on the class of global \mathbb{Z}_2 symmetry. In particular, we will consider the specific, but paradigmatic, example of the quantum XY models, and we will show that macroscopic entanglement survives until a quantum phase transition occurs. However, the central elements of this result hold in general and the remaining ones can be easily adapted to every instance of spontaneous symmetry breaking for different Hamiltonians and different classes of symmetry groups.

II. CLUSTERING OF TWO-RÉNYI ENTROPY AND LONG-RANGE ORDER

Macroscopic long-range total correlations are revealed by a nonvanishing mutual information \mathcal{I} between two arbitrary regions A and B separated by arbitrarily large distances. If \mathcal{I} vanishes when A and B are separated by a distance larger than what defines the macroscopic interaction scale, then we are assured that there is no macroscopic entanglement.

Let us first consider the case of fully factorized ground states. These states are realized at a precise and unique set of values of the Hamiltonian parameters in an ordered phase, the so-called factorization point or, in spin systems, the factorizing field, first discovered in Ref. [32]. The general theory of ground-state factorization in terms of the response to local unitary perturbations was fully developed much later, in Refs. [33–35]. Factorized ground states can only occur in an ordered phase of strongly interacting many-body systems. Indeed, they are always maximally symmetry-breaking ground states with degeneracy equal to the dimension of the symmetry group, and, being fully product states, they have a trivially vanishing mutual information \mathcal{I} . Remarkably, at the factorization point, all other ground states can always be expressed as coherent linear superpositions of the fully factorized and maximally symmetry-breaking ground states [42]. Then, by construction, symmetric ground states feature a nonvanishing \mathcal{I} , no matter how large the distance $d(A, B)$ between the A and B regions. Moreover, there are no further types of entanglement and quantum correlations involved. Therefore, the classicality of symmetry-breaking states is immediately verified at the factorization point.

As the Hamiltonian parameters are changed adiabatically and move away from the factorization point, ground-state entanglement does not come solely from the macroscopic superposition of disconnected sectors, but also from the fact that fully factorized states are no longer ground states. They can be represented as $U(s)|0\rangle^{\otimes N}, U(s)|1\rangle^{\otimes N}$, where $U(s)$ is the unitary operator that connects the instantaneous ground states of the Hamiltonian $H(s)$. The operator $U(s)$ is an entangling operator, showing that the symmetry-breaking ground states are now entangled. This raises the problem whether they are still only locally entangled or if they have developed some macroscopic entanglement. Similarly, for the symmetric states will be $U(s)\Psi_{\pm}$, it is not immediately obvious to what extent their entanglement is macroscopic or not for a generic value of s . In the following, we show that the entanglement and the correlations due to the adiabatic continuation $U(s)$ are local, i.e., they vanish in the limit of arbitrarily large spatial separations $d(A, B)$.

In order to proceed, we need to discriminate clearly between the macroscopic and the local contributions to \mathcal{I} . We first recall that by $S_2(A)$ we mean the quantity $S_2(A) = -\log_2 \mathcal{Q}_A$ where \mathcal{Q}_A is the purity defined as $\mathcal{Q}_A = \text{Tr}[\rho_A^2]$, and ρ_A is the reduced density matrix from the ground state of the total system to the subsystem contained in the finite region A . To compute the Rényi entropy of order 2 we will use the identity $\mathcal{Q}_A := \text{Tr}\rho_A^2 = \text{Tr}(\mathcal{S}_A\rho^{\otimes 2})$, where $\rho^{\otimes 2}$ represents two copies of the original full state on the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. The operator \mathcal{S}_A is the permutation operator (swap operator) of order 2 with support on $\mathcal{H}_{AA'}$ only: $\mathcal{S}_A = \tilde{\mathcal{S}}_A \otimes I_{\bar{A}}$ where $\tilde{\mathcal{S}}_A = \otimes_{a \in A} \tilde{\mathcal{S}}_a$ and $\tilde{\mathcal{S}}_a$ is the permutation operator on the a th spin of the system, i.e., $\tilde{\mathcal{S}}_a|i_1, \dots, i_a, \dots, i_n\rangle \otimes |j_1, \dots, j_a, \dots, j_n\rangle = |i_1, \dots, j_a, \dots, i_n\rangle \otimes |j_1, \dots, i_a, \dots, j_n\rangle$.

Next, we exploit a locality result, stemming essentially from the Lieb-Robinson bound [43]. Indeed, for the mutual information based on the von Neumann entropy, a recent seminal contribution has shown that $\mathcal{I}(A|B)$ is an upper bound for the two-point correlation functions and a lower bound to exponentially decreasing functions of the ratio between the $d(A, B)$ and the correlation length [44].

Following Hastings and Wen [45], let us consider a many-body Hamiltonian sum of local terms, $H(s) = \sum_i h_i(s)$, with a finite gap ΔE above the low-energy sector for some finite interval of values of the Hamiltonian parameters s (then, out of this interval a quantum phase transition may occur). Moreover, the local operators are assumed to be bounded: $\|h_i(s)\| < \infty$. If the ground state of $H(s)$ is known for a particular fixed set of values of the Hamiltonian parameters, say s_0 , we may obtain it for any other generic set s by the quasiadiabatic continuation $U(s)$ induced by a continuous deformation of $H(s)$. A local operator O_A with support on A transforms as $O_A(s) = U^\dagger(s)O_A U(s)$. The new operator $O_A(s)$ has support on the whole Hilbert space. Nevertheless, the locality result implies that we can arbitrarily approximate it with an operator $O_{A'}(s)$ that has support only over the Hilbert space associated to a region with diameter $\text{diam}(A') = \text{diam}(A) + l$, as long as l is larger than the correlation length ξ induced by the gap ΔE , and by this making an error bounded in this way: $\|O_A(s) - O_{A'}(s)\| \leq K e^{-l/\xi}$. The constant K grows like l^D where D is the spatial dimension (e.g., the lattice dimension for localized spins).

Let now ρ be the ground state of the system for $s = s_0$. The purity of the restriction of the evolved state $\rho(s)$ to a spatial region C reads

$$\begin{aligned} \mathcal{Q}_C(s) &= \text{Tr}\{U(s)^{\otimes 2} \rho^{\otimes 2} [U^\dagger(s)]^{\otimes 2} \mathcal{S}_C\} \\ &= \text{Tr}\{\rho^{\otimes 2} [U^\dagger(s)]^{\otimes 2} \mathcal{S}_C U(s)^{\otimes 2}\} \\ &\simeq \text{Tr}[\rho^{\otimes 2} \mathcal{S}_{C+l}(s)]. \end{aligned} \quad (4)$$

Here \mathcal{S}_{C+l} denotes the permutation operator with support on spins that are at most at distance l from C , and the \simeq sign means that the error is exponentially small in l/ξ . If the subsystem C is the union of disjoint and macroscopically separated subsets, $C = A \cup B$, we have

$$\begin{aligned} \mathcal{Q}_C &= \text{Tr}\{\rho^{\otimes 2} [U^\dagger(s)]^{\otimes 2} \mathcal{S}_A \mathcal{S}_B U(s)^{\otimes 2}\} \\ &= \text{Tr}[\rho^{\otimes 2} \mathcal{S}_{A+l}(s) \mathcal{S}_{B+l}(s)]. \end{aligned} \quad (5)$$

If the distance separating A and B is much larger than l , i.e., $d(A, B) \gg l$, we can write

$$\mathcal{Q}_C \simeq \text{Tr}[\rho^{\otimes 2} \mathcal{S}_{A+l}(s) \otimes \mathcal{S}_{B+l}(s) \otimes I_E], \quad (6)$$

where E is the complement to A and B together (see Fig. 1). Assume first that the initial state at $s = s_0$ is one of the completely factorized ground states (which, we recall, are also maximally symmetry-breaking ground states). In this case, $\rho(s_0) = \rho_A \otimes \rho_B \otimes \rho_E$ and we obtain

$$\mathcal{Q}_C \simeq \text{Tr}[\rho^{\otimes 2} \mathcal{S}_{A+l}(s)] \text{Tr}[\rho^{\otimes 2} \mathcal{S}_{B+l}(s)]. \quad (7)$$

Therefore, the purity in the region $C = A \cup B$ is the product of the purities in the two separated regions A and B from which it follows immediately that $\mathcal{I}_2(A|B) \simeq 0$.

Let us next consider the opposite case in which the ground state for $s = s_0$ is a macroscopic coherent superposition (Schrödinger cat) of the fully factorized ground states. We show that any such superposition leads to a nonvanishing $\mathcal{I}_2(A|B)$. Since we need to consider two copies of the ground state, we will label the factorized states by $|a\rangle$ and $|b\rangle$, respectively, on each copy. So, we can write $|a\rangle = |a\rangle_A \otimes |a\rangle_B \otimes |a\rangle_E$ and similarly for $|b\rangle$. Therefore, $|\rho\rangle^{\otimes 2} = \sum_{a,b} \alpha_a \alpha_b |a\rangle_A |a\rangle_B |a\rangle_E \otimes |b\rangle_A |b\rangle_B |b\rangle_E$, where $\sum_a |\alpha_a|^2 = \sum_b |\alpha_b|^2 = 1$. For the sake of an explicit evaluation, consider the case of a doubly degenerate ground-state manifold and subsystems A and B with equal size (number of spins). For symmetric superpositions we then obtain

$$\begin{aligned} \mathcal{I}_2(A|B) &\simeq \log_2 4 - 2 \log_2 \sum_{aba'b'} \langle a'b' | \mathcal{S}_{A+l}(s) | ab \rangle \\ &\quad + \log_2 \sum_{aba'b'} \langle a'b' | \mathcal{S}_{A+l}(s) | ab \rangle^2 \\ &= \log_2 4 + \log_2 \frac{\sum_{aba'b'} \langle a'b' | \mathcal{S}_{A+l}(s) | ab \rangle^2}{\left(\sum_{aba'b'} \langle a'b' | \mathcal{S}_{A+l}(s) | ab \rangle \right)^2}. \end{aligned} \quad (8)$$

Since for $s = 0$ the expectation values of $\langle a'b' | \mathcal{S}_A(0) | ab \rangle$ are positive definite, by continuity they are still positive for a small enough value of s . So we see that, for small s , the second term in Eq. (8) is negative but that the $\mathcal{I}_2(A|B)$ stays strictly positive, for arbitrary $d(A, B)$. This is also true for any nontrivial superposition of the symmetry-breaking sectors given by the

amplitudes α_a , so that only the maximally symmetry-breaking ones have exactly zero $\mathcal{I}_2(A|B)$ [as $d(A, B) \rightarrow \infty$].

In the following, we determine the exact value of $\mathcal{I}_2(A|B)$ for macroscopic coherent superpositions with arbitrarily large $d(A, B)$ in the entire ordered phase, beyond the perturbative case of small s in the case in which A and B are made by a single spin. We prove this result explicitly for models with \mathbb{Z}_2 symmetry, but the central elements of the proof are valid for arbitrary symmetry groups and arbitrary dimension of A and B . In fact, our proof implies that for a maximally symmetry-broken ground state all the correlation functions between two very far subsystems factorize in the product of the expectation value of the local operators. And this implies that the mutual information between A and B must vanish. When we turn to consider symmetric ground states, some of these local expectation values must vanish because the local operator does not commute with at least one of the parity operators that define the symmetry group of the Hamiltonian. As result the mutual information is expected to be different from zero. A comprehensive analysis on the dependence on the size of the subsystems and/or the symmetry group of the Hamiltonian is in progress [46].

III. LONG-RANGE MUTUAL INFORMATION IN MODELS WITH \mathbb{Z}_2 SYMMETRY

In the following, we focus on spin-1/2 systems with a global \mathbb{Z}_2 symmetry, thus described by Hamiltonians that commute with the parity operator along a fixed spin direction, i.e., $\mathbb{P}_\mu = \otimes_i \sigma_i^\mu$. In such systems spontaneous symmetry breaking is associated to the presence of a twofold degenerate ground state and an off-diagonal long-range order along a spin direction σ_i^ν that is orthogonal to σ_i^μ . We show that throughout the entire ordered phase the long-range mutual information vanishes identically on states that maximally break the symmetry, while it remains strictly positive on any macroscopic coherent superposition of the two broken symmetry sectors. For the sake of simplicity we focus on the case in which subsystems A and B are each made by a single spin but the results can be extended straightforwardly to more general choices.

The two orthogonal symmetric ground states $|e\rangle$ and $|o\rangle$ (respectively, the even-symmetric and the odd-symmetric ground states) form a convenient basis that allows us to write all other ground states $|g\rangle$ in the ordered phase as their linear superpositions: $|g(u, v)\rangle = u|e\rangle + v|o\rangle$. The reduced density matrix ρ_C of the two-spin block $C = A \cup B$ can be expressed in terms of the two-point correlation functions as follows [9]:

$$\rho_C(u, v) = \frac{1}{4} \sum_{i_A, i_B} G^{i_A, i_B}(u, v) \sigma_A^{i_A} \sigma_B^{i_B}, \quad (9)$$

where the expectations $G^{i_A, i_B}(u, v) = \langle g(u, v) | \sigma_A^{i_A} \sigma_B^{i_B} | g(u, v) \rangle$ are on products of Pauli matrices $\sigma_A^{i_A}$ and $\sigma_B^{i_B}$. As shown in Ref. [8], all correlation functions can be associated to spin operators that either commute or anticommute with the parity operator \mathbb{P}_μ . Therefore, the reduced density matrix $\rho_C(u, v)$ can be expressed as the sum of a symmetric part that coincides with the density matrix of the symmetric ground state, $\rho_C^{(s)}(u, v)$, and an antisymmetric part, that is a traceless matrix, $\rho_C^{(a)}(u, v)$. Taking into account the fact that the two

symmetric ground states fall in two orthogonal eigenspaces of the parity, it is straightforward to verify that $\rho_C^{(s)}(u, v)$ is independent of the superposition amplitudes and hence $\rho_C^{(s)}(u, v) \equiv \rho_C^{(s)}$.

The reduced density matrix of a symmetric ground state thus reads

$$\rho_C^{(s)} = \frac{1}{4} [\mathbb{1} + m_\mu (\sigma_A^\mu + \sigma_B^\mu) + m_\mu^2 \sigma_A^\mu \sigma_B^\mu + m_\nu^2 \sigma_A^\nu \sigma_B^\nu]. \quad (10)$$

In Eq. (10) m_μ is the expectation value of the local operator that commutes with the parity while m_ν is the local order parameter. Exploiting Eq. (10) one can derive the mutual information $\mathcal{I}_2^{(s)}$ and evaluate its asymptotic expression for $d(A, B) \rightarrow \infty$:

$$\mathcal{I}_2^{(s)}(\infty) = \log_2 \left[1 + \frac{m_\nu^4}{(1 + m_\mu^2)^2} \right]. \quad (11)$$

As $m_\nu \neq 0$ throughout the entire ordered phase, the above relation shows the presence of macroscopic entanglement and correlations in the symmetric coherent superposition ground states throughout the entire phase. Up to this point, this result is valid for any model with \mathbb{Z}_2 symmetry. The actual values of m_ν , of course depend on the specific model considered. Here, we analytically compute the result for the quantum XY model.

The one-dimensional spin-1/2 quantum XY Hamiltonian with ferromagnetic nearest-neighbor interactions in a transverse field with periodic boundary conditions reads

$$H = - \sum_{i=1}^N \left[\left(\frac{1+\gamma}{2} \right) \sigma_i^x \sigma_{i+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_i^y \sigma_{i+1}^y + h \sigma_i^z \right], \quad (12)$$

where $\sigma_i^\mu, \mu = x, y, z$, are the Pauli spin-1/2 operators acting on site i , γ is the anisotropy parameter in the xy plane, h is the transverse magnetic field along the z direction, and the periodic boundary conditions $\sigma_{N+1}^\mu \equiv \sigma_1^\mu$ ensure invariance of the model Hamiltonian under spatial translations.

Such a model can be solved analytically [6,47,48] and, hence, the phase diagram can be determined exactly and in great detail. In the thermodynamic limit, for any $\gamma \in (0, 1]$, a quantum phase transition occurs at the critical value $h_c = 1$ of the transverse field. For $h < h_c = 1$ the system is ferromagnetically ordered and is characterized by a twofold ground-state degeneracy such that the \mathbb{Z}_2 parity symmetry under inversions along the spin- z direction is broken by some elements of the ground space. Using the analytical solution, in Fig. 2 we have plotted the behavior of $\mathcal{I}_2^{(s)}(\infty)$ in the ferromagnetic phase.

Given the two symmetric ground states, the so-called even $|e\rangle$ and odd $|o\rangle$ states belonging to the two orthogonal subspaces associated to the two possible distinct eigenvalues of the parity operator, any symmetry-breaking linear superposition of the form

$$|g(u, v)\rangle = u|e\rangle + v|o\rangle \quad (13)$$

is also an admissible ground state, with the complex superposition amplitudes u and v constrained by the normalization

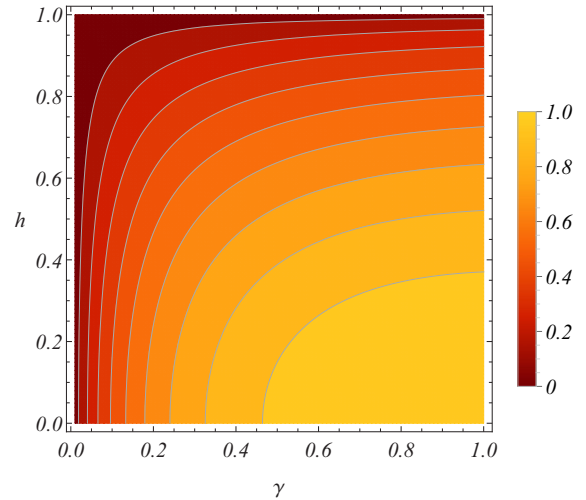


FIG. 2. Behavior of the mutual information $[\mathcal{I}_2^{(s)}(\infty)]$ between two spins at infinite distance in the symmetric ground state of the one-dimensional XY model (thermodynamic limit) as a function of the anisotropy γ and transverse field h in the ferromagnetic phase $0 \leq h \leq h_c = 1$. Within the ferromagnetic phase, it is always $m_\nu > 0$, and hence $\mathcal{I}_2^{(s)}(\infty) > 0$. On the other hand, $m_\nu = 0$ either at $h_c = 1$ or at $\gamma = 0$. Only at these points $\mathcal{I}_2^{(s)}(\infty) = 0$.

condition $|u|^2 + |v|^2 = 1$. Taking into account that the even and odd ground states are orthogonal, the expectation values of operators that commute with the parity operator are independent of the superposition amplitudes u and v . On the other hand, spin operators that do not commute with the parity may have nonvanishing expectation values on such linear combinations and hence break the symmetry of the Hamiltonian Eq. (12).

In the asymptotic macroscopic regime $d(A, B) \rightarrow \infty$, the general two-spin reduced density matrix for an arbitrary ground state reads

$$\rho_C(u, v) = \rho_C^{(s)} + \frac{1}{4} (uv^* + vu^*) [m_\nu (\sigma_A^\nu + \sigma_B^\nu) + m_\mu m_\nu (\sigma_A^\mu \sigma_B^\nu + \sigma_A^\nu \sigma_B^\mu)]. \quad (14)$$

The corresponding expression for the mutual information $\mathcal{I}_2(\infty)$ reads

$$\mathcal{I}_2(\infty) = \log_2 \left\{ 1 + \frac{m_\nu^4 [1 - (uv^* + vu^*)^4]}{[1 + m_\mu^2 + (uv^* + vu^*)^2 m_\nu^2]^2} \right\}. \quad (15)$$

Due to the normalization constraint, $|u|^2 + |v|^2 = 1$, the fraction in Eq. (15) is semipositive defined and vanishes only either at $m_\nu = 0$, i.e., in the disordered classical paramagnetic phase, or when $(uv^* + vu^*) = 1$. Therefore, in the ordered phase the only ground states with vanishing long-range mutual information, and hence vanishing macroscopic entanglement and correlations, are the maximally symmetry-breaking ground states. At the other end of the spectrum, it is easy to verify that the maximum of \mathcal{I}_2 , for a fixed value of the parameters m_μ and m_ν , is always achieved in the totally symmetric (even) and antisymmetric (odd) states, the absolute maximum being obtained for $m_\mu = 0$ and $m_\nu = 1$. Finally, since the one-dimensional XY model allows for the exact evaluation of all entropies in the Rényi hierarchy, in Fig. 3

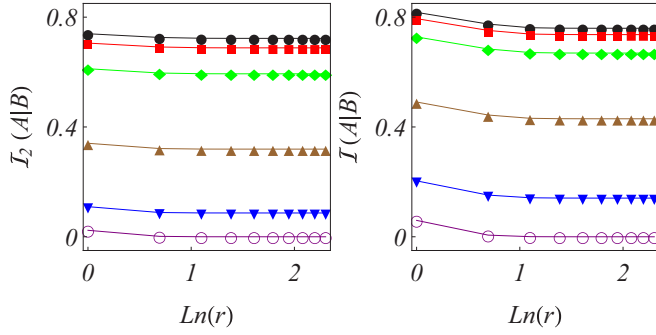


FIG. 3. Behavior of the mutual information based on the two-Rényi entropy, $\mathcal{I}_2(A|B)$ (left), and the von Neumann entropy, $\mathcal{I}(A|B)$ (right), as functions of the logarithm of the interspin distance r for different superpositions in the ferromagnetic phase of the one-dimensional XY model at ($\gamma = h = 0.5$). Black circles, $u = 1, v = 0$ (symmetric state); red squares, $u = \cos(0.05\pi), v = \sin(0.05\pi)$; green diamonds, $u = \cos(0.1\pi), v = \sin(0.1\pi)$; brown up-triangles, $u = \cos(0.15\pi), v = \sin(0.15\pi)$; blue down-triangles, $u = \cos(0.2\pi), v = \sin(0.2\pi)$; and violet empty circles, $u = \cos(0.25\pi), v = \sin(0.25\pi)$ (maximally symmetry-breaking ground state). The two definitions feature the same qualitative behavior; in particular, they both vanish in and only in the maximally symmetry-breaking ground states.

we compare the mutual information based on the two-Rényi and the one based on the von Neumann entropy, finding complete qualitative and quantitative agreement. In particular, they both vanish in, and only in, the maximally symmetry-breaking ground states. Finally, using the parametrization $u = \cos \theta, v = \sin \theta$, in Fig. 4 we also report and compare for completeness the behavior of the two-Rényi based and the von Neumann based mutual information as functions of the superposition parameter θ for the two extreme cases of nearest-neighbor distance $r = 1$ and asymptotic distance $r = \infty$, finding again perfect agreement.

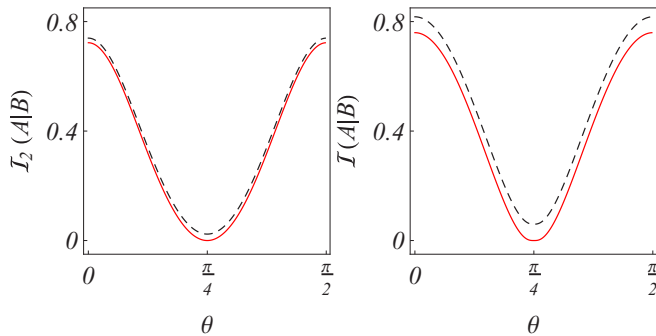


FIG. 4. Behavior of the mutual information based on the two-Rényi entropy, $\mathcal{I}_2(A|B)$ (left), and the von Neumann entropy, $\mathcal{I}(A|B)$ (right), as functions of the ground-state superposition parameter θ , with the parametrization $u = \cos \theta, v = \sin \theta$, for two extreme values of the distance r between A and B . Dashed black curve: $r = 1$. Solid red curve: $r = \infty$. The absolute minimum is always realized at $\theta = \pi/4$, corresponding to the maximally symmetry-breaking ground state. At this point, both measures of mutual information vanish exactly for $r = \infty$.

IV. COMPARISON WITH OTHER INDICATORS OF MACROSCOPIC COHERENT SUPERPOSITIONS

Quantum discord [31,49] is a measure of quantum correlations more general than entanglement that may exist in mixed quantum states, including separable ones. It is defined as the difference between mutual information—which accounts for all correlations, both classical and quantum—and the optimal classical correlations between A and B , by maximizing over all the measurement on B : $\mathcal{C}_{AB} = \max_{\{\hat{B}_k\}} [\mathcal{S}(\hat{\rho}_A) - \mathcal{S}_C(\hat{\rho}_{AB}|\{\hat{B}_k\})]$. It is therefore sensible to verify whether it may be a good quantifier of macroscopic quantum coherence. The long-range pairwise quantum discord between two spins in the ground state of the one-dimensional XY chain has been recently investigated in Refs. [8,12–14]. It turns out that such a quantity features a long-range behavior quite analogous to that of the mutual information, with the crucial difference that it vanishes identically in all possible ground states as $m_\mu \rightarrow 0$. From a mathematical point of view this can be easily explained considering that, in such a case, the two-spin reduced density matrix in the symmetric ground states at asymptotically large interspin distance is indistinguishable from the one obtained by the symmetry-breaking Gibbs states at zero temperature.

The localizable entanglement in a many-body pure state is defined as the maximal amount of pairwise entanglement between two spins i, j at arbitrary distance that can be achieved, on average, by performing generalized measurements on all other spins [50,51]. This naturally defines an entanglement length χ that diverges in symmetric states. Numerics show that in the maximally symmetry-breaking ground states of the XY chain the localizable entanglement behaves like the connected spin-spin correlation functions Q_{xx} that are bound to decay exponentially at long distance. Therefore long-range pairwise entanglement defined via the localizable entanglement features a behavior quite similar to that of the mutual information. However, this is just pairwise entanglement (although at long distance), so it does not quite capture the notion of a macroscopic superposition. Moreover, the mutual information is more readily generalized to thermal mixed states, the Gibbs states at finite temperature, for which the evaluation of the mutual information presents no particular difficulty.

V. CONCLUSIONS AND OUTLOOK

We investigated macroscopic entanglement [52] through the behavior of the quantum mutual information between two macroscopically separated blocks of dynamical variables in the ground state of many-body systems featuring spontaneous symmetry breaking. This quantity detects macroscopic total correlations, including entanglement. The main result of this paper is that in the *entire phase with broken symmetry* the symmetry-breaking states have vanishing long-distance mutual information, while the latter remains finite for any nonmaximally symmetry-breaking superposition, attaining a maximum for the totally symmetric states. This fact is easy to prove when considering symmetry-breaking states that are completely unentangled (fully factorized), the symmetric superpositions of which are Greenberger-Horne-Zeilinger states. In order to prove this feature in the entire ordered phase, a much more challenging task, we followed a strategy

based on two ingredients: (i) adopting measures of mutual information based on the two-Rényi and on the von Neumann entropies and (ii) exploiting locality results about quasiadiabatic continuation of quantum states derived by using the Lieb-Robinson bounds [43,53,54]. In this way we were able to prove that spontaneous symmetry breaking selects the many-body states with vanishing long-distance mutual information, and thus macroscopically least entangled, and therefore most classical.

In perspective, we are concerned with the investigation of several open problems. In particular, it would be interesting to extend our analysis to the case of subsystems of arbitrarily variable size, in order to observe possible threshold effects, and to generic classes of symmetry groups [46]. Moreover, we are interested in studying the case of globally mixed states [24] and, in particular, equilibrium thermal states of models featuring spontaneous symmetry breaking below a critical temperature. At thermal equilibrium the system will be described by the Gibbs state $\rho_{eq} = Z^{-1} \sum_{i,a} e^{-\beta E_i} |E_i\rangle\langle E_i|_a$, where with a we have explicitly labeled different sectors. Below a critical temperature T_c , if a labels the sectors with broken symmetry, spontaneous symmetry breaking means that in every single realization the label a will be fixed by the initial conditions. Therefore, our statement is that Gibbs states obtained by fixing a feature the least long-range mutual information compared to all other nonmaximally symmetry-breaking Gibbs states [46].

From a different perspective, we remark that many-body localization has recently become a subject of great interest for the condensed-matter community. Systems with strong disorder featuring many-body localization fail to thermalize and to obey the eigenstate thermalization hypothesis [55], because their eigenstates are more weakly entangled than in typical nonintegrable systems. It would be interesting to see how the techniques developed in the present work might help in providing rigorous results regarding the clustering of mutual information in such systems. In fact, our techniques can be used to investigate also systems that involve long-range entanglement and correlations, such as topologically ordered states [56,57] and their resilience in the presence of perturbations or at finite temperature [26,58,59].

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