

Quantum Benchmarks for Pure Single-Mode Gaussian States

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Teleportation and storage of continuous variable states of light and atoms are essential building blocks for the realization of large-scale quantum networks. Rigorous validation of these implementations require identifying, and surpassing, benchmarks set by the most effective strategies attainable without the use of quantum resources. Such benchmarks have been established for special families of input states, like coherent states and particular subclasses of squeezed states. Here we solve the longstanding problem of defining quantum benchmarks for general pure Gaussian single-mode states with arbitrary phase, displacement, and squeezing, randomly sampled according to a realistic prior distribution. As a special case, we show that the fidelity benchmark for teleporting squeezed states with totally random phase and squeezing degree is $1/2$, equal to the corresponding one for coherent states. We discuss the use of entangled resources to beat the benchmarks in experiments.

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Quantum teleportation [1–3] is the emblem of long-distance quantum communication [4] and provides a powerful primitive for quantum computing [5]. Similarly, quantum state storage [6] is a central ingredient for quantum networks [7]. In the past two decades, the experimental progress in teleporting and storing quantum states realized on different physical systems has been impressive [8–27]. Particularly groundbreaking are the demonstrations involving continuous variable (CV) systems [28,29], where states having an infinite-dimensional support, such as coherent and squeezed states, have been unconditionally teleported and stored between light modes and atomic ensembles in virtually all possible combinations [10,22–25,30,31]. These experiments might be reckoned as stepping stones for the quantum Internet [32].

Ideally, teleportation and storage aim at the realization of a perfect identity channel between an unknown input state $|\psi\rangle_{\text{in}}$, issued to the sender Alice, and the output state received by Bob. In principle, this is possible if Alice and Bob share a maximally entangled state, supplemented by classical communication [1–3]. In practice, limitations on the available entanglement and technical imperfections lead to an output state ρ_{out} which is not, in general, a perfect replica of the input. It is then customary to quantify the success of the protocol in terms of the input-output fidelity [33,34] $\mathcal{F} = \int_{\Lambda} \langle \psi | \rho_{\text{out}} | \psi \rangle_{\text{in}}$, averaged over an ensemble $\Lambda = \{|\psi\rangle_{\text{in}}, p_{\psi}\}$ of possible input states, sampled according to a prior distribution known to Alice and Bob. To assess whether the execution of transmission protocols takes advantage of genuine quantum resources, it is mandatory to establish benchmarks for the average fidelity [35]. A benchmark is given in terms of a threshold $\bar{\mathcal{F}}_c$, corresponding to the maximum average fidelity that can be reached without sharing any entanglement. Indeed, in a

classical procedure Alice might just attempt to estimate $|\psi\rangle_{\text{in}}$ through an appropriate measurement, and communicate the outcome to Bob, who could then prepare an output state based on such an outcome: this defines a “measure-and-prepare” strategy. For a given ensemble Λ , the classical fidelity threshold (CFT) $\bar{\mathcal{F}}_c$ amounts then to the highest average fidelity achievable by means of measure-and-prepare strategies. If an actual implementation attains an average fidelity $\bar{\mathcal{F}}_q$ higher than $\bar{\mathcal{F}}_c$, then it is certified that no classical procedure could have reproduced the same results, and the quantumness of the implemented protocol is therefore validated. This is, in a sense [36–40], similar to observing a violation of Bell inequalities to testify the nonlocality of correlations in a quantum state [41,42].

In recent years, an intense activity has been devoted to devising appropriate benchmarks for teleportation and storage of relevant sets of input states [35,36,40,43–49]. In particular, if the ensemble Λ contains arbitrary pure states of a d -dimensional system drawn according to a uniform distribution, then $\bar{\mathcal{F}}_c = 2/(d+1)$ [44]. In the limit of a CV system, $d \rightarrow \infty$, the CFT goes to zero, as it becomes impossible for Alice to guess a particular input state with a single measurement. However, for a quantum implementation it is meaningless to assume that the laboratory source can produce arbitrary input states from an infinite-dimensional Hilbert space with nearly uniform probability distribution. To benchmark CV implementations, one thus needs to restrict to ensembles of input states that can be realistically prepared and are distributed according to probability distributions with finite width.

In the majority of CV protocols [28], Gaussian states have been employed as the preferred information carriers [50]. Gaussian states enjoy a privileged role as, on one hand, their mathematical description only requires a finite

number of variables (first and second moments of the canonical mode operators) [51], and on the other, they represent the set of states which can be reliably engineered and manipulated in a multitude of laboratory setups [29]. High-fidelity teleportation and storage architectures involving Gaussian states [2,3,10,22,23,30] can be scaled up to realize networks [13,52,53] and hybrid teamworks [54], and cascaded to build nonlinear gates for universal quantum computation [30,50]. The problem of benchmarking the transmission of Gaussian states is thus of pressing relevance for quantum technology.

This problem has so far only witnessed partial solutions. Here and in the following, we shall focus on pure single-mode Gaussian states. Any such state can be written as (we drop the subscript “in”) [50,51]

$$|\psi_{\alpha,s,\theta}\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle, \quad (1)$$

where $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator, $\hat{S}(\xi) = \exp[\frac{1}{2}(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2)]$ is the squeezing operator with $\xi = se^{i\theta}$, \hat{a} and \hat{a}^\dagger are respectively the annihilation and creation operators obeying the relation $[\hat{a}, \hat{a}^\dagger] = 1$, and $|k\rangle$ denotes the k th Fock state, $|0\rangle$ being the vacuum. Pure single-mode Gaussian states are thus entirely specified by their displacement vector $\alpha \in \mathbb{C}$, their squeezing degree $s \in \mathbb{R}^+$, and their squeezing phase $\theta \in [0, 2\pi]$. A widely employed teleportation benchmark is available for the ensemble Λ_C of input coherent states [10,35,45], for which $s, \theta = 0$ and the displacement α is sampled according to a Gaussian distribution $p_\lambda^C(\alpha) = (\lambda/\pi)e^{-\lambda|\alpha|^2}$ of width λ^{-1} . In this case, the CFT reads [45]

$$\bar{\mathcal{F}}_c^C(\lambda) = \frac{1 + \lambda}{2 + \lambda}, \quad (2)$$

converging to $\lim_{\lambda \rightarrow 0} \bar{\mathcal{F}}_c^C(\lambda) = \frac{1}{2}$ in the limit of infinite width. More recently, benchmarks were obtained for particular subensembles of squeezed states [46–48], specifically either for known s and totally unknown α, θ [47], or for totally unknown s with $\alpha, \theta = 0$ [46,55]. However, a fundamental question has remained unanswered in CV quantum communication: What is the general benchmark for teleportation and storage of arbitrary pure single-mode Gaussian states?

In this Letter we solve this longstanding open problem. We build on a recent method for the evaluation of quantum benchmarks proposed in Ref. [39], and develop group-theoretical techniques to calculate the CFT for the following two classes of input single-mode states: (a) the ensemble Λ_S , containing pure Gaussian squeezed states with no displacement ($\alpha = 0$), totally random phase θ , and unknown squeezing degree s drawn according to a realistic distribution with width β^{-1} and (b) the ensemble Λ_G , containing arbitrary pure Gaussian states with totally random phase θ and α, s drawn according to a joint distribution with finite widths λ^{-1}, β^{-1} , respectively. By properly selecting the prior distributions, we

obtain analytical results for the benchmarks, which eventually take the following simple and intuitive form:

$$\bar{\mathcal{F}}_c^S(\beta) = \frac{1 + \beta}{2 + \beta}; \quad (3a)$$

$$\bar{\mathcal{F}}_c^G(\lambda, \beta) = \frac{(1 + \lambda)(1 + \beta)}{(2 + \lambda)(2 + \beta)}. \quad (3b)$$

These benchmarks are probabilistic [39]: they give the maximum of the fidelity over arbitrary measure-and-prepare strategies, even including probabilistic strategies based on postselection of some measurement outcomes. By definition, probabilistic benchmarks are stronger than deterministic ones: beating a probabilistic benchmark means having an implementation whose performance cannot be achieved classically, even with a small probability of success.

Case (a) shows that for input squeezed states with totally unknown complex squeezing ξ , the benchmark reaches $\lim_{\beta \rightarrow 0} \bar{\mathcal{F}}_c^S(\beta) = \frac{1}{2}$ just like the case of coherent states; we provide a nearly optimal measure-and-prepare deterministic strategy which saturates the benchmark of Eq. (3a) for $\beta \gg 0$. On the other hand, the general result of case (b) encompasses the previous partial findings providing an elegant and useful prescription to validate experiments involving transmission of Gaussian states, with input distribution widths λ^{-1}, β^{-1} tunable depending on the capabilities of actual implementations.

Mathematical formulation of quantum benchmarks.— Suppose that Alice and Bob want to teleport or store a state chosen at random from an ensemble $\{|\varphi_x\rangle, p_x\}_{x \in X}$ using a measure-and-prepare strategy, where Alice measures the input state with a positive operator-valued measure (POVM) $\{P_y\}_{y \in Y}$ and, conditionally on outcome y , Bob prepares an output state ρ'_y . In a probabilistic strategy, Alice and Bob have the extra freedom to discard some of the measurement outcomes and to produce an output state only when the outcome y belongs to a set of favorable outcomes Y_{yes} . The fidelity of their strategy is

$$\bar{\mathcal{F}} = \sum_{x \in X} \sum_{y \in Y_{\text{yes}}} p(x|y_{\text{yes}}) p(y|x, \text{yes}) \langle \varphi_x | \rho'_y | \varphi_x \rangle, \quad (4)$$

where $p(x|y_{\text{yes}})$ is the conditional probability of having the state $|\varphi_x\rangle$ given that a favorable outcome was observed and $p(y|x, \text{yes}) = \langle \varphi_x | P_y | \varphi_x \rangle / \sum_{y \in Y_{\text{yes}}} \langle \varphi_x | P_y | \varphi_x \rangle$. Then the CFT is the supremum of Eq. (4) over all possible measure-and-prepare strategies. Using a result of [39], we have

$$\bar{\mathcal{F}}_c = \|(I \otimes \tau^{-1/2})\rho(I \otimes \tau^{-1/2})\|_X, \quad (5)$$

where $\tau = \sum_x p_x |\varphi_x\rangle \langle \varphi_x|$ is the average state of the ensemble, $\rho = \sum_x p_x |\varphi_x\rangle \langle \varphi_x| \otimes |\varphi_x\rangle \langle \varphi_x|$, and, for a positive operator A , $\|A\|_X = \sup_{\|\varphi\|=\|\psi\|=1} \langle \varphi | A | \varphi \rangle \langle \psi | A | \psi \rangle$.

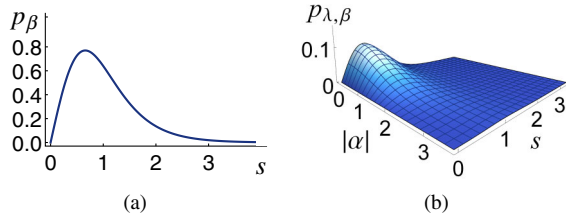


FIG. 1 (color online). Marginal probability distributions for (a) the subset of input squeezed states with arbitrary squeezing degree s and arbitrary phase θ , and (b) the complete set of input Gaussian states with arbitrary displacement α , arbitrary squeezing degree s , and arbitrary random phase θ . The plots show the marginal distributions after integrating over θ , having set $\beta = 2$, $\lambda = \frac{1}{2}$.

Case (a): Benchmark for arbitrary squeezed states.— We consider the ensemble Λ_S of squeezed vacuum states $|\xi\rangle \equiv |\psi_{0,s,\theta}\rangle$ with arbitrary complex squeezing [see Eq. (1)] distributed according to the prior

$$p_\beta^S(\xi) = \frac{p_\beta(s)}{2\pi}, \quad \text{with} \quad p_\beta(s) = \frac{\beta \sinh s}{(\cosh s)^{\beta+1}}, \quad (6)$$

where $\beta^{-1} > 0$ regulates the width of the squeezing distribution, while the phase θ is uniformly distributed, which is natural for CV experiments [48]. The marginal prior $p_\beta(s)$ is plotted in Fig. 1(a). For squeezed states, the prior $p_\beta^S(\xi)$ is the analogue of the Gaussian $p_\lambda^C(\alpha)$ for coherent states: indeed, the Gaussian can be expressed as $p_\lambda^C(\alpha)d^2\alpha = |\langle 0|\alpha\rangle|^{2\lambda}d^2\alpha/\pi$, where the measure $d^2\alpha$ is invariant under the action of displacements, while $p_\beta^S(\xi)$ can be expressed as $p_\beta^S(\xi)dsd\theta = |\langle 0|\xi\rangle|^{2(2+\beta)}\mu(d^2\xi)$, where the measure $\mu(d^2\xi) = \sinh s \cosh s dsd\theta/(2\pi)$ is invariant under the action of the squeezing transformations. For integer β , the prior $p_\beta^S(\xi)$ can be generated by preparing $2 + \beta$ modes in the vacuum and performing the optimal measurement for the estimation of squeezing [56,57].

Using Eq. (5), the CFT can be written as $\bar{\mathcal{F}}_c^S(\beta) = \|A_\beta\|_\times$, $A_\beta = (I \otimes \tau_\beta^{-1/2})\rho_\beta(I \otimes \tau_\beta^{-1/2})$, where $\tau_\beta =$

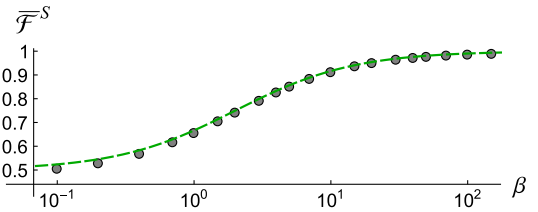


FIG. 2 (color online). Average classical fidelities for input squeezed states versus the distribution parameter β . The dots correspond to the fidelity $\bar{\mathcal{F}}_{\text{sr}}^S$ for the best square-root measurement, while the dashed line depicts the optimal probabilistic CFT $\bar{\mathcal{F}}_c^S$.

$\int dsd\theta p_\beta^S(\xi)|\xi\rangle\langle\xi|$ and $\rho_\beta = \int dsd\theta p_\beta^S(\xi)|\xi\rangle\langle\xi| \otimes |\xi\rangle\langle\xi|$. To obtain the benchmark announced in Eq. (3a), we compute explicitly the states τ_β and ρ_β and show that the eigenvalues of A_β are all equal to $(1 + \beta)/(2 + \beta)$ [57]. Observing that $(1 + \beta)/(2 + \beta) = \langle 0|\langle 0|A_\beta|0\rangle|0\rangle$, we then get $\|A_\beta\|_\times = (1 + \beta)/(2 + \beta)$, thus concluding the proof of Eq. (3a). This benchmark allows one to certify the quantumness of experiments involving teleportation and storage of squeezed states with arbitrary amount of squeezing and arbitrary phase [14–17,24], bypassing the limitations of [46,47].

We highlight the similarity of our result to the case of input coherent states [45]. In that case, the probabilistic benchmark of Eq. (5) coincides with the maximum over deterministic strategies, given by Eq. (2) [39]. Precisely, the CFT of Eq. (2) is achievable with heterodyne detection and reparation of coherent states [35,45]. Since the heterodyne detection can be interpreted as a square-root measurement [58,59] for a suitable Gaussian prior, in the case of squeezed states it is natural to wonder whether a deterministic square-root measurement strategy suffices to saturate the probabilistic CFT given by Eq. (3a). For an ensemble of the form $\{|\xi\rangle, p_\eta^S(\xi)\}$ the square-root measurement has POVM elements $P_\eta(\xi) = p_\eta^S(\xi)\tau_\eta^{-1/2}|\xi\rangle\langle\xi|\tau_\eta^{-1/2}$ (here we allow η to be different from β). Performing the square-root measurement and reparing the state $|\xi\rangle$ conditional on outcome ξ gives the average fidelity

$$\bar{\mathcal{F}}_{\text{sr}}^S(\beta, \eta) = \frac{\beta}{\beta + 2\eta + 2} \frac{\eta + 1}{\eta + 2} \sum_{k=0}^{\infty} \left| \sum_{n=0}^k \binom{k-n-\frac{1}{2}}{k-n} \sqrt{\binom{n-\frac{1}{2}}{n} \binom{\frac{\eta+1}{2} + n}{n}} \right|^2 / \left[\binom{\frac{\beta+2}{2} + k}{k} \binom{\frac{\eta+2}{2} + k}{k} \right]$$

[57], where we are using the notation

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$$

for a general $x \in \mathbb{R}$. In Fig. 2 we compare $\sup_\eta \bar{\mathcal{F}}_{\text{sr}}^S(\beta, \eta)$, maximized numerically over η , with the CFT $\bar{\mathcal{F}}_c^S$ of Eq. (3a), for a range of values of β . We find that the

square-root measurement is a nearly optimal classical strategy, which reaches the CFT asymptotically for large values of β , when the input squeezing distribution becomes more and more peaked.

Case (b): Benchmark for general Gaussian states.— Consider now the ensemble Λ_G of arbitrary pure Gaussian states $|\alpha, \xi\rangle \equiv |\psi_{\alpha,s,\theta}\rangle$ [Eq. (1)] distributed according to the prior

$$p_{\lambda,\beta}^G(\alpha, s, \theta) = \frac{\lambda\beta}{2\pi^2} \frac{e^{-\lambda|\alpha|^2 + \lambda \operatorname{Re}(e^{-i\theta}\alpha^2) \tanh s} \sinh s}{(\cosh s)^{\beta+2}}. \quad (7)$$

We note that in this case the prior can be written as $p_{\lambda,\beta}^G(\xi) \propto |\langle 0|\lambda\alpha, \xi\rangle|^2 |\langle 0|\xi\rangle|^{2(4+\beta)} \nu(d^2\alpha, d^2\xi)$ where $\nu(d^2\alpha, d^2\xi) = d^2\alpha \sinh s (\cosh s)^3 ds d\theta$ is the invariant measure under the joint action of displacement and squeezing. For integer β , the prior can be generated by performing an optimal measurement of squeezing and displacement on $5 + \beta$ modes prepared in the vacuum [57]. The marginals of this prior correctly reproduce the previous subcases, namely the distribution of Eq. (6) for the squeezing, $\int d^2\alpha p_{\lambda,\beta}^G(\alpha, s, \theta) = p_\beta^S(\xi)$, and the Gaussian distribution of [45] for the displacement, $\lim_{\beta \rightarrow \infty} \int d^2\xi p_{\lambda,\beta}^G(\alpha, s, \theta) = p_\lambda^C(\alpha)$. The marginal probability distribution after integrating over the phase θ , $p_{\lambda,\beta}(\alpha, s) = \int_0^{2\pi} d\theta p_{\lambda,\beta}^G(\alpha, s, \theta) = \pi^{-1} \lambda \beta e^{-\lambda|\alpha|^2} \sinh s (\cosh s)^{-\beta-2} I_0[\lambda|\alpha|^2 \tanh s]$, where I_0 is a modified Bessel function [60], is plotted in Fig. 1(b).

To compute the benchmark, we observe that the pure Gaussian states of Eq. (1) are instances of the generalized coherent states introduced by Gilmore and Perelomov for arbitrary Lie groups [61–64]. Here we consider Gilmore-Perelomov coherent states of the form $|\varphi_g\rangle = \hat{U}_g|\varphi\rangle$, where $\hat{U}: g \mapsto \hat{U}_g$ is an irreducible representation of a Lie group \mathbf{G} and $|\varphi\rangle$ is a lowest weight vector for the representation $\hat{U}: g \mapsto \hat{U}_g$. This general setting includes the cases of coherent and squeezed states, and the present case of pure Gaussian states, where the group is the Jacobi group, the group element g is the pair $g = (\alpha, \xi)$, $\hat{U}_g \equiv \hat{D}(\alpha)\hat{S}(\xi)$, and $|\varphi\rangle \equiv |0\rangle$ [65]. In the Supplemental Material [57], we solve the benchmark problem for arbitrary sets of Gilmore-Perelomov coherent states, randomly drawn with a prior probability of the form $p_\gamma(g) dg \propto |\langle \varphi_\gamma | \varphi_{\gamma,g} \rangle|^2 dg$, where dg is the invariant measure on the group and $|\varphi_{\gamma,g}\rangle = \hat{U}_g^{\gamma'}|\varphi_\gamma\rangle$ is the Gilmore-Perelomov coherent state for a given irreducible representation $\hat{U}^{\gamma'}: g \mapsto \hat{U}_g^{\gamma'}$. Our key result is a powerful formula for the probabilistic CFT for Gilmore-Perelomov coherent states, given by [57]

$$\bar{\mathcal{F}}_c(\gamma) = \frac{\int dg p_\gamma(g) |\langle \varphi | \varphi_g \rangle|^4}{\int dg p_\gamma(g) |\langle \varphi | \varphi_g \rangle|^2}. \quad (8)$$

Using this general expression in the cases of coherent and squeezed states, it is immediate to retrieve the benchmarks of Eqs. (2) and (3a). We now use this result to find the benchmark for the transmission of arbitrary input Gaussian states with prior distribution given by Eq. (7), which now reads

$$\bar{\mathcal{F}}_c^G(\lambda, \beta) = \frac{\int d^2\alpha \times \times ds d\theta p_{\lambda,\beta}^G(\alpha, s, \theta) |\langle 0 | \psi_{\alpha,s,\theta} \rangle|^4}{\int d^2\alpha ds d\theta p_{\lambda,\beta}^G(\alpha, s, \theta) |\langle 0 | \psi_{\alpha,s,\theta} \rangle|^2}. \quad (9)$$

The integrals can be evaluated analytically [57]. The final result yields the general benchmark announced in Eq. (3b), which is the main contribution of this Letter. Notice how

the previous partial findings are contained in this result. For coherent states, $\lim_{\beta \rightarrow \infty} \bar{\mathcal{F}}_c^G(\lambda, \beta) = \bar{\mathcal{F}}_c^C(\lambda)$; for squeezed states, $\lim_{\lambda \rightarrow \infty} \bar{\mathcal{F}}_c^G(\lambda, \beta) = \bar{\mathcal{F}}_c^S(\beta)$. The benchmark for teleporting Gaussian states in the limit of completely random α, s, θ , is finally established to be $\lim_{\lambda,\beta \rightarrow 0} \bar{\mathcal{F}}_c^G(\lambda, \beta) = \frac{1}{4}$.

Discussion.—We now investigate how well an actual implementation of quantum teleportation can fare against the benchmarks derived above. We focus on the conventional Braunstein–Kimble CV quantum teleportation protocol [3] using as a resource a Gaussian two-mode squeezed vacuum state with squeezing r , $|\phi\rangle_{AB} = (\cosh r)^{-1} \sum_{k=0}^{\infty} (\tanh r)^k |k\rangle_A |k\rangle_B$, also known as a twin beam. We assume that the input is an arbitrary pure single-mode Gaussian state $|\psi_{\alpha,s,\theta}\rangle$, Eq. (1), drawn according to the probability distribution of Eq. (7). The output state received by Bob will be a Gaussian mixed state whose fidelity with the input can be written as [46] $\mathcal{F}_q(s; r) = \{2e^{-2r} [\cosh(2r) + \cosh(2s)]\}^{-1/2}$. Notice that it depends neither on the phase θ nor the displacement α by construction of the CV protocol [3] (for unit gain [30,66]). Averaging this over the input set Λ_G , we get the average quantum teleportation fidelity

$$\begin{aligned} \bar{\mathcal{F}}_q^G(\beta; r) &= \int d\theta d^2\alpha ds p_{\lambda,\beta}^G(\alpha, s, \theta) \mathcal{F}_q(s; r) \\ &= \frac{\beta}{2\beta+2} e^r {}_2F_1\left(\frac{1}{2}, \frac{\beta+1}{2}; \frac{\beta+3}{2}; -\sinh^2(r)\right), \end{aligned} \quad (10)$$

where ${}_2F_1$ is a hypergeometric function [60]. The average quantum fidelity is obviously independent of λ ; i.e., in particular, it is the same for the ensemble of all Gaussian states Λ_G and for the ensemble of squeezed states Λ_S . In Fig. 3, we compare $\bar{\mathcal{F}}_q^G(\beta; r)$ with the CFT $\bar{\mathcal{F}}_c^G(\lambda, \beta)$, in particular with the case $\lambda \rightarrow 0$ (totally random displacement) and with

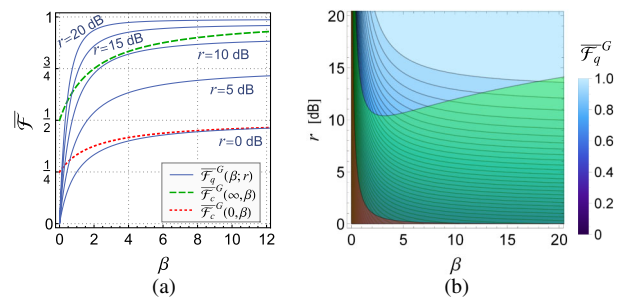


FIG. 3 (color online). Performance of the CV quantum teleportation protocol for general input Gaussian states $|\psi_{\alpha,s,\theta}\rangle$ using a two-mode squeezed entangled resource with squeezing r . (a) Plot of the quantum teleportation fidelity $\bar{\mathcal{F}}_q^G(\beta; r)$, averaged over the input set Λ_G according to a prior distribution $p_{\lambda,\beta}$, against the benchmark $\bar{\mathcal{F}}_c^G(\lambda, \beta)$ for $\lambda \rightarrow 0$ (dotted red) and $\lambda \rightarrow \infty$ (dashed green). (b) Contour plot of $\bar{\mathcal{F}}_q^G(\beta; r)$ as a function of β and r ; the lower (red) and upper (green) shadings correspond to parameter regions where the quantum fidelity does not beat the benchmark $\bar{\mathcal{F}}_c^G(0, \beta)$ and $\bar{\mathcal{F}}_c^G(\infty, \beta)$, respectively.

the case $\lambda \rightarrow \infty$ (undisplaced squeezed states, whose CFT reduces to $\bar{\mathcal{F}}_c^S(\beta)$). In the latter case, we see that the shared entangled state needs to have a squeezing r above 10 dB, which is at the edge of current technology [67,68], in order to beat the benchmark for the ensemble Λ_S . For general input Gaussian states in Λ_G with random displacement, squeezing, and phase, less resources are instead needed to surpass the CFT of Eq. (3b), especially if the input squeezing distribution is not too broad ($\beta \gg 0$), which is the realistic situation in experimental implementations (where e.g., s can fluctuate around a set value which depends on the specifics of the nonlinear crystal used for optical parametric amplification [29,30]). For the case of coherent input states with totally random displacement ($\lambda \rightarrow 0, \beta \rightarrow \infty$), the CFT converges to $\frac{1}{2}$ and we recover the known result that any $r > 0$ is enough to beat the corresponding benchmark [3,10,33,35,45].

Summarizing, we have derived exact analytical quantum benchmarks for teleportation and storage of arbitrary pure single-mode Gaussian states, which can be readily employed to validate current and future implementations. The mathematical techniques developed here to obtain the presented results are of immediate usefulness to analyze a much larger class of problems, such as the determination of benchmarks for cloning, amplification [39] and other protocols involving multimode Gaussian states and other classes of Gilmore-Perelomov coherent states, including finite-dimensional states. We will explore these topics in forthcoming publications.

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