Optimal Mechanisms with Simple Menus

ZIHE WANG, IIIS, Tsinghua University
PINGZHONG TANG, IIIS, Tsinghua University

We consider revenue-optimal mechanism design for the case with one buyer and two items. The buyer's valuations towards the two items are independent and additive. In this setting, optimal mechanism is unknown for general valuation distributions. We obtain two categories of structural results that shed light on the optimal mechanisms. These results can be summarized into one conclusion: under certain conditions, the optimal mechanisms have simple menus.

The first category of results state that, under a centain condition, the optimal mechanism has a monotone menu. In other words, in the menu that represents the optimal mechanism, as payment increases, the allocation probabilities for both items increase simultaneously. This theorem complements Hart and Reny's recent result regarding the nonmonotonicity of menu and revenue in multi-item settings. Applying this theorem, we derive a version of revenue monotonicity theorem that states stochastically superior distributions yield more revenue. Moreover, our theorem subsumes a previous result regarding sufficient conditions under which bundling is optimal [Hart and Nisan 2012].

The second category of results state that, under certain conditions, the optimal mechanisms have few menu items. Our first result in this category says that, for certain distributions, the optimal menu contains at most 4 items. The condition admits power (including uniform) density functions. Our second result in this category works for a weaker (hence more general) condition, under which the optimal menu contains at most 6 items. This condition is general enough to include a wide variety of density functions, such as exponential functions and any function whose Taylor series coefficients are nonnegative. Our last result in this category works for unit-demand setting. It states that, for uniform distributions, the optimal menu contains at most 5 items. All these results are in sharp contrast to Hart and Nisan's recent result that finite-sized menu cannot guarantee any positive fraction of optimal revenue for correlated valuation distributions.

Categories and Subject Descriptors: I.2.11 [**Distributed Artificial Intelligence**]: Multiagent systems; J.4 [**Social and Behavioral Sciences**]: Economics

Additional Key Words and Phrases: Optimal mechanism design; menu representation; revenue maximization

1. INTRODUCTION

Optimal mechanism design has been a topic of intensive research over the past thirty years. The general problem is, for a seller, to design a revenue-maximizing mechanism for selling k items to n buyers, given the buyers' valuations distributions but not the actual values. A special case of the problem, where there is only one item ($k = 1$) and buyers have independent valuation distributions towards the item, has been resolved by Myerson's seminal work Myerson [1981]. Myerson's approach has turned out to be quite general and has been successfully applied to a number of similar settings, such as [Maskin and Riley 1989; Jehiel et al. 1996; Levin 1997; Ledyard 2007; Deng and Pekeč 2011].

Authors' address: Institute of Interdisciplinary Information Sciences, Tsinghua University, Beijing, China. Emails: Zihe Wang, wangzihe11@mails.tsinghua.edu.cn; Pingzhong Tang, kenshin@tsinghua.edu.cn

EC'14, June 8–12, 2014, Stanford, CA, USA.
http://dx.doi.org/10.1145/2600057.2602863

While this line of work has flourished, it does not deepen our understanding of the cases with more than one items ($k > 1$). In fact, even for the simplest multi-item case, where there are two independent items ($k = 2$) and one buyer ($n = 1$) with additive valuations, a direct characterization of the optimal mechanism is still open for general, especially continuous, valuation distributions.

When the distributions are discrete, Daskalakis and Weinberg [2011]; Cai et al. [2012a,b] show that the general optimal mechanism ($k > 1$) is the solution of a linear program. They provide different methods to solve the linear program efficiently. For continuous distributions, Chawla et al. [2010]; Cai and Huang [2013] study the possibility of using simple auctions to approximate optimal auctions. In addition, Daskalakis and Weinberg [2012]; Cai and Huang [2013] provide PTAS of the optimal auction under various assumptions on distributions.

Zoom in and look at the case with two independent items and a single buyer, significant progresses have been made in this particular setting lately. Hart and Nisan [2012] investigate two simplest forms of auctions: selling the two items separately and selling them as a bundle. They prove that selling separately obtains at least one half of the optimal revenue while bundling always returns at least one half of separate sale revenue. They further extend these results to the general case with k independent items: separate sale guarantees at least a $\frac{c}{log^2 k}$ fraction of the optimal revenue; for identically distributed items, bundling guarantees at least a $\frac{c}{log\ k}$ fraction of the optimal revenue. Li and Yao [2013] tighten these lower bounds to $\frac{c}{log\ k}$ and c respectively. Under some technical assumptions, [Daskalakis et al. 2013] show close relation between mechanism design and transport problem and use techniques there to solve for optimal mechanisms in a few special distributions. Hart and Nisan [2013] investigate how the "menu size" of an auction can affect the revenue and show that revenue of any finite menu-sized auction can be arbitrarily far from optimal (thus confirm an earlier consensus that restricting attention to deterministic auctions, which has an exponentially-sized (at most) menu, indeed loses generality). In the economic literature, Manelli and Vincent [2006, 2007]; Pavlov [2011a,b] obtain the optimal mechanisms in several specific distributions (such as both items are distributed according to uniform [0,1]). We will discuss these results in much more detail as we proceed to relevant sections.

In this paper, we study the case with one buyer and two independent items, in hopes of a direct characterization of *exact* optimal mechanisms. We obtain several exciting structural results. *Our conclusion is that, under fairly reasonable conditions, optimal mechanisms has "simple" menus.* We summarize our results below into two parts, based on the conditions under which the results hold, as well as different interpretations of simplicity.

For ease of presentation, we need the following definition: for a density function h, the *power rate* of h is $PR(h(x)) = \frac{xh'(x)}{h(x)}$.

— Part I (Section 4). If density functions f_1 and f_2 satisfy $PR(f_1(x)) + PR(f_2(y)) \leq -3,\ \forall x, y$. The optimal mechanism has a monotone menu – sort the menu items in ascending order of payments, the allocation probabilities of both items increase simultaneously – a desirable property that fails to hold in general (cf [Hart and Reny 2012]). Our result complements Hart and Reny's observation and has two important implications.
  (1) [Hart and Nisan 2012, Theorem 28]. Hart and Nisan show that, if two item distribution are further identical (i.e., $f_1 = f_2$), bundling sale is optimal. Our result subsumes this theorem as a corollary.

(2) A revenue monotonicity theorem. Based on menu monotonicity theorem, we are able to prove that, stochastically superior distributions yield higher revenue, another desirable property that fails to hold in general.

Our proof is semi-constructive in the sense that we fix part of the buyer utility function (for this part, relation between revenue and buyer utility is unknown/undesirable) and construct the remainder of the utility function (for this part, relation between revenue and buyer utility is known/desirable). This technique might be of potential interest.

— Part II. (Section 5). If the density functions f_1 and f_2 satisfy $PR(f_1(x))+PR(f_2(y)) \geq -3,\ \forall x, y$. The optimal mechanisms often contain few menu items. In particular,

(1) If both $PR(f_1(x))$ and $PR(f_2(y))$ are constants, the optimal mechanism contains at most 4 menu items. The result is tight. Constant power rate is satisfied by a few interesting classes of density functions, including power functions $h(x) = ax^b$ and uniform density as a special case. This is consistent with earlier results for uniform distributions [Manelli and Vincent 2006; Pavlov 2011a]: the optimal mechanisms indeed contain four menu items.

(2) — If $PR(f_1(x)) + PR(f_2(y)) = -3\ \forall x, y$, the optimal mechanism contains at most 3 menu items.

— If $-2 \leq PR(f_1(x)) \leq y_A f_2(y_A) - 2$ and $-2 \leq PR(f_2(y)) \leq x_A f_1(x_A) - 2$, the optimal mechanism contains 3 menu items. Here x_A and y_A are the lowest possible valuations for item one and two respectively, Consequently, under either condition, bundling sale gives a 2-approximation.

(3) If we relax the condition to be that both $PR(f_1(x))$ and $PR(f_2(y))$ are monotone, then the optimal mechanism contains at most 6 menu items. This condition is sufficiently mild to include density functions such as exponential density and any function whose Taylor series coefficients are nonnegative.

(4) Our last result requires the buyer demands at most one item. Under this condition, for uniform densities, the optimal mechanism contains at most 5 menu items. The result is tight.

These results are in sharp contrast to Hart and Nisan's recent result that there is some distribution where finite number of menu items cannot guarantee any fraction of revenue [Hart and Nisan 2013]. Here we show that, for several wide classes of distributions, the optimal mechanisms have a finite and even extremely simple menus. Our proofs for this part are based on Pavlov's characterization and careful analyses of how the revenue changes as a function of the buyer's utility. A rough line of reasoning is as follows, the "extreme points" in the set of convex utility functions on the boundary values are piecewise linear functions. Since the utility on the boundaries contains only few linear pieces and the utility on inner values are linearly related to that on the boundary, it must be the case that the utility function on the inner points contains only few linear pieces as well. In other words, the mechanism only contains few menu items. We expect similar insight can be applied to cases with more items.

Our results not only offer original insights of "what do optimal mechanisms look like", but are also in line with the "simple versus optimal" literature (cf [Hartline and Roughgarden 2009; Hart and Nisan 2012]): in our case, simple mechanisms are exactly optimal.

All of the missing proofs are deferred to the full version of the paper.

2. THE SETTING

We consider a setting with one seller who has two distinct items for sale, and one buyer who has private valuation x for item 1, y for the item 2, and $x + y$ for both items. The seller has zero valuation for any subset of items.

As usual, x and y are unknown to the seller and are treated as independent random variables according to density functions f_1 on $[x_A, x_B] \subset \mathbb{R}$ and f_2 on $[y_A, y_C] \subset \mathbb{R}$ respectively. The valuation (aka. type) space of the buyer is then $V = [x_A, x_B] \times [y_A, y_C]$. To visualize, we sometimes refer to V as rectangle $ABDC$, where A represents the lowest possible type (x_A, y_A) and D represents the highest possible type (x_B, y_C). Let $f(x, y) = f_1(x)f_2(y)$ be the joint density on V. We assume the f_1 and f_2 are positive, bounded, and differentiable densities.

The seller sells the items through a mechanism that consists of an allocation rule q and a payment rule t. In our two-item setting, an allocation rule is conveniently represented by $q = (q_1, q_2)$, where q_i is the probability that buyer gets item $i \in \{1, 2\}$. Given valuation (x, y), buyer's utility is

$$u(x, y) = xq_1(x, y) + yq_2(x, y) - t(x, y)$$

In other words, buyer has a quasi-linear, additive utility function. It is sometimes convenient to view a mechanism as a (possibly infinite) set of *menu items* $\{(q_1(x, y), q_2(x, y), t(x, y))|(x, y) \in [x_A, x_B] \times [y_A, y_C]\}$. Given a mechanism, the expected revenue of the seller is $R = \mathbb{E}_{(x,y)}[t(x, y)]$.

A mechanism must be *Individually Rational (IR)*:

$$\forall (x, y), u(x, y) \geq 0.$$

In other words, a buyer cannot get negative utility by participation.

By revelation principle, it is without loss of generality to focus on the set of mechanisms that are *Incentive Compatible (IC)*:

$$\forall (x, y), (x', y'), u(x, y) \geq xq_1(x', y') + yq_2(x', y') - t(x', y').$$

This means, it is the buyer's (weak) dominant strategy to report truthfully. Equivalently, an IC mechanism presents a set of menu items and let the buyer do the selection (aka. the taxation principle). As a result,

$$u(x, y) = sup_{(x',y')}\{xq_1(x', y') + yq_2(x', y') - t(x', y')\},$$

which is the supremum of a set of linear functions of (x, y). Thus, u must be convex. Fixing y, by IC, we have

$$\begin{aligned}
& u(x', y) - u(x, y) - q_1(x, y)(x' - x) \\
= \; & x'q_1(x', y) + yq_2(x', y) - t(x', y) - xq_1(x, y) - yq_2(x, y) + t(x, y) - x'q_1(x, y) + xq_1(x, y) \\
= \; & x'q_1(x', y) + yq_2(x', y) - t(x', y) - (x'q_1(x, y) + yq_2(x, y) - t(x, y)) \geq 0
\end{aligned}$$

Substitute x' twice by $x^- = x - \epsilon$ and $x^+ = x + \epsilon$ respectively, for any arbitrarily small positive ϵ, we have

$$pu_x(x^-, y) \leq q_1(x, y) \leq u_x(x^+, y),$$

where u_x denotes the partial derivative of u on the x dimension. The inequality above implies u is differentiable almost everywhere on x and $u_x = q_1(x, y)$. Similarly, u is differentiable almost everywhere on y and $u_y = q_2(x, y)$. As a result u_x and u_y must be within interval $[0, 1]$. This means, the seller can never allocate more than one pieces of either item. Now, payment function t can be represented by utility function u, $t(x, y) = xu_x(x, y) + yu_y(x, y) - u(x, y)$.

The seller's problem is to design a non-negative, convex utility function, whose partial derivatives on both x and y are within $[0, 1]$, that maximizes expected revenue R (cf. [Hart and Nisan 2012, Lemma 5]).

3. REPRESENTING REVENUE AS A FUNCTION OF UTILITY

Let Ω denote any area in V and R_Ω be the revenue obtained within Ω. Let $z = (x, y)^T$ and $\mathbf{T}(z) = zu(z)f(z)$. By Green's Theorem, we have $\int_\Omega \nabla \cdot \mathbf{T} dz = \oint_{\partial\Omega} \mathbf{T} \cdot \hat{\mathbf{n}} ds$.

$$\begin{aligned}
\nabla \cdot \mathbf{T} &= 2u(z)f(z) + (\nabla u(z))^T z f(z) + u(z) z^T \nabla f(z) \\
&= [(\nabla u(z))^T z - u(z)]f(z) + [3f(z) + z^T \nabla f(z)]u(z) \\
&= t(z)f(z) + \triangle(z)u(z)
\end{aligned}$$

where $\triangle(x, y) = 3f_1(x)f_2(y) + xf_1'(x)f_2(y) + yf_2'(y)f_1(x)$. Seller's revenue formula within Ω is as follows:

$$\begin{aligned}
R_\Omega &= \int_\Omega t(z)f(z)dz = \int_\Omega (\nabla \cdot \mathbf{T} - \triangle(z)u(z))dz \\
&= \oint_{\partial\Omega} \mathbf{T} \cdot \hat{\mathbf{n}} ds - \int_\Omega \triangle(z)u(z)dz
\end{aligned}$$

Set Ω to be the rectangle $ABDC$, the seller's total revenue R_{ABDC} is

$$\begin{aligned}
&\int_{y_A}^{y_C} x_B u(x_B, y) f_1(x_B) f_2(y) dy + \int_{x_A}^{x_B} y_C u(x, y_C) f_1(x) f_2(y_C) dx \\
&- \int_{y_A}^{y_C} x_A u(x_A, y) f_1(x_A) f_2(y) dy - \int_{x_A}^{x_B} y_A u(x, y_A) f_1(x) f_2(y_A) dx \\
&- \int_{x_A}^{x_B} \int_{y_A}^{y_C} u(x, y) \triangle(x, y) dy dx
\end{aligned} \tag{1}$$

Formula (1) consists of 5 terms. The first term represents the part of seller's revenue that depends on utilities on edge BD only. Moreover, this part is increasing as utilities on edge BD increase. Similarly, the second term represents the part of seller's revenue that depends positively on utilities on edge CD. The third and fourth terms represent respectively the parts of seller's revenue that depend negatively on utilities on edges AC and AB. The fifth term represents the part of revenue that depends on the utilities on the inner points of the rectangle. Under different conditions, $\triangle(x, y)$ can be either positive or negative, which suggests this part can either increase or decrease as the utilities on inner points increases. We now define these conditions.

Definition 3.1. For any density $h(x)$, let $PR(h(x)) = \frac{xh'(x)}{h(x)}$ be the *power rate* of h.

Consider the following two conditions regarding power rate.

Condition 1: $PR(f_1(x)) + PR(f_2(y)) \leq -3, \forall (x, y) \in V$.

Condition 2: $PR(f_1(x)) + PR(f_2(y)) \geq -3, \forall (x, y) \in V$.

Clearly, under Condition 1, we have $\triangle(x, y) \leq 0$. This means seller's revenue depends positively on utilities of the inner points. Similarly, under Condition 2, seller's revenue depends negatively on utilities of the inner points.

Remark 1. *To understand the intuition behind power rate, consider an example of selling one item, where the valuation distribution is uniform on* $[0, 1]$ *(in this example, we have a relatively high power rate PR=0). Consider a mechanism with 2 menu items:* $(0, 0)$ *and* $(1, 0.5)$ *(take-it-or-leave-it on price 0.5). Now let us consider the effect of adding a new menu item* $(0.5, 0.2)$.

— *Case 1. When the buyer's valuation is within* $[0.4, 0.5)$*, buyer's utility weakly increases (compared to the old mechanism) by switching to this new menu item. Seller's revenue also increases because of positive sale probability.*
— *Case 2. When the buyer's valuation is within* $[0.5, 0.6)$*, buyer's utility will also weakly increase by switching to this new menu item. However, seller's revenue decreases because the buyer now chooses a lower payment menu item.*

Intuitively, high power rate $(PR \geq -3)$ *places sufficiently high density on large valuations, which ensures that the revenue increment in Case 1 is less than the revenue decrement in Case 2. In other words, adding more menu items hurts revenue. This explains, under Condition 2 (the high power rate case), we only need few menu items.*

Based on the two conditions above, we obtain two parts of results: under Condition 1, the optimal mechanisms have simple menus in the sense that their menus are monotone – allocation probabilities and payment are increasing in the same order; under Condition 2, the optimal mechanisms also have simple menus, but in a different sense, that their menus only contain a few items.

[Daskalakis et al. 2013] consider the same problem but restrict to the case where

$$y_A f_2(y_A) = 0, x_A f_1(x_A) = 0;$$

$$\lim_{x \to x_B} x^2 f_1(x) = 0, \lim_{y \to y_C} y^2 f_2(y) = 0.$$

These assumptions ignore the effect of utilities on the edges of the rectangle. We do not have any of these constraints. As a result, their techniques (such as reduction to the transportation problem) do not apply to our more general case. In fact, one of our main techniques is to conduct sensitivity analysis on how revenue changes as a function of the utilities on the edges.

4. PART I: MENU MONOTONICITY AND REVENUE MONOTONICITY

In this section, we consider the case where power rates of both density functions satisfy Condition 1. When this condition is not met, Hart and Reny [2012] give several interesting counter-examples of revenue monotonicity: the optimal revenue for stochastically inferior valuation distributions may be greater than that of stochastically superior distributions. When this condition is met, for identical item distributions, Hart and Nisan [2012] prove that, bundling sale is the optimal mechanism. In this section, we show that, under Condition 1, the optimal menu can be sorted so that both allocations as well as payment monotonically increase. We coin this result *menu monotonicity theorem*. The theorem has two immediate consequences. First, it yields a version of revenue monotonicity theorem, thus complements the Hart-Reny result above. Second, it subsumes the above Hart-Nisan result as a corollary.

Our analysis starts from a simple observation: any optimal mechanism must extract all of the buyer's valuation when he is in the lowest type.

LEMMA 4.1. *In the optimal mechanism,* $u(x_A, y_A) = 0$.

PROOF. Suppose otherwise that $u(x_A, y_A) > 0$, one can revise every menu item from $(q_1(x,y), q_2(x,y), t(x,y))$ to $(q_1(x,y), q_2(x,y), t(x,y) + u(x_A, y_A))$ and obtain a mechanism with strictly higher revenue, contradiction. □

THEOREM 4.2. ***Menu Monotonicity***
Under Condition 1, menu items of the optimal mechanism can be represented in the form of $(q_1(t), q_2(t), t)$*, such that allocation probabilities* $q_1(t)$ *and* $q_2(t)$ *are weakly increasing in payment* t*.*

Roughly speaking, Theorem 4.2 suggests that, among the menu items of the optimal mechanism, higher payment t corresponds to higher allocation probabilities q_1 and q_2. Note that allocation and payment monotonicity are well understood in single-item optimal auction (i.e., Myerson auction) but in general fail to hold in two item settings [Hart and Reny 2012].

In the following, we give a semi-constructive proof. By Formula (1), under Condition 1, we know that seller's revenue is increasing as the utilities of the buyer increases on V, except on edges AB and AC. Our idea is then, to fix the utility function on AB and AC and construct the (largest possible) remainder of the utility function subject to convexity. In the appendix, we give another proof.

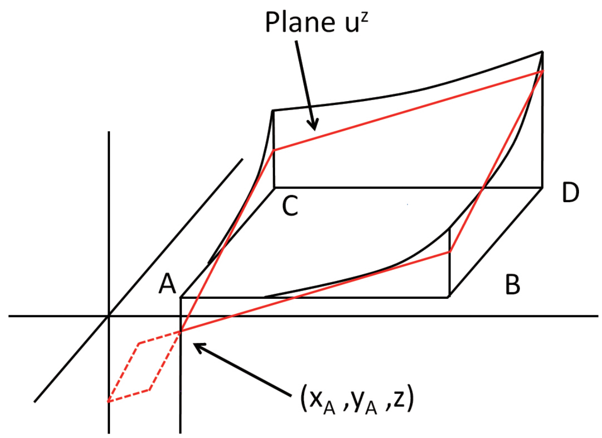

Fig. 1. Optimal mechanism in 3D

PROOF. Let's look at Fig. 1. Fixing $u(AB)$ and $u(AC)$ (not necessarily optimal), consider any point (x_A, y_A, z) lower than A on the vertical line $(x = x_A, y = y_A)$, draw a plane going through point and touching $u(AC)$ and $u(AB)$(subject to gradient no greater than 1). Because the gradients in two directions are unique. This plane is unique, we call it u^z. We directly claim that $u^*(x,y) = \sup_{z\in(-\infty,0]}\{u^z(x,y)\}$ is the optimal utility function subject to fixed $u(AB)$ and $u(AC)$.

First, we prove $(x,y) \in AB \cup AC$, $u^*(x,y) = u(x,y)$. Pick any point $(x_0,y_0) \in AB$. We have $u^z(x_0,y_0) \le u(x_0,y_0)$ for $z \in (-\infty, 0]$, so $u^*(x_0,y_0) = \sup_{z\in(-\infty,0]}\{u^z(x_0,y_0)\} \le u(x_0,y_0)$. Since $u(AB)$ is convex, there always exists a plane u^{z_0}, where $z_0 = u(x_0,y_0) - q_1(x_0,y_0)(x_0 - x_A)$, that passes through $(x_0, y_0, u(x_0,y_0))$. So $u^*(x_0,y_0) \ge u^{z_0}(x_0,y_0) = u(x_0,y_0)$, i.e. $u^*(x_0,y_0) = u(x_0,y_0)$. Similar for points on AC.

Second, we prove $(x,y) \in V\backslash\{AB\cup AC\}$, $u^*(x,y)$ is its largest possible value subject to fixed $u(AB)$ and $u(AC)$. Pick any point $(x_1,y_1) \in V\backslash\{AB\cup AC\}$. Let the largest possible utility on point (x_1,y_1) be $\tilde{u}(x_1,y_1)$, achieved by utility function $\tilde{u}$. Let $(x_1,y_1,\tilde{u}(x_1,y_1))$ be in some plane $\tilde{u}^{(x_1,y_1),0}(x,y) = x\tilde{q}_1(x_1,y_1) + y\tilde{q}_2(x_1,y_1) - \tilde{t}(x_1,y_1)$.

In other words, $\tilde{u}^{(x_1,y_1),0}(x,y)$ is the buyer's utility at type (x,y) but chooses menu item $(\tilde{q}_1(x_1,y_1),\ \tilde{q}_2(x_1,y_1), \tilde{t}(x_1,y_1))$. By IC, $\tilde{u}^{(x_1,y_1),0}(x,y) \le \tilde{u}(x,y)$. Think of $\tilde{u}^{(x_1,y_1),0}(x,y)$ as a plane that is always weakly below $\tilde{u}(x,y)$ but touches $\tilde{u}$ at the point of (x_1,y_1).

Plane $\tilde{u}^{(x_1,y_1),0}$ passes through point (x_A, y_A, z_1), where $z_1 = px_A\tilde{q}_1(x_1,y_1) + y_A\tilde{q}_2(x_1,y_1) - \tilde{t}(x_1,y_1)$. By definition, u^{z_1} goes through point (x_A,y_A,z_1) too! By our construction, u^{z_1} has the largest possible gradients in both directions subject to the fixed $u(AB)$ and $u(AC)$. Since we fix $u(AB)$ and $u(AC)$, $\tilde{u}(AB) = u(AB)$, $\tilde{u}(AC) = u(AC)$, so u^{z_1} has weakly larger gradients than $\tilde{u}^{(x_1,y_1),0}$. Hence, we have $u^*(x_1, y_1) \geq u^{z_1}(x_1, y_1) \geq \tilde{u}^{(x_1,y_1),0}(x_1, y_1)$. Since (x_1, y_1) is arbitrarily chosen and $\tilde{u}^{(x_1,y_1),0}$ is the largest at (x_1, y_1), we conclude that $u^*(x, y)$ is the largest possible value on any point (x, y) subject to fixed $u(AB)$ and $u(AC)$.

Finally, according to Formula (1), u^* gives the optimal revenue subject to fixed $u(AB)$ and $u(AC)$. By our construction u^* consists of a set of monotone planes u^z. As $|z|$ increases, allocation probabilities weakly increase and payment strictly increases. This completes the proof. □

Theorem 4.2 implies the aforementioned Hart-Nisan result as a corollary.

COROLLARY 4.3. *[Hart and Nisan 2012, Theorem 28] For two i.i.d. items, $PR(f_1) = PR(f_2) \leq -\frac{3}{2}$, bundling sale is optimal.*

PROOF. It is without loss to restrict attention to symmetric mechanisms [Maskin and Riley 1984]. Thus, $u(AB)$ is identical to $u(AC)$. $u^1(AB)$ and $u^2(AC)$ have the same slope (in fact, plane u' touches both $u(AB)$ and $u(AC)$ simultaneously). So, $q_1(v) = q_2(v)$ $\forall v \in V$. In other words, the two items are always sold with the same probability. The seller's revenue of this optimal mechanism is equivalent to a mechanism that sells two items as a bundle with the same probability. So bundling is optimal as well. □

As another application of Theorem 4.2, we obtain a *revenue monotonicity* theorem in this setting.

THEOREM 4.4. ***(Revenue Monotonicity)***
Under Condition 1, optimal revenue is monotone: let F_i, G_i be the cumulative distribution function of density functions $f_i, g_i, i = 1, 2$, respectively. If G_1 and G_2 first-order stochastically dominate F_1 and F_2[1] respectively, optimal revenue obtained for (G_1, G_2) is no less than that of (F_1, F_2).

PROOF. Consider any two points (x_2, y_2) and (x_3, y_2), where $x_3 > x_2$. If $q_1(x_2, y_2) < q_1(x_3, y_2)$, by Theorem 4.2, we must have $t(x_2, y_2) < t(x_3, y_2)$. If $q_1(x_2, y_2) = q_1(x_3, y_2)$, then $q_1(x, y_2) = q_1(x_2, y_2), \forall x \in [x_2, x_3]$. $u(x_3, y_2) = u(x_2, y_2) + q_1(x_2, y_2)(x_3 - x_2)$, which can be achieved by choosing the same menu as (x_2, y_2) chooses. While buyer at type (x_3, y_2) has several menu items that all achieve the highest utility, we can assume, WLOG, that the buyer chooses the menu with the highest payment ([Hart and Reny 2012]). Thus there is an optimal choice guarantees $t(x_2, y_2) \leq t(x_3, y_2)$.

To sum up, $t(x_2, y_2) \leq t(x_3, y_2)$ when $x_2 \leq x_3$. For the same reason, $t(x_3, y_2) \leq t(x_3, y_3)$ when $y_2 \leq y_3$. Hence $t(x, y)$ is a weakly monotone function in both directions. Suppose G_1 and G_2 first-order stochastically dominates F_1 and F_2 respectively. Let $R(F_1 \times F_2)$ denote the optimal revenue when item 1 and 2 distributes independently according to F_1 and F_2. When distribution $G_1 \times G_2$ chooses the same mechanism as $F_1 \times F_2$ does, let the revenue be $R^*(G_1 \times G_2)$. We have $R^*(G_1 \times G_2) \geq R(F_1 \times F_2)$, since t is weakly monotone. By transitivity, $R(G_1 \times G_2) \geq R^*(G_1 \times G_2) \geq R(F_1 \times F_2)$. □

5. PART II: OPTIMAL MECHANISM WITH SMALL MENUS

In this section, we investigate optimal mechanisms under Condition 2. We obtain several results saying that the optimal mechanism contains only few menu items. All these results are built upon Pavlov's characterization [Pavlov 2011a] and an important lemma introduced in the next subsection.

---

[1] G_i first-order stochastically dominates F_i if $G_i(x) \leq F_i(x)$ for all x and $G_i(x) > F_i(x)$ for some x.

5.1. Pavlov's characterization and graph representation lemma

If both f_1 and f_2 satisfy Condition 2, Pavlov [2011a, Proposition 2] states that, in the optimal mechanism, the seller either keeps both items (i.e., $q = (0, 0)$), or sells one of the items at probability 1 (i.e., $q_1 = 1$ or $q_2 = 1$).

For graphic representation, let the buyer's valuation be within rectangle $ABDC$, we have the following lemma.

LEMMA 5.1. ***Graph Representation Lemma***

Under Condition 2, the optimal mechanism can be represented by one of the rectangles shown in Fig. 2 or Fig. 3. More precisely, the optimal mechanism divides the valuation rectangle into four regions, where

(*1*) *in the bottom left region (region $ASME$ in both figures), it assigns $q = (0, 0)$ and $u(x, y) = 0$ to any point (x, y) in the region. Furthermore, region $ASME$ is convex.*
(*2*) *in the top right region, it assigns $q = (1, 1)$ to any point in the region.*
(*3*) *in the top left region, it assigns $q = (*, 1)$ to any point in the region, where $*$ is a variable. Thus this region represents a set of menu items, each of which is a vertical slice.*
(*4*) *Symmetrically, in the bottom right region, it assigns $q = (1, *)$ to any point in the region. This region represents a set menu items, each of which is a horizontal slice.*
(*5*) *The boundary between the top left and right regions is vertical (QL in both figures); the boundary of the top right and bottom right regions is horizontal (MN in Fig. 2 or LI in Fig. 3). The boundary between $(1, *)$ region and $(*, 1)$ region is in the upper right direction.*

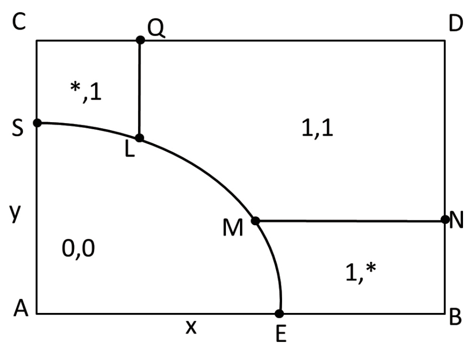

Fig. 2. The optimal allocation that there is a point on curve SME chooses allocation menu (1,1).

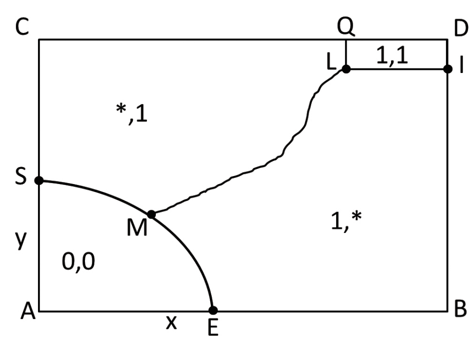

Fig. 3. The optimal allocation that there is no point on curve SME chooses allocation menu (1,1).

PROOF. We first determine the relative positions of the four possible regions.

If the seller keeps both items, the buyer's utility is zero. Since $u(x, y)$ is an increasing function, it assigns $q = (0, 0)$ in the bottom left region, i.e. $ASME$. Since $u(x, y)$ is convex, the convex combination of any two zero-utility points must also be zero. Therefore, $ASME$ is a convex region.

If for a type (x_0, y_0) with $q_1(x_0, y_0) = q_2(x_0, y_0) = 1$, for any point (x_1, y_1), IC requires that

$$x_0 + y_0 - t(x_0, y_0) \geq x_0 q_1(x_1, y_1) + y_0 q_2(x_1, y_1) - t(x_1, y_1),$$
$$x_1 q_1(x_1, y_1) + y_1 q_2(x_1, y_1) - t(x_1, y_1) \geq x_1 + y_1 - t(x_0, y_0).$$

Summing the two inequalities, we get $(q_1(x_1,y_1)-1)(x_1-x_0)+(q_2(x_1,y_1)-1)(y_1-y_0) \geq 0$. If $x_1 > x_0, y_1 > y_0$, we must have $q_1(x_1,y_1) = q_2(x_1,y_1) = 1$.

Let (x_2,y_2) be a point where some positive proportions of the items are sold, then according to Pavlov's characterization [Pavlov 2011a], one of the items must be sold deterministically. Consider two types (x_2,y_2) and (x_3,y_3) where $q_1(x_2,y_2) = 1, q_2(x_2,y_2) < 1$ and $q_1(x_3,y_3) < 1, q_2(x_3,y_3) = 1$. By IC, we must have

$$x_2 + y_2q_2(x_2,y_2) - t(x_2,y_2) \geq x_2q_1(x_3,y_3) + y_2 - t(x_3,y_3),$$
$$x_3q_1(x_3,y_3) + y_3 - t(x_3,y_3) \geq x_3 + y_3q_2(x_2,y_2) - t(x_2,y_2).$$

Summing up the two inequalities, we get $(1-q_1(x_3,y_3))(x_2-x_3)+(1-q_2(x_2,y_2))(y_3-y_2) \geq 0$. So, one of $x_2 < x_3$ and $y_2 > y_3$ does not hold. This implies the second part of (5).

To sum up, $(1,1)$ must be assigned to the upper right corner, $(1, q_2(x,y))$ is assigned to the bottom right corner, $(q_1(x,y), 1)$ is assigned to the upper left corner, and $(0,0)$ is assigned to the bottom left corner, (some regions may be empty).

Let the allocation vector at (x_4,y_4) be $(1, q_2(x_4,y_4))$. For any $x \in [x_4, x_B]$, by IC, we must have $q_1(x,y_4) = 1$, so $u(x,y_4) = u(x_4,y_4) + x - x_4$. It is still in the buyer's best interest to choose menu item $(1, q_2(x_4,y_4))$ at (x,y_4). This implies the first part of (5): the boundary between for different q_2 in $(1, q_2(x,y))$ is horizontal. In particular, in Fig. 2, MN is horizontal. Similarly, LQ is vertical.

If there is a point on curve SE that chooses menu item $(1,1)$, the mechanism is of the form shown in Fig. 2, otherwise it is of the form shown in Fig. 3. $\square$

5.2. Optimal mechanisms for constant power rate

To describe our first theorem under Condition 2, we need the following condition on density functions.

Condition 3: $PR(f_i(x)), \quad i = 1, 2, \quad$ is constant.

THEOREM 5.2. *Under Conditions 2 and 3, there is an optimal mechanism such that it has at most 4 menu items.*

The result is tight: one can find instances where optimal mechanism contains exactly 4 menu items [Pavlov 2011a, Example 3].

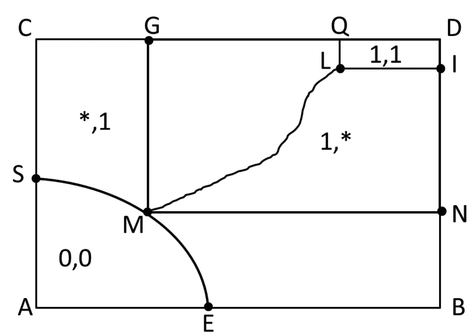

Fig. 4. The optimal allocation that there is no point on curve SME chooses (1,1) allocation menu.

We prove the theorem for the case shown in Fig. 4.(All figures except Fig. 2 and 3 are used as intermediate graphs to illustrate the proofs, not the final shapes of the optimal mechanisms. For example, in this proof, we starts from an arbitrary diagram that only has the properties listed in Lemma 5.1. For every theorem, the final optimal mechanism is listed as a table in the appendix.) The other case related to Fig. 2 follows from an almost identical proof. First, draw a horizontal line through M and it intersects BD at N. Then draw a vertical line through M crossing CD at G. We have the following two lemmas.

LEMMA 5.3. *There is an optimal utility function such that $u(BN)$ is piecewise linear with at most 2 pieces.*

LEMMA 5.4. *There is an optimal utility function such that $u(ND)$ is piecewise linear with at most 2 pieces.*

With these two lemmas, we are able to prove Theorem 5.2 (in the Appendix). Hart and Nisan [2012, Theorem 1 and Lemma 14] state that bundling 4-approximates optimal revenue for general two-item setting. As an application of Theorem 5.2, we obtain a better lower bound for bundling sale.

COROLLARY 5.5. *Under Conditions 2 and 3, bundling 3-approximates optimal revenue.*

PROOF. Revenue of an optimal mechanism with 3 non-zero menu items is less than or equal to the sum of revenues of 3 mechanisms, each of which has only 1 non-zero menu items. Since bundling is optimal among all mechanisms that contains only 1 non-zero menu item, thus no worse than any of these three mechanisms. Consequently bundling gives a 3-approximation of the optimal revenue. □

Two cases where the optimal mechanism contains ≤ 3 items

In fact, Conditions 2 and 3 have intersections. When both conditions are satisfied, revenue does not depend on utilities on inner points any more. In this case, we obtain a condition under which there is an optimal mechanism that contains at most 3 menu items.

COROLLARY 5.6. *For $f_1(x) = s_1 x^{i_1}, f_2(y) = s_2 y^{i_2}, s_1, s_2 > 0, i_1 + i_2 = -3$, there is an optimal mechanism such that it contains at most 3 menu items, thus bundling gives a 2-approximation of the optimal revenue.*

PROOF. By Formula (1), we can see that revenue only depends on the utility of the boundaries. According to Theorem 5.2, there are at most 4 menu items and both $u(BD)$ and $u(CD)$ are piecewise linear with two pieces. Suppose otherwise that there are 4 different menu items, then one could rise up the plane of the $(1,1)$-item uniformly, i.e. expand the top right region, until one of $u(BD), u(CD)$ becomes a straight line, i.e. the top right region covers part of AB or AC. Note that this procedure will not change $u(AB)$ or $u(AC)$ since it will terminate as long as the $(1,1)$-item reaches AB and AC. More over, this procedure increases $u(BD)$ and $u(CD)$ while maintains convexity. So the new utility function corresponds to a strictly higher revenue, which contradicts to the fact that u is optimal. □

Following a similar proof of Theorem 5.2, we obtain another condition under which 3 menu items are enough. Note that, this condition does not impose constant power rate, thus is not a special case of Condition 3.

Condition 4: $-2 \leq PR(f_1(x)) \leq y_A f_2(y_A) - 2, \forall x$ and $-2 \leq PR(f_2(y)) \leq x_A f_1(x_A) - 2, \forall y$.

THEOREM 5.7. *Under Conditions 2 and 4, there is an optimal mechanism such that it contains at most 3 menu items, thus bundling gives a 2-approximation of the optimal revenue.*

5.3. Optimal mechanisms for i.i.d. monotone power rate

The requirement of power rate to be constant might be restrictive. If one relaxes this requirement to be *monotone power rate*, one only needs to add two more menu items.

Condition 5: $PR(f_i(x)), i = 1, 2,$ is weakly monotone.

THEOREM 5.8. *Under Conditions 2 and 5, if $f_1 = f_2$, there is an optimal mechanism such that it consists of at most 6 menu items.*

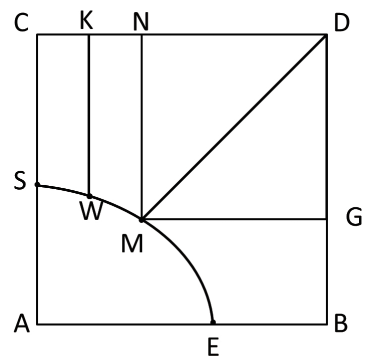

Fig. 5. Optimal allocation under symmetric value distribution that satisfies Condition 5.

The general form of optimal mechanism is shown in Fig. 5. It is without loss to restrict attention on symmetric mechanisms [Maskin and Riley 1984, section 1]. Let AD intersects SE at point M. In region $ASME$, seller keeps both items. Item 2 is sold deterministically in $CSMD$ and item 1 is sold determinately in $MEBD$. Let the allocation rule on point (x, y) in $CSMD$ be $(q_1(x), 1)$. Similar to the proof of Theorem 5.2, we start with the following lemma.

LEMMA 5.9. *There is an optimal utility function such that $u(ND)$ is piecewise linear with at most 2 pieces.*

Similarly, we show that $u(CN)$ is piecewise linear as well.

LEMMA 5.10. *There is an optimal utility function such that $u(CN)$ is piecewise linear with at most 2 pieces.*

With the two lemmas above, we are able to prove Theorem 5.8.

Condition 5 is general enough to admit a large variety of density functions. To have a sense of what these functions are, we have the following two propositions.

COROLLARY 5.11. *If $h(x), x \in [x_A, x_B]$ is a convex, weakly monotone density function, and $x_B h'(x_B) \leq h(x_B)$, then $h(x)$ satisfies Condition 5.*

It is easy to check $h(x) = a_n x^n, a_n \geq 0$ also satisfies Condition 5. In fact, there are many other functions satisfy Condition 5.

COROLLARY 5.12. *If h_1 and h_2 both satisfy Condition 5, then $h_1 + h_2$ and $h_1 \cdot h_2$ both satisfy Condition 5. Particularly, for all nonnegative-coefficient polynomial $h(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, a_i \geq 0, i = 0, ..., n$, including those whose coefficients in its Taylor series are nonnegative. Examples are, $(1-x)^{-1}$,e^x,e^{e^x},$\ln \frac{1}{1-x}$, $\frac{x}{1-x-x^2}$,$\tan x (|x| < \frac{\pi}{2})$, hyperbolic sine $\sinh x = \frac{e^x - e^{-x}}{2}$, and $\frac{F_n x}{1-(F_{n-1}+F_{n+1})x-(-1)^n x^2}$ where F_n denotes the Fibonacci numbers.*

5.4. Optimal mechanism for uniform distributions under unit-demand constraint

A buyer has *unit-demand* if $q_1(x,y)+q_2(x,y)\le 1$. Under unit-demand model, Pavlov [2011b, Proposition 2] states that, if distribution functions satisfy Condition 2, it is without loss to restrict attention on mechanisms such that

$$q_1(x,y)+q_2(x,y)\in\{0,1\}\qquad \forall(x,y)$$

Pavlov solves the optimal mechanism for two items with identical uniform distributions. The resulting mechanism contains 5 menu items for uniform distribution on $[c,c+1]*[c,c+1], c\in(1,\bar{c})$ (where $\bar{c}\approx 1.372$). We show that in nonidentical settings, the optimal mechanism also contains at most 5 menu items. It follows trivially that our result is tight.

THEOREM 5.13. *In unit-demand model, if both f_1 and f_2 are uniform distributions, there is an optimal mechanism such that it consists of at most 5 menu items.*

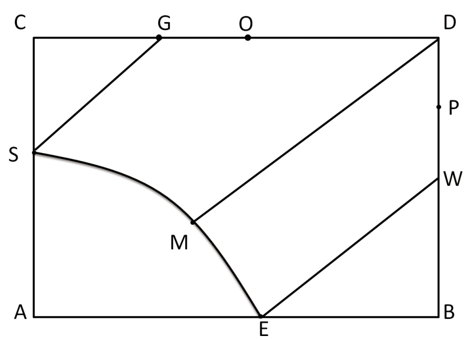

Fig. 6. Optimal unit demand allocation.

Let ASE denote the zero utility region and $CSEBD$ the non-zero utility region. For the same reason in Lemma 5.1, ASE is convex. For points in ASE, allocation $(0,0)$ is the best. For $(x,y)\in CSEBD$, $(q_1(x,y),q_2(x,y))\neq(0,0)$, so $q_1(x,y)+q_2(x,y)=1$. The mechanism is shown in Fig. 6. Draw a 45 degree line across E, intersecting BD or CD at W. Draw a 45 degree line across S, intersecting BD or CD at G. We consider here the case that W is on BD and G is on CD. Other cases follow from similar arguments.

The theorem can be similarly proved via the following two lemmas.

LEMMA 5.14. *There is an optimal utility function such that $u(BW)$ is piecewise linear with at most 2 pieces.*

LEMMA 5.15. *There is an optimal utility function such that $u(WD)$ is piecewise linear with at most 2 pieces.*

6. ACKNOWLEDGEMENT

The authors would like to acknowledge the EC-14 reviewers for their thoughtful comments, especially one reviewer who pointed out a mistake in a previous proof of Theorem 4.2. The authors also benefit from discussions with Gregory Pavlov and participants of the second workshop of "New trend in mechanism design II". This work was supported in part by the National Basic Research Program of China Grant 2011CBA00300, 2011CBA00301, the National Natural Science Foundation of China Grant 61033001, 61361136003, 61303077, and a Tsinghua University Initiative Scientific Research Grant.

REFERENCES

CAI, Y., DASKALAKIS, C., AND WEINBERG, S. M. 2012a. An algorithmic characterization of multi-dimensional mechanisms. In *STOC*. 459–478.

CAI, Y., DASKALAKIS, C., AND WEINBERG, S. M. 2012b. Optimal multi-dimensional mechanism design: Reducing revenue to welfare maximization. In *FOCS*. 130–139.

CAI, Y. AND HUANG, Z. 2013. Simple and nearly optimal multi-item auctions. In *SODA*. 564–577.

CHAWLA, S., HARTLINE, J. D., MALEC, D. L., AND SIVAN, B. 2010. Multi-parameter mechanism design and sequential posted pricing. STOC '10.

DASKALAKIS, C., DECKELBAUM, A., AND TZAMOS, C. 2013. Mechanism design via optimal transport. In *ACM Conference on Electronic Commerce*. 269–286.

DASKALAKIS, C. AND WEINBERG, S. M. 2011. On optimal multi-dimensional mechanism design. *Electronic Colloquium on Computational Complexity (ECCC) 18*, 170.

DASKALAKIS, C. AND WEINBERG, S. M. 2012. Symmetries and optimal multi-dimensional mechanism design. In *ACM Conference on Electronic Commerce*. 370–387.

DENG, C. AND PEKEČ, S. 2011. Money for nothing: exploiting negative externalities. In *Proceedings of the ACM Conference on Electronic commerce (EC)*.

HART, S. AND NISAN, N. 2012. Approximate revenue maximization with multiple items. In *ACM Conference on Electronic Commerce*. 656.

HART, S. AND NISAN, N. 2013. The menu-size complexity of auctions. In *ACM Conference on Electronic Commerce*.

HART, S. AND RENY, P. 2012. Maximal revenue with multiple goods: Nonmonotonicity and other observations. Tech. rep.

HARTLINE, J. D. AND ROUGHGARDEN, T. 2009. Simple versus optimal mechanisms. In *ACM EC*.

JEHIEL, P., MOLDOVANU, B., AND STACCHETTI, E. 1996. How (not) to sell nuclear weapons. *American Economic Review 86,* 4, 814–29.

LEDYARD, J. O. 2007. Optimal combinatoric auctions with single-minded bidders. In *Proceedings of the ACM Conference on Electronic Commerce (EC)*.

LEVIN, J. 1997. An optimal auction for complements. *Games and Economic Behavior 18,* 2, 176–192.

LI, X. AND YAO, A. C.-C. 2013. On revenue maximization for selling multiple independently distributed items. *Proceeding of National Academy of Science*.

MANELLI, A. M. AND VINCENT, D. R. 2006. Bundling as an optimal selling mechanism for a multiple-good monopolist. *Journal of Economic Theory 127,* 1, 1–35.

MANELLI, A. M. AND VINCENT, D. R. 2007. Multi-dimensional mechanism design: Revenue maximization and the multiple good monopoly. *Journal of Economic Theory 137,* 1, 153–185.

MASKIN, E. AND RILEY, J. 1989. Optimal multi-unit auctions. In *The Economics of Missing Markets, Information, and Games*. Chapter 14, 312–335.

MASKIN, E. S. AND RILEY, J. G. 1984. Optimal auctions with risk averse buyers. *Econometrica 52,* 6, 1473–1518.

MYERSON, R. B. 1981. Optimal auction design. *Mathematics of Operations Research 6,* 1, 58–73.

PAVLOV, G. 2011a. Optimal mechanism for selling two goods. *The B.E. Journal of Theoretical Economics 11,* 1.

PAVLOV, G. 2011b. A property of solutions to linear monopoly problems. *The B.E. Journal of Theoretical Economics 11,* 4.