



# The antimagicness of the Cartesian product of graphs

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## ABSTRACT

An *antimagic labeling* of a graph with  $M$  edges and  $N$  vertices is a bijection from the set of edges to the set  $\{1, 2, 3, \dots, M\}$  such that all the  $N$  *vertex-sums* are pairwise distinct, where the *vertex-sum* of a vertex  $v$  is the sum of labels of all edges incident with  $v$ . A graph is called antimagic if it has an antimagic labeling. The antimagicness of the Cartesian product of graphs in several special cases has been studied [Tao-Ming Wang, Toroidal grids are anti-magic, in: Proc. 11th Annual International Computing and Combinatorics Conference, COOCOON'2005, in: LNCS, vol. 3595, Springer, 2005, pp. 671–679, Yongxi Cheng, A new class of antimagic cartesian product graphs, Discrete Mathematics 308 (24) (2008) 6441–6448]. In this paper, we develop new construction methods that are applied to more general cases. We prove that the Cartesian product of paths is antimagic, if one of them has at least three edges. This (almost) answers the open problems in [Yongxi Cheng, Lattice grids and prisms are antimagic, Theoretical Computer Science 374 (2007) 66–73]. We also prove that the Cartesian product of an antimagic regular graph and a connected graph is antimagic, which extends the results of the latter of the two references, where several special cases are studied.

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## 1. Introduction

In 1990, Hartsfield and Ringel [5] introduced the concept of the *antimagic* graph. An *antimagic labeling* of a graph with  $M$  edges and  $N$  vertices is a bijection from the set of edges to the set  $\{1, 2, 3, \dots, M\}$  such that all the  $N$  *vertex-sums* are pairwise distinct, where the *vertex-sum* of a vertex  $v$  is the sum of labels of all edges incident with  $v$ . A graph is called antimagic if it has an antimagic labeling. Hartsfield and Ringel showed that paths  $P_n$  ( $n \geq 3$ ), cycles, wheels, and complete graphs  $K_n$  ( $n \geq 3$ ) are antimagic. They conjectured that all trees except  $K_2$  are antimagic. Moreover, all connected graphs except  $K_2$  are antimagic. These two conjectures are unsettled. Significant progress was made by Alon et al. [1], which states that if  $G$  is a  $n$ -vertex graph with minimum degree  $\Omega(\log n)$  (or even further  $\Omega(\log n / \log \log n)$ ), then  $G$  is antimagic. In [6] Hefetz proved several special cases and variants of the latter conjecture. In particular, he proved that for any integers  $k > 0$ , a graph with  $3^k$  vertices is antimagic if it admits a  $K_3$ -factor. In [4] Cranston showed that regular bipartite graphs are antimagic.

Wang [7] showed that the Cartesian products of two or more cycles are antimagic. The antimagicness of Cartesian product of two paths and Cartesian products of two or more regular graphs were proved in [2,3] by Cheng, respectively.

In this paper, we develop new construction methods which are applied to more general cases. First we prove that (almost) all Cartesian products of paths are antimagic. The proof relies on several construction strategies. Our main result is the following theorem.

**Theorem 1.** *The Cartesian product of two or more paths is antimagic, if there is at least one path with three or more edges.*

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**Remark.** For the case that every path has length one or two, we were not able to give a canonical labeling method.

Using these strategies in similar ways, we can deal with the Cartesian product of regular graphs and arbitrary connected graphs.

**Theorem 2.** *If  $R$  is an antimagic regular graph, and  $G$  is a connected graph, the Cartesian product of  $R$  and  $G$  is antimagic.*

This paper is organized as follows: In Section 2 we introduce some definitions and useful lemmas, in Section 3 we give the proof of Theorem 1, and we give the proof of Theorem 2 in Section 4.

## 2. Preliminaries

Given a graph  $G = (V, E)$ , we color the edges with two colors: red and blue. For each vertex  $u \in V(G)$ , let  $R(u)$  and  $B(u)$  indicates the number of red and blue edges incident to  $u$  respectively. A vertex  $u$  is called  $k$ -balance-colored if  $|R(u) - B(u)| \leq k$ . A graph  $G$  can be  $k$ -balance-colored if, and only if, there exist a coloring for  $G$  so that every vertex in  $G$  is  $k$ -balance-colored.

**Lemma 1.** *Every connected graph  $G$  can be 2-balance-colored. Further more, if  $G$  contains at least one vertex whose degree is odd, then it can be 1-balance-colored.*

**Proof.** First, we construct a new graph  $G'$  from  $G$  that contains only even-degree vertices. We do this by pairing all the odd-degree vertices in  $G$  and then adding one virtual-edge between the two vertices of each pair. It is easy to see that  $G'$  has an Eulerian circuit  $u_1u_2 \cdots u_tu_1$ . If  $G$  contains odd-degree vertices, then  $G'$  contains virtual-edges. We may assume that  $u_1u_2$  is such a virtual-edge.

Then we color the edges of  $u_1u_2 \cdots u_tu_1$  with the two colors. We begin with red for  $u_1u_2$ , then alternatively use blue and red for  $u_2u_3, u_3u_4, \dots$  until all the edges have been colored; in this way every vertex in  $G'$  has an equal number of red and blue incident edges, except  $u_1$ . In the case that  $u_1u_2 \cdots u_tu_1$  contains an odd number of edges,  $u_1$  would have two more red incident edges than blue incident edges.

Finally, we remove the virtual-edges of  $G'$  and maintain the coloring, unchanged. At this time, we have already obtained a 2-balance-colored coloring for  $G$ . To see this, we note that every vertex except  $u_1$  is 0-balance-colored in  $G'$ , thus they are 0-balance-colored or 1-balance-colored in  $G$ ;  $u_1$  would also be 0-balance-colored in  $G'$  if  $u_1u_2 \cdots u_tu_1$  contains an even number of edges. In this case it is 0-balance-colored or 1-balance-colored in  $G$  for the same reason, so  $G$  can be 1-balance-colored. Otherwise,  $u_1u_2 \cdots u_tu_1$  has an odd number of edges thus  $u_1$  has two more red incident edges, i.e.  $u_1$  is 2-balance-colored in  $G'$ . In this case, if  $G$  contains at least one odd-degree vertex, then  $u_1u_2$  would be a virtual-edge so  $u_1$  is 1-balance-colored in  $G$ , which means that  $G$  can be 1-balance-colored. But if  $G$  contains only even-degree vertices, then there is no virtual-edge incident to  $u_1$  so  $u_1$  is 2-balance-colored in  $G$ , which means that  $G$  can be 2-balance-colored only by our coloring method.  $\square$

The Cartesian product  $G_1 \times G_2 \times \cdots \times G_n$  of  $n$  graphs  $G_i = (V_i, E_i)$  is a graph with vertex set  $V_1 \times V_2 \times \cdots \times V_n$ , and vertex  $(u_1, u_2, \dots, u_n)$  is adjacent to vertex  $(v_1, v_2, \dots, v_n)$  if, and only if, there is an index  $t$  such that  $(u_t, v_t) \in E_t$  and  $u_i = v_i$  for  $i \neq t$ . The Cartesian product of paths is represented by  $P_1[m_1 + 1] \times P_2[m_2 + 1] \times \cdots \times P_n[m_n + 1]$ , where  $P_i$  is a path with  $m_i$  edges. For the sake of simplicity, we use  $(x_1, x_2, \dots, x_n)$  to represent a corresponding vertex in the product graph, where  $x_i$  corresponds to the  $x_i$ -th vertex of  $P_i$ ,  $1 \leq x_i \leq m_i + 1$  ( $i = 1, \dots, n$ ).

**Corollary 1.** *Cartesian product of paths can be 1-balance-colored.*

**Proof.** Given an arbitrary Cartesian product of  $n$  paths, suppose that there are  $m$  paths whose lengths are larger than 1. Then, all integers from  $n$  to  $n + m$  can hold some vertices of the graph as their degrees. If  $m > 1$ , we can say that there must be at least one vertex whose degree is odd, which leads to the conclusion by Lemma 1. However, if  $m = 0$ , there may be no odd-degree vertex in the graph when  $n$  is even. In this special case, the graph contains an even number of edges, so it can also be 1-balance-colored. The proof of this assertion is included in the proof of Lemma 1.  $\square$

To continue our discussion, we also need some labeling templates which may simplify our later constructions. The template will be built on the Cartesian product of paths.

**Lemma 2.** *Every Cartesian product of  $n$  ( $n \geq 2$ ) paths contains such a circuit in which there are at least  $2n$  edges; when this circuit is removed, at most one virtual-edge is necessary to be added between a remaining pair of odd-degree vertices to keep the graph connected.*

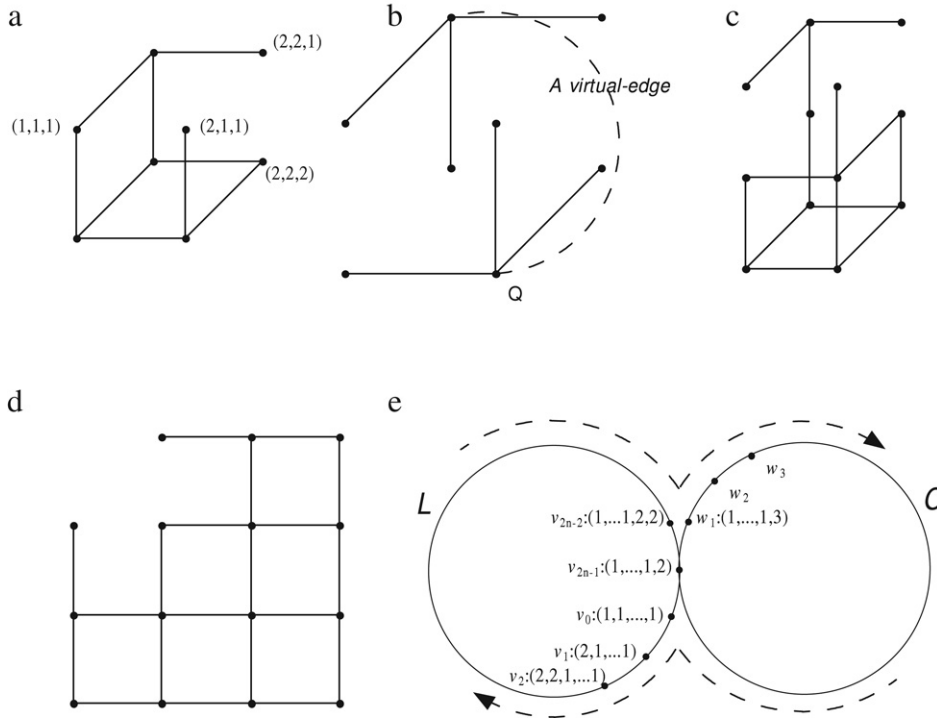
**Proof.** Let  $G$  be the Cartesian product of  $n$  paths,  $G = P_1[m_1 + 1] \times P_2[m_2 + 1] \times \cdots \times P_n[m_n + 1]$ ,  $n \geq 2$ .  $P_i[2]$  denotes a unit path of  $P_i$  which is formed by the first and the second vertices of  $P_i$  and the edge between them (A unit path is a path that contains only one edge). Let  $H^k = P_1[2] \times P_2[2] \times \cdots \times P_k[2]$ ,  $3 \leq k \leq n$ . Vertices in  $H^k$  are represented by  $k$ -dimension coordinates  $(x_1, x_2, \dots, x_k)$ . Notice that the  $k$ -dimension path  $T_1^k$ :

$$(1, 1, 1, \dots, 1)(2, 1, 1, \dots, 1)(2, 2, 1, \dots, 1) \cdots (2, 2, 2, \dots, 2)$$

is a trail in  $H^k$ . In case of  $k \geq 3$ , we first prove by induction that when  $T_1^k$  is removed from  $H^k$ ,  $H^k$  remains connected.

Case 1:  $k = 3$ , the conclusion holds for the unit cube  $H^3$ , which is shown in Fig. 1.

Case 2:  $k \geq 4$ , we assume that the conclusion holds for  $H^{k-1}$ . Let  $H_i^k$  ( $i = 1, 2$ ) be the subgraph of  $H^k$  whose last coordinate is  $i$ , which means that we let  $V(H_i^k)$  be the set of vertices in  $H^k$  whose last coordinates are equal to  $i$ , and let  $E(H_i^k)$  be the set of edges in  $H^k$  whose two endpoints are both in  $V(H_i^k)$ , thus  $H_i^k$  is formed by  $(V(H_i^k), E(H_i^k))$ . Considering the values of the last coordinate of  $H_1^k$  and  $T_1^k$ , we can see that  $H_1^k$  can be reduced to  $H^{k-1}$ ; at the same time  $H_1^k \cap T_1^k$  can be reduced



**Fig. 1.** The connectedness of  $H^3$  without  $T_1^3$  is showed in (a). Several examples that verify Lemma 2 are given in (b)–(d), in which (b) shows the only case in which a virtual-edge must be added. (e) shows the way to link  $L$  and  $C$  up in the proof of Lemma 3.

to  $T_1^{k-1}$ . Thus by the inductive assumption, which states that  $H^{k-1}$  remains connected without  $T_1^{k-1}$ , we can conclude that  $H_1^k$  remains connected without  $H_1^k \cap T_1^k$ . So  $H_1^k$  is connected without  $T_1^k$ . Subgraph  $H_2^k$  is also connected without  $T_1^k$  since  $E(H_2^k) \cap E(T_1^k) = \emptyset$ . Moreover,  $H_1^k$  and  $H_2^k$  can be linked by edge  $(1, 2, 1, \dots, 1)(2, 2, 1, \dots, 1)$ , which does not appear in  $T_1^k$ . Thus  $H^k$  is connected without  $T_1^k$ .

For the same reason, trail  $T_2^k : (2, 2, 2, \dots, 2) (1, 2, 2, \dots, 2) (1, 1, 2, \dots, 2) \cdots (1, 1, 1, \dots, 1)$  is of the same nature. So we can say that  $H^{n-1}$  is connected without  $T_1^{n-1}$ , and  $H^{n-1}$  is connected without  $T_2^{n-1}$ , when  $n \geq 4$ .

Returning to the proof of Lemma 2, we can merge the two  $n$ -dimension trails  $T_1^n$  and  $T_2^n$  to create a  $2n$ -long circuit  $L$ :

$$\begin{aligned} & \underline{(1, 1, \dots, 1, 1)(2, 1, \dots, 1, 1)(2, 2, \dots, 1, 1) \cdots (2, 2, \dots, 2, 1)} \rightarrow \\ & \rightarrow \underline{(2, 2, \dots, 2)(1, 2, \dots, 2)(1, 1, \dots, 2) \cdots (1, 1, \dots, 1, 2)(1, 1, \dots, 1, 1)} \end{aligned}$$

Look at the two underlined trails in  $L$ . They can be reduced to  $T_1^{n-1}$  and  $T_2^{n-1}$  respectively, since the last coordinates within these two trails themselves are all the same. Obviously,  $L$  is a subgraph of  $G$ . Now we show that  $L$  is precisely the circuit that the proof needs.

For  $n = 2$  and  $n = 3$ , it is quite straightforward to verify that  $L$  meets the requirements, so we omit the proof here (Several examples are given in Fig. 1(b) shows the only one case when a virtual-edge must be added). For  $n \geq 4$ , we use the definition of  $H_i^k$  again, by creating two subgraphs  $H_1^n$  and  $H_2^n$  from  $H^n$ . Considering the values of the last coordinate of  $H_1^n$  and  $L$ , we can see that  $H_1^n$  can be reduced to  $H^{n-1}$ ; and  $H_1^n \cap L$ , which is precisely the first underlined trail of  $L$ , can be reduced to  $T_1^{n-1}$ . Because that  $H^{n-1}$  is connected without  $T_1^{n-1}$ , which we have stated before, we conclude that  $H_1^n$  is connected without  $H_1^n \cap L$ . Thus  $H_1^n$  is connected without  $L$ . For the same reason,  $H_2^n$  is connected without  $L$ .  $H_1^n$  and  $H_2^n$  can be linked by edge  $(1, 2, 1, 1, \dots, 1)(2, 2, 1, 1, \dots, 1)$ , which does not appear in  $L$ . Moreover, none of the edges that link these two subgraphs with other parts of  $G$  appear in  $L$ , so the connectedness of  $G$  would not change if  $L$  is removed.  $\square$

**Lemma 3.** Every Cartesian product of  $n$  ( $n \geq 2$ ) paths can be labeled by  $1, 2, 3, \dots, M$  ( $M$  is the total number of edges in the graph) so that

- (1) for any vertex  $u$  in this graph,  $s(u) \leq n(M + 2)$ ;
- (2) for any two vertices  $u$  and  $v$ , if  $\deg(u) < \deg(v)$ , then  $s(u) - \deg(u) \leq s(v) - \deg(v)$ .<sup>1</sup>

here the **vertex-sum**  $s(w)$  for a vertex  $w$  is the sum of labels of all edges incident with  $w$ .

<sup>1</sup>  $\deg(u) < \deg(v)$  and  $s(u) - \deg(u) \leq s(v) - \deg(v)$  means that  $s(u) < s(v)$ , so we can conclude  $\deg(u) < \deg(v) \Rightarrow s(u) < s(v)$  from (2), its converse-negative proposition is  $s(u) \geq s(v) \Rightarrow \deg(u) \geq \deg(v)$ , which is equivalent to  $s(u) \leq s(v) \Rightarrow \deg(u) \leq \deg(v)$ .

**Proof.** Let  $G$  be the Cartesian product of  $n$  paths. If  $G$  is a Cartesian product of several unit paths, then  $s(u) < nM$ , and the degrees of all vertices are the same. So the two requirements are obviously met.

Otherwise,  $n \geq 2$  and there is a path within  $P_1, P_2, \dots, P_n$  that contains more than one edge, which also means  $M > 2n$ . Without loss of generality, assuming that  $P_n$  is such a path, then vertex  $(1, 1, \dots, 1, 3)$  exists in  $G$ . By Lemma 2, we can remove the  $2n$ -long circuit  $L$ :

$$(1, 1, 1, \dots, 1)(2, 1, 1, \dots, 1)(2, 2, 1, \dots, 1) \cdots (2, 2, 2, \dots, 1) \rightarrow \\ \rightarrow (2, 2, 2, \dots, 2)(1, 2, 2, \dots, 2)(1, 1, 2, \dots, 2) \cdots (1, 1, 1, \dots, 2)(1, 1, 1, \dots, 1)$$

from  $G$  and then add virtual-edges between appropriate pairs of odd-degree vertices to construct a connected and all-even-degree graph  $G'$ . An Eulerian circuit  $C'$  of  $G'$  must exist.

$(1, 1, \dots, 1, 2)(1, 1, \dots, 1, 1)$  is an edge of  $L$ , and  $(1, 1, \dots, 1, 2)(1, 1, \dots, 1, 3)$  is an edge of  $C'$ . These two edges are both incident edges of  $(1, 1, \dots, 2)$ , and means that  $(1, 1, \dots, 2)$  appears in  $L$  and in  $C'$ . So we can link up the two circuits at this vertex to form a new Eulerian circuit, say  $C^*$ .  $C^*$  contains all original edges in  $G$  and all virtual-edges we added. We represent  $C^*$  by  $v_{2n-1}v_0v_1v_2 \cdots v_{2n-1}w_1w_2 \cdots v_{2n-1}$ , in which  $v_{2n-1}v_0v_1v_2 \cdots v_{2n-1}$  is the segment that comes from  $L$ , and  $v_{2n-1}w_1w_2 \cdots v_{2n-1}$  is the segment that comes from  $C'$ ,  $v_{2n-1}$  is the link vertex. More precisely,  $v_{2n-1}$  corresponds to  $(1, 1, \dots, 2)$ ,  $v_0$  corresponds to  $(1, 1, \dots, 1)$ , and  $v_1$  corresponds to  $(2, 1, \dots, 1)$ , etc. The structure of  $C^*$  is shown in detail in Fig. 1. Notice that  $v_0$  holds degree  $n$ , which is the smallest degree among all possible degrees.

Now we give labels to circuit  $C^*$ : We label the original edges of  $G$  by  $1, 2, 3, \dots, M$  and label all of the virtual-edges by  $0$ . Begin with  $1$  at edge  $v_0v_1$  and followed by  $M, 2, M - 1, 3, \dots, \lceil \frac{M}{2} \rceil + 1$  around the circuit  $C^*$ :  $v_1v_2, v_2v_3 \cdots$ . The incident edges of each vertices of  $G'$  can be paired so that each pair consists of two adjacent edges in  $C^*$ . According to this labeling, if the two edges in a pair are both original edges of  $G$ , then the sum of their labels is  $M + 1$  or  $M + 2$  (however,  $v_0$  is an exception, and this will be discussed at the end of this proof). If there are virtual-edges within the pair, in which case the pair is called to be a *virtual-pair*, then the label of another original edge in this pair can only be numbers between  $n$  and  $M - n + 1$ . This is because that numbers between  $1$  and  $n - 1$  and numbers between  $M - n + 2$  and  $M$  are all labeled at the edges in trail  $v_0v_1v_2 \cdots v_{2n-2}$ , and these edges cannot be adjacent to a virtual-edge in  $C^*$ .

When virtual-edges are removed, the vertex-sums of vertices would not change since virtual-edges are all labeled with  $0$ . However, vertex-sums of vertices rely heavily on their original degrees in  $G$ , because it is the degree of a vertex that determines the number of edge-pairs incident to this vertex, and also determines whether there is an incident virtual-pair. In conclusion, we summarize the range of vertex-sums for all possible degrees as follow,

$$s(u) \in \begin{cases} \left[ \frac{\deg(u)}{2}(M + 1), \frac{\deg(u)}{2}(M + 2) \right] & \deg(u) \text{ is even} \\ \left[ \frac{\deg(u) - 1}{2}(M + 1) + n, \frac{\deg(u) - 1}{2}(M + 2) + M - n + 1 \right] & \deg(u) \text{ is odd} \end{cases}$$

Thus  $s(u) \leq n(M + 2)$  can be concluded directly from above, since  $\deg(u) \leq 2n$ .

The rest thing is to prove (2):  $\deg(u) < \deg(v) \Rightarrow s(u) - \deg(u) \leq s(v) - \deg(v)$ . Notice that  $\frac{\deg(u)}{2}(M + 1) > \frac{(\deg(u)-1)-1}{2}(M + 2) + M - n + 1$  and  $\frac{\deg(u)-1}{2}(M + 1) + n > \frac{(\deg(u)-1)}{2}(M + 2)$  always hold, since  $\deg(u) \leq 2n$ . Thus the range of vertex-sums of different degrees has no intersection. It means that if  $\deg(u) < \deg(v)$ , then  $s(u) \leq s(v) - 1$ . According to the nature of graph  $G$ , we have  $\deg(1, 1, 1, \dots, 1) \leq \deg(2, 1, 1, \dots, 1) \leq \deg(2, 2, 1, \dots, 1) \leq \cdots \leq \deg(2, 2, 2, \dots, 2)$ . The difference between every two adjacent degrees in this inequality sequence is, at most,  $1$ ;  $\deg(1, 1, 1, \dots, 1)$  and  $\deg(2, 2, 2, \dots, 2)$  are the smallest degree and the largest degree among all possible degrees, respectively. This means that the degrees of  $G$  have covered every integer between the smallest degree of  $G$  and the largest degree of  $G$ . If  $\deg(u) < \deg(v)$ , we can say that, for every integer  $k$  between  $\deg(u)$  and  $\deg(v)$ , there exists a vertex  $x_k$  such that  $\deg(x_k) = k$ . Therefore,

$$\deg(u) < \deg(x_{\deg(u)+1}) < \deg(x_{\deg(u)+2}) < \cdots < \deg(x_{\deg(v)-1}) < \deg(v).$$

By this inequality and the result shown above, we have  $s(u) \leq s(x_{\deg(u)+1}) - 1 \leq s(x_{\deg(u)+2}) - 2 \leq \cdots \leq s(v) - \deg(v) + \deg(u)$ . So we have proved that  $s(u) - \deg(u) \leq s(v) - \deg(v)$ .

However, there is an exception  $v_0$  whose vertex-sum may not fall into the corresponding range, because the pair of original edges  $v_{2n-1}v_0$  and  $v_0v_1$  is the only exceptional pair whose label sum is  $\lceil \frac{M}{2} \rceil + 2$ , which is smaller than  $M + 1$ . But it doesn't matter because  $v_0$  has the smallest degree  $n$  and its vertex-sum is also smaller than normal. Thus, it meets the two requirements as well.  $\square$

### 3. Proof of Theorem 1

Let the graph be  $P_0[m_0 + 1] \times P_1[m_1 + 1] \times \cdots \times P_n[m_n + 1]$ , the case when  $n \leq 1$  is already solved before [5, 2], so we assume  $n \geq 2$  here. Let  $P_0$  be the longest path and contains at least three edges, i.e.  $m_0 \geq 3$ . Let graph  $G = P_1[m_1 + 1] \times \cdots \times P_n[m_n + 1]$ ,  $M$  and  $N$  denote the total number of edges and vertices in  $G$  respectively. Thus we need to prove that  $P_0 \times G$  is antimagic. Our general approach is to construct labeling templates for  $P_0$  and  $G$ , then copy and adjust them to label the Cartesian product of these two graphs.

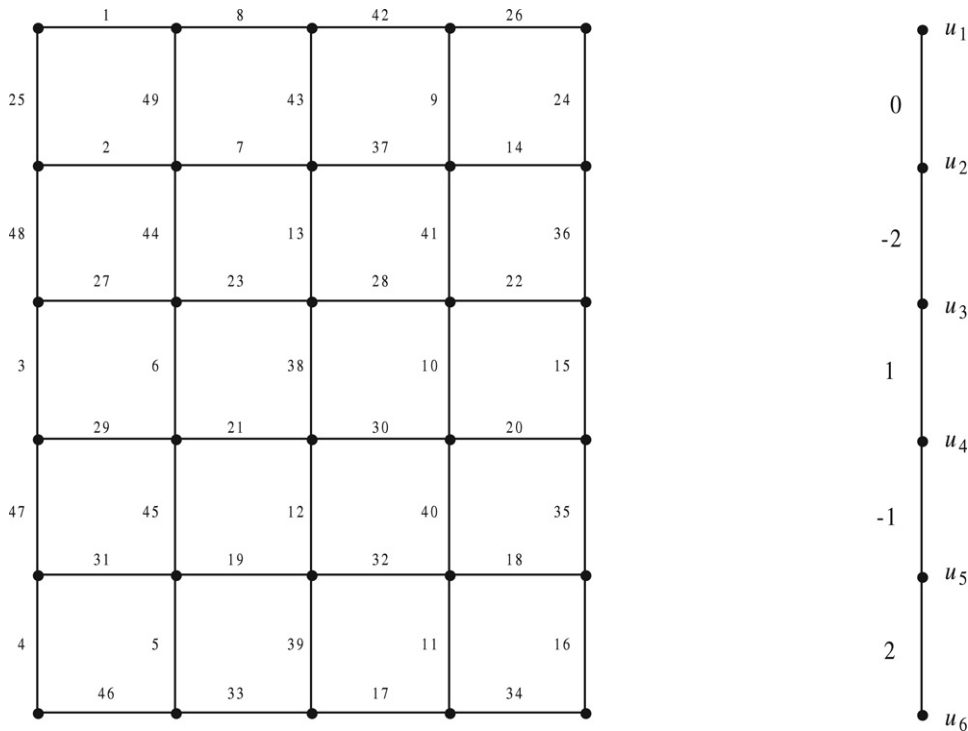


Fig. 2. Labeling templates for  $G = P_1[5] \times P_2[6]$  and  $P_0 = P_0[6]$ .

Colorings

We color  $G$  with red and blue and let  $G$  be 1-balance-colored. This is feasible by Corollary 1. Let  $C(v) = R(v) - B(v)$ , where  $v \in V(G)$ .  $G$  is 1-balance-colored which means that there are three possible values of  $C(v)$ :  $-1, 0$  and  $1$ .

Labeling templates

We construct labeling templates for  $G$  and  $P_0$  respectively. The template of  $G$  is a function  $f : E(G) \rightarrow \{1, 2, \dots, M\}$ . It is constructed by the methods of Lemma 3. Let us denote the vertices sums of  $G$  under function  $f$  by  $w(v) (\forall v \in V(G))$ . As a result of Lemma 3, we have

1.  $w(u) \leq n(M + 2)$ .
2. If  $\deg(u) < \deg(v)$ , then  $w(u) - \deg(u) \leq w(v) - \deg(v)$ .

The labeling template of  $P_0$  is defined by function  $h$ . Let  $k = \lfloor \frac{m_0+1}{2} \rfloor$ ,  $u_i$  denotes the  $i$ -th vertex of  $P_0$ , then

$$h(u_i u_{i+1}) = \begin{cases} k + \frac{i+1}{2} - \frac{m_0+1}{2}, & i \text{ is odd;} \\ \frac{i}{2} - \frac{m_0+1}{2}, & i \text{ is even.} \end{cases}$$

The following proposition can be derived directly from the definition above.

- Proposition 1.1** Function  $h$  is a bijection from  $E(P_0)$  to the values  $\{1 - \frac{m_0+1}{2}, 2 - \frac{m_0+1}{2}, \dots, \frac{m_0+1}{2} - 1\}$ .
- 1.2 For  $2 \leq i \leq m_0$ ,  $h(u_{i-1}u_i) + h(u_i u_{i+1}) = k + i - (m_0 + 1)$ . It is an monotone increasing function of  $i$ , whose values belong to  $[k - m_0 + 1, k - 1]$ .
  - 1.3  $h(u_1 u_2) - h(u_{m_0} u_{m_0+1}) = \begin{cases} -k, & m_0 \text{ is odd;} \\ 1, & m_0 \text{ is even.} \end{cases}$

Examples of labeling templates for  $G$  and  $P_0$  are given in Fig. 2.

Distribution function

Distribution function  $g(i)$  is defined to adjust the template of  $G$  in the labeling process.  $g(1)$  is defined to be  $(-1)^{m_0} \frac{1}{2}$  and  $g(m_0 + 1) = -g(1)$ . For  $2 \leq i \leq m_0$ ,  $g(i) = i - 1 - \frac{m_0}{2}$ . The following proposition can be deduced directly from the definition.

**Proposition 2. 2.1** For  $2 \leq i \leq m_0$ , function  $g$  is a bijection from integers  $\{2, 3, \dots, m_0\}$  to the values  $\{1 - \frac{m_0}{2}, 2 - \frac{m_0}{2}, \dots, \frac{m_0}{2} - 1\}$ .

2.2 For  $1 \leq i \leq m_0 + 1$ ,  $g(i) = -g(m_0 + 2 - i)$ .

2.3 For  $2 \leq i \leq m_0$ ,  $h(u_{i-1}u_i) + h(u_iu_{i+1}) + g(i) = k + 2i - \frac{3}{2}m_0 - 2$ . It is an monotone increasing function of  $i$ , whose values belong to  $[k - \frac{3}{2}m_0 + 2, k + \frac{1}{2}m_0 - 2]$ .

2.4  $h(u_1u_2) + g(1) \neq h(u_{m_0}u_{m_0+1}) + g(m_0 + 1)$ .

*Labeling rules*

Without loss of generality, we can rename the vertices of  $G$  as  $v_0, v_1, \dots, v_{N-1}$  such that  $w(v_0) \leq w(v_1) \leq \dots \leq w(v_{N-1})$ . On the other hand, vertices of  $P_0$  is represented by  $u_1, u_2, \dots, u_{m_0+1}$  as above, where  $u_i$  denotes the  $i$ -th vertex of  $P_0$ . So  $(u_i, v_j)$  is a vertex of  $P_0 \times G$ , corresponding to two vertices based on  $P_0$  and  $G$ , respectively. The edges of  $P_0 \times G$  can be partitioned into two groups: one group is copied from  $P_0$  and another is copied from of  $G$ . Our labeling rules are also divided into these two cases, as follows.

We assign  $1, 2, 3, \dots, (m_0 + 1)M + m_0N$  to elements of  $E(P_0 \times G)$ :

1. For  $u_iu_{i+1} \in E(P_0)$  and  $v_j \in V(G)$ ,  $(u_i, v_j)(u_{i+1}, v_j)$  is labeled with

$$\begin{cases} (m_0 + 1)M + \frac{m_0 + 1}{2} + m_0j + h(u_iu_{i+1}), & \text{if } C(v_j) \geq 0; \\ (m_0 + 1)M + \frac{m_0 + 1}{2} + m_0j + h(u_{m_0+2-i}u_{m_0+1-i}), & \text{if } C(v_j) < 0. \end{cases}$$

For a fixed  $j$ , both  $\frac{m_0+1}{2} + h(u_iu_{i+1})$  and  $\frac{m_0+1}{2} + h(u_{m_0+2-i}u_{m_0+1-i})$  are bijections from  $\{1, 2, \dots, m_0\}$  to  $\{1, 2, \dots, m_0\}$ . This can be learned from Proposition 1.1. So the conclusion is that, when  $j$  runs over every integer from 0 to  $N - 1$ , every integer in  $[(m_0 + 1)M + 1, (m_0 + 1) + m_0N]$  is used once and only once in the labeling process.

2. For  $u_i \in V(P_0)$  and  $v_jv_k \in E(G)$ ,  $(u_i, v_j)(u_i, v_k)$  is labeled with

$$\begin{cases} 2M - \frac{m_0}{2} + 1 + (m_0 - 1)f(v_jv_k) + g(i), & \text{if } v_jv_k \text{ is red;} \\ 2M - \frac{m_0}{2} + 1 + (m_0 - 1)f(v_jv_k) + g(m_0 + 2 - i), & \text{if } v_jv_k \text{ is blue} \end{cases}$$

when  $2 \leq i \leq m_0$ . For fixed  $(j, k)$ , we know from Proposition 2.1 that both  $-\frac{m_0}{2} + 1 + g(i)$  and  $-\frac{m_0}{2} + 1 + g(m_0 + 2 - i)$  are bijections from  $\{2, \dots, m_0\}$  to  $\{-m_0 + 2, -m_0 + 1, \dots, 0\}$ . Hence, when  $f(v_jv_k)$  runs over all its value range, every integer in  $[2M + 1, (m_0 + 1)M]$  is used once and only once in the labeling process for  $2 \leq i \leq m_0$ .

When  $i = 1$  or  $i = m_0 + 1$ ,  $(u_i, v_j)(u_i, v_k)$  is labeled with

$$\begin{cases} -\frac{1}{2} + 2f(v_jv_k) + g(i), & \text{if } v_jv_k \text{ is red;} \\ -\frac{1}{2} + 2f(v_jv_k) + g(m_0 + 2 - i), & \text{if } v_jv_k \text{ is blue.} \end{cases}$$

We know from the definition of  $g$  that  $\{g(i) - \frac{1}{2}, g(m_0 + 2 - i) - \frac{1}{2}\} = \{-1, 0\}$  for  $i = 1$  and  $i = m_0 + 1$ .  $f(v_jv_k)$  is a bijection from  $E(G)$  to  $\{1, 2, \dots, M\}$ . So it is obvious to see that every integer in  $[1, 2M]$  is used once and only once in the labeling process for  $i = 1$  and  $i = m_0 + 1$ .

It has been verified, above, that our labels precisely cover every integer between 1 and  $(m_0 + 1)M + m_0N$ , and we never assign the same value to different edges.

*Analysis*

We show two corollaries derived from the labeling rules.

**Corollary 2.** For  $v_j, v_k \in V(G)$ , if  $j < k$ , then

$$\begin{aligned} \sum_{x \in \Gamma(v_j)} \left( \left( 2M - \frac{m_0}{2} + 1 \right) + (m_0 - 1)f(v_jx) \right) &\leq \sum_{x \in \Gamma(v_k)} \left( \left( 2M - \frac{m_0}{2} + 1 \right) + (m_0 - 1)f(v_kx) \right), \\ \sum_{x \in \Gamma(v_j)} \left( -\frac{1}{2} + 2f(v_jx) \right) &\leq \sum_{x \in \Gamma(v_k)} \left( -\frac{1}{2} + 2f(v_kx) \right), \end{aligned}$$

where  $\Gamma(v)$  is the set of vertices that are adjacent to  $v$ .

**Proof.** Recalling the structure of graph  $G$ , we have

$$\sum_{x \in \Gamma(v_j)} (a + bf(v_jx)) = bw(v_j) + a \deg(v_j),$$

$$\sum_{x \in \Gamma(v_k)} (a + bf(v_kx)) = bw(v_k) + a \deg(v_k),$$

for any  $a, b, j < k$  implies  $w(v_j) \leq w(v_k)$ , which also means  $\deg(v_j) \leq \deg(v_k)$  (this appears as a note for Lemma 3). By the labeling template of  $G$ ,  $w(v_j) - \deg(v_j) \leq w(v_k) - \deg(v_k)$  can be concluded. Thus when  $b > 0$  and  $a + b \geq 0$ ,  $bw(v_j) + a \deg(v_j) \leq b(w(v_k) + \deg(v_j) - \deg(v_k)) + a \deg(v_j) = bw(v_k) + a \deg(v_k) + (a + b)(\deg(v_j) - \deg(v_k)) \leq bw(v_k) + a \deg(v_k)$ .  $\square$

The vertex-sum of a vertex  $(u_i, v_j)$  is expressed by function  $s((u_i, v_j))$ , and  $s((u_i, v_j))$  can be represented as a sum of four independent expressions  $s((u_i, v_j)) = s_1 + s_2(j) + s_3(j) + s_4(i, j)$ , in which  $s_1$  is a constant,  $s_2, s_3$  relies only on  $j$ , and  $s_4$  relies on both  $i$  and  $j$ :

**Proposition 3.** Considering the labeling rules and the definition of vertex-sums altogether, we can write the particular expressions of  $s_1, s_2, s_3$  and  $s_4$ . For  $2 \leq i \leq m_0$ ,

$$s_1 = (m_0 + 1)(2M + 1),$$

$$s_2(j) = \sum_{x \in \Gamma(v_j)} \left( 2M - \frac{m_0}{2} + 1 + (m_0 - 1)f(v_jx) \right),$$

$$s_3(j) = 2m_0j,$$

$$s_4(i, j) = \begin{cases} h(u_{i-1}u_i) + h(u_iu_{i+1}) + g(i), & \text{if } C(v_j) = 1; \\ h(u_{m_0+3-i}u_{m_0+2-i}) + h(u_{m_0+2-i}u_{m_0+1-i}) + g(m_0 + 2 - i), & \text{if } C(v_j) = -1; \\ h(u_{i-1}u_i) + h(u_iu_{i+1}), & \text{if } C(v_j) = 0. \end{cases}$$

By Proposition 1 and Proposition 2, the values of  $s_4(i, j)$  are restricted in the range of  $[k - m_0 + 1, k - 1] \cup [k - \frac{3}{2}m_0 + 2, k + \frac{1}{2}m_0 - 2]$ . We also know that  $\frac{m_0-1}{2} \leq k \leq \frac{m_0}{2}$ , thus

$$-m_0 < \min \left\{ -\frac{m_0}{2} + \frac{1}{2}, -m + \frac{3}{2} \right\} \leq s_4(i, j) \leq \max \left\{ \frac{m_0}{2} - 1, m_0 - 2 \right\} < m_0 - 1.$$

So the largest difference between the values of  $s_4(i, j)$  is less than  $2m_0$ .

For  $i = 1$  or  $i = m_0 + 1$ ,

$$s_1 = (m_0 + 1) \left( M + \frac{1}{2} \right),$$

$$s_2(j) = \sum_{x \in \Gamma(v_j)} \left( -\frac{1}{2} + (m_0 - 1)f(v_jx) \right),$$

$$s_3(j) = m_0j,$$

$$s_4(i, j) = \begin{cases} h(u_iu_{i\pm 1}) + g(i \pm 1), & \text{if } C(v_j) = 1; \\ h(u_{m_0+2-i\mp 1}u_{m_0+2-i}) + g(m_0 + 2 - i), & \text{if } C(v_j) = -1; \\ h(u_iu_{i\pm 1}), & \text{if } C(v_j) = 0. \end{cases}$$

By Proposition 1 and the definition of  $g(i)$ , the largest difference between the values of  $s_4(i, j)$  is, at most,  $\max \{2, k + 1\}$ , which is less than  $m_0$  when  $m_0 \geq 3$ .

Prove the antimagicness of the labeling

Two vertices conflict if they have the a same vertex-sum. We partition the vertices  $(u_i, v_j)$  into two groups: one group consists of vertices with  $2 \leq i \leq m_0$  and another group consists of vertices with  $i = 1$  or  $m_0 + 1$ . First we prove that there is no conflict within these two groups themselves. Then we show that all vertex-sums of the first group are larger than every vertex-sum of the second group.

Given two arbitrary different vertices of  $P_0 \times G$  belonging to the same group:  $(u_i, v_j)$  and  $(u_{i'}, v_{j'})$ , suppose  $j \leq j'$ . Let us compare  $s_1, s_2, s_3$  and  $s_4$  respectively.  $s_1$  is a constant for vertices in the same group. If  $j = j'$ , then  $s_2(j) = s_2(j')$  and  $s_3(j) = s_3(j')$ , while  $s_4(i, j) \neq s_4(i', j')$ , by Proposition 1.2 and Proposition 2.3, since  $i \neq i'$ . Thus, the two vertices have different vertex-sums. If  $j < j'$ , then  $s_2(j) \leq s_2(j')$  (by Corollary 2). Notice that the smallest possible difference between  $s_3(j)$  and  $s_3(j')$  is larger than the largest possible difference between  $s_4(i, j)$  and  $s_4(i', j')$  (proved in Proposition 3). Thus, the vertex-sum of  $(u_i, v_j)$  is less than that of  $(u_{i'}, v_{j'})$ , and means that they have no conflict.



Consider the smallest possible vertex-sum of  $(u_i, v_j)$  for  $2 \leq i \leq m_0$  and arbitrary  $j$ .  $s(u_i, v_j)$  can be partitioned into two parts: one is the partial sum of edges copied from  $P_0$ , which is at least  $\sum_{i=1}^2 ((m_0 + 1)M + i)$ , another is the partial sum of edges copied from  $G$ , which is at least  $\sum_{i=1}^n (2M + i)$ , thus

$$s((u_i, v_j)) \geq (2m_0 + 2n + 2)M + \frac{n(n+1)}{2} + 3.$$

The largest possible vertex-sum of  $(u_{i'}, v_{j'})$  when  $i' = 1$  or  $m_0 + 1$  can be estimated in the same way. The labeling sum of edges copied from  $P_0$  cannot be larger than the total number of edges, which is  $(m_0 + 1)M + m_0N$ . By labeling rules, the labeling sum of edges copied from  $G$  cannot be larger than  $2w(v_{j'})$ , which is, at most,  $2n(M + 2)$ . Thus

$$s((u_{i'}, v_{j'})) \leq (m_0 + 2n + 1)M + m_0N + 4n.$$

So we have

$$\begin{aligned} s((u_i, v_j)) - s((u_{i'}, v_{j'})) &\geq (2m_0 + 2n + 2)M + \frac{n(n+1)}{2} + 3 - ((m_0 + 2n + 1)M + m_0N + 4n) \\ &\geq (m_0 + 1)M - m_0N + \frac{n(n+1)}{2} - 4n + 3 > 0. \end{aligned}$$

The last statement is always true when  $n \geq 2$  (thus  $M \geq N \geq 4$ ). So there is no conflict between the two groups of vertices.  $\square$

#### 4. Proof of Theorem 2

We assume that  $R$  is a  $k$ -regular antimagic graph ( $k \geq 2$ ).  $T$  and  $N$  denotes the total number of vertices in  $R$  and  $G$  respectively, while the total number of edges in  $G$  is denoted by  $M$ .

The strategies used to prove the antimagicness of  $R \times G$  are quite similar to those of Section 3. We also construct colorings and templates, and then copy and adjust them to label the product graph. Fortunately, the construction process in this section can be much more concise than Section 3.

##### Colorings

We color  $G$  with red and blue and let  $G$  be 2-balance-colored. It is feasible by Lemma 1. Let  $C(v) = R(v) - B(v)$ , where  $v \in V(G)$ .  $G$  is 2-balance-colored means that  $-2 \leq C(v) \leq 2$ .

##### Labeling templates

We assign the antimagic labeling  $h : E(R) \rightarrow \{1, 2, \dots, \frac{kT}{2}\}$  to  $R$ . The vertex-sum of vertex  $v \in V(R)$  is expressed by function  $w_1(v)$ .

For  $G$ , we have  $M \geq N - 1$  since  $G$  is connected. Assign an arbitrary labeling  $f : E(G) \rightarrow \{(\frac{k}{2} + 1)Ti - a | i = 1, 2, \dots, N - 1\} \cup \{Tj + \frac{kT}{2}N - a | j = N, N + 1, \dots, M\}$  to  $G$ , where  $a = \frac{T-1}{2}$ . The vertex sum of vertex  $v \in V(G)$  is expressed by function  $w_2(v)$ .

##### Labeling rules

First, we rename the vertices of  $R$  as  $u_0, u_1, \dots, u_{T-1}$  such that  $w_1(u_0) < w_1(u_1) < \dots < w_1(u_{T-1})$ , the strict inequality relies on the antimagicness of  $R$ . We also rename the vertices of  $G$  as  $v_0, v_1, \dots, v_{N-1}$  such that  $w_2(v_0) \leq w_2(v_1) \leq \dots \leq w_2(v_{N-1})$ . Vertex of  $R \times G$  is represented by  $(u_i, v_j)$  where  $u_i \in V(R)$  and  $v_j \in V(G)$ . The labeling rules for  $R \times G$  are given as follow.

We assign  $1, 2, 3, \dots, TM + \frac{kT}{2}N$  to elements of  $E(R \times G)$ .

1. For  $u_i u_{i'} \in E(R)$  and  $v_j \in V(G)$ ,  $(u_i, v_j)(u_{i'}, v_j)$  is labeled with

$$\begin{cases} \left(\frac{k}{2} + 1\right) Tj + h(u_i u_{i'}), & \text{if } C(A) \geq 0; \\ \left(\frac{k}{2} + 1\right) Tj + \left(\frac{kT}{2} + 1 - h(u_i u_{i'})\right), & \text{if } C(A) < 0. \end{cases}$$



2. For  $u_i \in V(R)$  and  $v_j v_{j'} \in E(G)$ ,  $(u_i, v_j)(u_i, v_{j'})$  is labeled with

$$\begin{cases} f(v_j v_{j'}) + (i - a), & \text{if } v_j v_{j'} \text{ is red;} \\ f(v_j v_{j'}) + (a - i), & \text{if } v_j v_{j'} \text{ is blue.} \end{cases}$$

It is quite easy to verify that our labels have precisely covered every integer between 1 and  $TM + \frac{kT}{2}N$ , and we never assign the same value to different edges. The details of the verification are omitted, considering its similarity with Section 3.

*Prove the antimagicness of the labeling*

For  $R \times G$ , the vertex sum  $s((u_i, v_j))$  can also be represented as a sum of four independent expressions  $s((u_i, v_j)) = s_1 + s_2(j) + s_3(j) + s_4(i, j)$ , in which

$$\begin{aligned} s_1 &= \frac{k}{2} \left( \frac{kT}{2} + 1 \right) \\ s_2(j) &= w_2(v_j) \\ s_3(j) &= \left( \frac{k}{2} + 1 \right) kTj \\ s_4(i, j) &= \begin{cases} \frac{C(v_j)}{|C(v_j)|} \left( w_1(u_i) - \frac{k}{2} \left( \frac{kT}{2} + 1 \right) \right) + C(v_j)(i - a) & C(v_j) \neq 0 \\ w_1(u_i) - \frac{k}{2} \left( \frac{kT}{2} + 1 \right) & C(v_j) = 0. \end{cases} \end{aligned}$$

To estimate the largest possible difference between the values of  $s_4(i, j)$ , we have

$$|s_4(i, j)| \leq \left| w_1(u_i) - \frac{k}{2} \left( \frac{kT}{2} + 1 \right) \right| + 2|i - a| \leq \left( \frac{k^2 T}{4} - \frac{k^2}{2} \right) + (T - 1) < \left( \frac{k^2}{4} + 1 \right) T.$$

Thus the largest possible difference cannot be larger than  $2|s_4(i, j)| < \left( \frac{k^2}{2} + 2 \right) T \leq \left( \frac{k}{2} + 1 \right) kT$ . The last term is the smallest possible difference between the values of the  $s_3(j)$  for different  $j$ , in case of  $k \geq 2$ .

Given two arbitrary different vertices of  $R \times G$ ,  $(u_i, v_j)$  and  $(u_{i'}, v_{j'})$ , suppose  $j \leq j'$ . We compare  $s_1, s_2, s_3$  and  $s_4$  respectively.  $s_1$  is a constant. If  $j = j'$ , then  $s_2(j) = s_2(j')$  and  $s_3(j) = s_3(j')$ , but  $s_4(i, j) \neq s_4(i', j')$ . This is because  $i \neq i'$ , and both  $\frac{C(v_j)}{|C(v_j)|} \left( w_1(u_i) - \frac{k}{2} \left( \frac{kT}{2} + 1 \right) \right) + C(v_j)(i - a)$  and  $w_1(u_i) - \frac{k}{2} \left( \frac{kT}{2} + 1 \right)$  are monotonic functions of  $i$ . Thus, the two vertex sums cannot be the same. If  $j < j'$ , then  $s_2(j) \leq s_2(j')$ . Notice that the smallest possible difference between  $s_3(j)$  and  $s_3(j')$  is larger than the largest possible difference between  $s_4(i, j)$  and  $s_4(i', j')$  (proved above). Thus the vertex sum of  $(u_i, v_j)$  is less than that of  $(u_{i'}, v_{j'})$ , which means that they have no conflict.  $\square$

### 5. Open problems

The challenge still remains to prove the antimagicness of some other cases of Cartesian products of graphs. For example, Cartesian products of 2-edge paths cannot be solved by the methods in this paper. It would also be interesting to find a general approach to deal with the Cartesian products of two arbitrary antimagic graphs or, even further, the Cartesian products of two arbitrary connected graphs. On the other hand, trying to prove the antimagicness of all regular graphs may also be interesting.

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