On Incentive Compatible Competitive Selection Protocol
(Extended Abstract)

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Abstract. The selection problem of m highest ranked out of n candidates is considered for a model while the relative ranking of two candidates is obtained through their pairwise comparison. Deviating from the standard model, it is assumed in this article that the outcome of a pairwise comparison may be manipulated by the two participants. The higher ranked party may intentionally lose to the lower ranked party in order to gain group benefit. We discuss incentive compatible mechanism design issues for such scenarios and develop both possibility and impossibility results.

1 Introduction

Ensuring truthful evaluation of alternatives in human activities has always been an important issue throughout the history. In sport, in particular, such an issue is vital and the practice of the fair play principle has been consistently put forth at the foremost priority. In addition to reliance on the code of ethics and professional responsibility of players and coaches, the design of game rules is an important measure to make fair play enforced. The problem of tournament design consists of issues such as ranking, round-robin scheduling, timetabling, home-away assignment, etc. Ranking alternatives through pairwise comparisons is the most common approach in sports tournaments. Its goal is to find out the 'true' ordering among alternatives through complete or partial pairwise comparisons, and it has been widely studied in the decision theory.

In [4], Harary and Moser gave an extensive review of the properties of round-robin tournaments, and introduced the concept of 'consistency'. In [7], Rubinstein proved that counting the number of winning matches is a good scheme to rank among alternatives in round-robin tournaments; it is also the only scheme that satisfies all the nice rationality properties of ranking. Jech [5] proposed a ranking procedure for incomplete tournaments, which mainly depended on transitivity. He proved that if all players are comparable, i.e. there exists a beating

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chain between each pair of players, then the ranking of players under a specific scheme uniquely exists. Chang et al. [1] investigated the ability of methods in revealing the true ranking in multiple incomplete round-robin tournaments. Works have also been done on evaluating the efficiency and efficacy of ranking methods. Steinhaus [8] proposed an upper bound for the number of matches required to reveal the overall ranking of all players. Mendonca et al. [6] developed a methodology for comparing the efficacy of ranking methods, and investigated their abilities of revealing the true ranking.

Such studies have been mainly based on the assumption that all the players play truthfully, i.e. with their maximal effort. It is, however, possible that some players cheat and seek for group benefit. For example, in the problem of choosing m winners out of n candidates, if the number of winning matches is the only parameter considered in selecting winners, some top players could intentionally lose some matches when confronting their 'friends', so the friends could earn a better ranking while the top players remain highly ranked. Such problems will be the focus of our study: Is there an ideal protocol which allows no cheating strategy under any circumstances, even when a majority of players, possibly many with high ranks, form a coalition to help lower ranked players in it?

The problem, that is, choosing m winners out of n players, is studied under two models. Under both models, a coalition will try to have more of its members be selected as winners than that under the true ranking. For the *collective incentive compatible model*, its only goal is to have more members be selected as winners, even by sacrificing some highly ranked players who ought to be winners. For the *alliance incentive compatible* model, it succeeds not only by having more winners, but also by ensuring the ones who ought to win remain winners, i.e. no players sacrifice their winning positions in order to bring in extra winners. Under both models, our objective is to find an incentive compatible protocol if it exists, or to prove the non-existence of such protocols.

We will formally introduce the models, notations and definitions in Section 2. In Section 3, we discuss the collective incentive compatible model and prove the non-existence of incentive compatible protocols under it. In Section 4, we present an incentive compatible selection protocol under the alliance incentive compatible model. Finally, we conclude with remarks and open problems.

2 Issues and Definitions

Firstly, we describe a protocol which is widely used in bridge tournaments, the *Swiss Team* Protocol. Using it as an example, we show collaboration is possible to improve the outcome of a subgroup of players, if the protocol is not properly designed.

2.1 Existence of Cheating Strategy Under the Swiss Team Protocol

The Swiss Team protocol chooses two winners out of four players. Let the four players $P_4 = \{p_1, p_2, p_3, p_4\}$ play according to the following arrangements. After all the three rounds, two of them will be selected as winners.

- Assign a distinct ID in $N_4 = \{1, 2, 3, 4\}$ to each player in P_4 by a randomly selected indexing function.
- In **round** 1, player (with ID) 1 vs. player 2, and player 3 vs. player 4.
- In **round** 2, two winners of the first round play against each other, and so as the two losers. The player continuously wins twice will be selected as the first winner of the whole game; the player continuously loses twice will be out. Therefore, there are only two players left.
- In **round** 3, the two remaining players play against each other. The winner will be selected as the second winner of the whole game.

Suppose the true capacity of the four players in P_4 is $p_1 > p_2 > p_3 > p_4$ and we consider the case in which p_1 and p_3 form a group. Their purpose is to get both winning positions by applying a cheating strategy, while the winners should be p_1 and p_2 according to the true ranking. Under the settings of the Swiss Team Protocol described above, the probability of this group $\{p_1, p_3\}$ having effective cheating strategies is non-negligible. Following is their strategy.

- Luckily, the IDs assigned to p_1, p_2, p_3 and p_4 are $1, 2, 3$ and 4 respectively.
- In **round** 1, p_1 plays against p_2 and p_3 plays against p_4. p_1 and p_3 win.
- In **round** 2, p_1 plays against p_3 and p_2 plays against p_4. In order to let p_3 be one of the winners, p_1 loses the match to p_3 intentionally. p_3 will then be selected as the first winner for winning twice. In the other match, both p_2 and p_4 play truthfully and p_2 wins.
- In **round** 3, p_1 and p_2 play against each other, and p_1 wins. Therefore, p_1 is selected as the second winner.

By applying the cheating strategy above, the group of bad players $\{p_1, p_3\}$ can break the Swiss Team protocol by letting p_1 confront p_2 twice, and earn an extra winning position.

2.2 Problem Description

Suppose a tournament is held among n players $P_n = \{p_1...p_n\}$ and m winners are expected to be selected by a selection protocol. Here a protocol $f_{n,m}$ is a predefined function to choose winners through pairwise competitions, with the intention of finding m players of highest capacity. When the tournament starts, a distinct ID in $N_n = \{1...n\}$ is assigned to each player in P_n by a randomly picked indexing function I. Then a match is played between each pair of players. The competition outcomes will form a tournament graph [2], whose vertex set is N_n and edges represent results of all the matches. Finally, the graph will be treated as input to $f_{n,m}$, and it will output a set of m winners.

Assume there exists a group of bad players play dishonestly, i.e. they might lose a match on purpose to gain overall benefit of the whole group, while all the other players always play truthfully, i.e. they try their best to win matches. We say that the group of bad players gains benefit if they are able to have more winning positions than that according to the true ranking. Given knowledge of the selection protocol $f_{n,m}$, the indexing function I and the true ranking of all players, the group of bad players tries to find a cheating strategy that can fool the selection protocol and gains benefit.

The problem is considered under two models in which the characterizations of bad players are different. Under the *collective incentive compatible model*, bad players are willing to sacrifice themselves to win group benefit; while the ones under the *alliance incentive compatible model* only cooperate if their individual interests are well maintained in the cheating strategy.

Our goal is to find an incentive compatible selection protocol, under which players or group of players maximize their benefits only by strictly following the fair play principle, i.e. always play with maximal effort. Otherwise, we prove the inexistence of such protocols.

2.3 Formal Definitions

When the tournament begins, an indexing function I is randomly picked and a distinct ID $I(p) \in N_n$ is assigned to each player $p \in P_n$. Then a match is played between each pair of players, and results are represented as a directed graph G. Finally, G is feeded to the predefined selection protocol $f_{n,m}$, to produce a set of m winners $W = f_{n,m}(G) \subset N_n$.

Definition 1 (Indexing Function). *An indexing function I for a tournament attended by n players $P_n = \{p_1, p_2, ...p_n\}$ is a one-to-one correspondence from P_n to the set of* IDs*: $N_n = \{1, 2, ...n\}$.*

Definition 2. *A tournament graph of size n is is a directed graph $G = (N_n, E)$ such that, for any $i \neq j \in N_n$, either edge $ij \in E$ (player with ID i beats player with ID j) or edge $ji \in E_n$. We use K_n to denote the set of all such graphs.*

A selection protocol $f_{n,m}$ which chooses m winners out of n candidates is a function from K_n to $\{ S \subset N_n$ and $|S| = m \}$.

The group of bad players not only know the selection protocol, but also the true ranking of players. We say a bad player group gains benefit if it has more members be selected as winners than that according to the true ranking.

Definition 3 (Ranking Function). *A ranking function R of is a one-to-one correspondence from P_n to N_n. $R(p) \in N_n$ represents the underlying true ranking of player p among the n players. The smaller, the stronger.*

Definition 4 (Tournament). *A tournament T_n among n players P_n is a pair $T_n = (R, B)$, where R is a ranking function from P_n to N_n and $B \subset P_n$ is the group of bad players.*

Definition 5 (Benefit). *Given a protocol $f_{n,m}$, a tournament $T_n = (R, B)$, an indexing function I and a tournament graph $G \in K_n$, the benefit of the group of bad players is*

$$\boldsymbol{Ben}(f_{n,m}, T_n, I, G) = \Big| \big\{ i \in f_{n,m}(G),\ I^{-1}(i) \in B \big\} \Big| - \Big| \big\{ p \in B,\ R(p) \leq m \big\} \Big|.$$

Given $f_{n,m}$, T_n and I, not every graph $G \in K_n$ is a feasible strategy for the group of bad players. First, it depends on the tournament $T_n = (R, B)$, e.g. a player $p_b \in B$ cannot win player $p_g \notin B$ if $R(p_b) > R(p_g)$. Second, it depends on the property of bad players which is specified by the model considered.

We now, for each model, characterize tournament graphs which are recognized as feasible strategies. The key difference is that a bad player in alliance incentive compatible model is not willing to sacrifice his own winning position, while a player in the other model fights for group benefit at all costs.

Definition 6. *Given $f_{n,m}$, $T_n = (R, B)$ and I, a graph $G \in K_n$ is* c-feasible *if*

1. *For every two players $p_i, p_j \notin B$, if $R(p_i) < R(p_j)$, then $I(p_i)I(p_j) \in E$;*
2. *For all $p_g \notin B$ and $p_b \in B$, if $R(p_g) < R(p_b)$, then edge $I(p_g)I(p_b) \in E$.*

Graph $G \in K_n$ is a-feasible *if it is c-feasible and also satisfies*

3. *For every bad player $p \in B$, if $R(p) \leq m$, then $I(p) \in f_{n,m}(G)$.*

A cheating strategy is then a feasible tournament graph G that can be employed by the group of bad players to gain positive benefit.

Definition 7 (Cheating Strategy). *Given $f_{n,m}$, T_n and I, a cheating strategy for the group of bad players under the collective incentive compatible (alliance incentive compatible) model is a graph $G \in K_n$ which is c-feasible (a-feasible) and satisfies $\boldsymbol{Ben}(f_{n,m}, T_n, I, G) > 0$.*

We ask the following two natural questions.

> **Q$_1$:** Is there a protocol $f_{n,m}$ such that for all T_n and I, no cheating strategy exists under the collective incentive compatible model?
>
> **Q$_2$:** Is there a protocol $f_{n,m}$ such that for all T_n and I, no cheating strategy exists under the alliance incentive compatible model?

In the following sections, we will present an impossibility proof for the first question, and design an incentive compatible protocol for the second model.

3 Incentive Compatible Protocol Under the Collective Incentive Compatible Model

In this section, we prove the inexistence of incentive compatible protocol under the collective incentive compatible model. For every $f_{n,m}$, we are able to find a large number of tournaments T_n where cheating strategy exists.

Definition 8. *For all integers n and m such that $2 \leq m \leq n-2$, we define a graph $G_{n,m} = (N_n, E) \in K_n$ which consists of 3 parts, T_1, T_2 and T_3.*

1. *$T_1 = \{1, 2, ... m-2\}$. For all $i < j \in T_1$, edge $ij \in E$;*
2. *$T_2 = \{m-1, m, m+1\}$. $(m-1)m$, $m(m+1)$, $(m+1)(m-1) \in E$;*

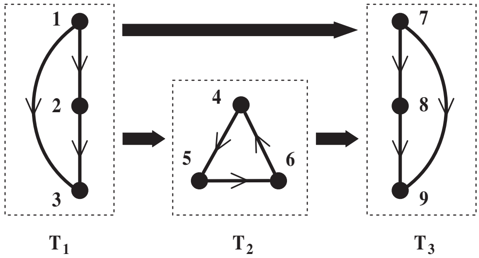

Fig. 1. Tournament Graph $G_{9,5}$

3. *$T_3 = \{m+2, m+3, ...n\}$. For all $i < j \in T_3$, edge $ij \in E$;*
4. *For all $i' \in T_i$ and $j' \in T_j$ such that $i < j$, edge $i'j' \in E$.*

Players in T_1 and T_3 are well ordered among themselves, but the ones in T_2 are not due to the existence of a cycle. All players in T_1 beat the ones in T_2 and T_3, and all players in T_2 beat the ones in T_3. Sample graph $G_{9,5}$ is shown in Figure 1. Proof of Lemma 1 can be found in the full version [3].

Lemma 1. *For every $f_{n,m}$ where $2 \le m \le n-2$, if $T_n = (R, B)$ satisfies that $B = \{p_{m-r+1}...p_{m+1}, p_{m+2}\}$ where $r \ge 2$ and $R(p_i) = i$ for all $1 \le i \le n$, then there exists an indexing function I such that $G_{n,m}$ is a cheating strategy.*

Corollary 1. *For every $f_{n,m}$ where $2 \le m \le n-2$, if $T_n = (R, B)$ satisfies that $B = R^{-1}(m-r+1...m+1, m+2)$ where $r \ge 2$, then there exists an indexing function I such that $G_{n,m}$ is a cheating strategy.*

Corollary 2 can be derived from Lemma 1 immediately. Figure 2 shows the true ranking of a tournament T_n in which a cheating strategy exists.

By Lemma 2, one can extend Corollary 2 to Theorem 1 below.

Lemma 2. *Given $f_{n,m}$ and I, if $G \in K_n$ is a cheating strategy for tournament $T_n = (R, B)$, and there exist players $p_b \in B$ and $p_g \notin B$ such that $R(p_b) = R(p_g)+1 \le m$, then graph G remains a cheating strategy of $T'_n = (R', B)$ where $R'(p_b) = R(p_g)$, $R'(p_g) = R(p_b)$ and $R'(p) = R(p)$ for every other player p.*

Theorem 1. *For every $f_{n,m}$ where $2 \le m \le n-2$, if $T_n = (R, B)$ satisfies:* **1)**. *at least one bad player ranks as high as $m-1$;* **2)**. *the ones ranked $m+1$ and $m+2$ are both bad players;* **3)**. *the one ranked m is a good player, then there always exists an indexing function I such that $G_{n,m}$ is a cheating strategy.*

Theorem 1 describes a much larger class of tournaments in which cheating strategy exists. An example of such tournaments is shown in Figure 3.

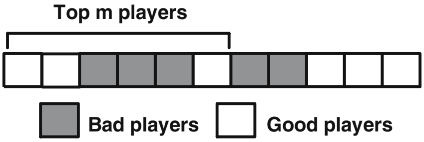

Fig. 2. An Example of Tournaments

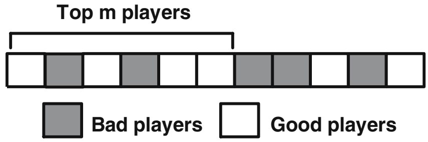

Fig. 3. An Example of Tournaments

4 Incentive Compatible Protocol Under the Alliance Incentive Compatible Model

In this section, we answer question $\mathbf{Q}_2$ for arbitrary n and m. We prove that whether a successful protocol exists is completely determined by the value of $n-m$. When $n-m \leq 2$, cheating strategies can always be constructed, and thus we prove the inexistence of ideal protocol. When $n-m \geq 3$, we present a selection protocol $f^*_{n,m}$ under which no cheating strategy exists.

4.1 Inexistence of Selection Protocol When $n - m \leq 2$

Definition 9. *We define two classes of tournament graphs, graph G^*_n for any $n \geq 3$ and graph G'_n for any $n \geq 4$. Their structures are similar to $G_{n,m}$.*

- *For G^*_n, $T_1 = \{1, 2, \ldots n-3\}$, $T_2 = \{n-2, n-1, n\}$ and $T_3 = \emptyset$ with edges $(n-2)(n-1)$, $(n-1)n$, $n(n-2) \in G^*_n$. Graph G^*_6 is shown in Figure 4.*
- *For G'_n, $T_1 = \{1, 2, \ldots n-4\}$, $T_2 = \{n-3, n-2, n-1\}$ and $T_3 = \{n\}$ with edges $(n-3)(n-2)$, $(n-2)(n-1)$, $(n-1)(n-3) \in G'_n$. Sample graph G'_7 is shown in Figure 5.*

By the following two lemmas, no ideal protocol exists when $n - m \leq 2$. The proofs can be found in the full version [3].

Lemma 3. *For every $f_{n,m}$ where $n-m=1$ and $m \geq 2$, if $T_n = (R, B)$ satisfies $B = \{p_1, p_2, \ldots p_{n-2}, p_n\}$ and $R(p_i) = i$ for all $1 \leq i \leq n$, then there exists an indexing function I such that graph G^*_n is a cheating strategy for the group of bad players under the alliance incentive compatible model.*

Lemma 4. *For every $f_{n,m}$ where $n-m=2$ and $m \geq 2$, if $T_n = (R, B)$ satisfies $B = \{p_1, p_2, \ldots p_{n-3}, p_{n-1}, p_n\}$ and $R(p_i) = i$ for all $1 \leq i \leq n$, then there exists an indexing function I such that graph G'_n is a cheating strategy for the group of bad players under the alliance incentive compatible model.*

4.2 Selection Protocol $f^*_{n,m}$ for Case $n - m \geq 3$

In this section, we'll first introduce some important properties of tournament graphs. Then a selection protocol $f^*_{n,m}$ will be described for case $n-m \geq 3$. Finally, we prove that for any tournament T_n and indexing function I, no cheating strategy exists for the group of bad players.

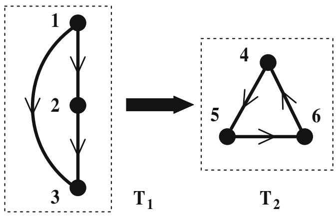

Fig. 4. Tournament graph G_6^*

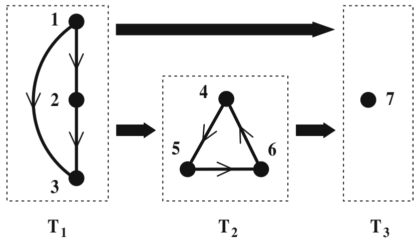

Fig. 5. Tournament graph G_7'

Definition 10. *A directed graph G is said to be strongly connected if there's a directed path between every pair of vertices. Any maximal subgraph of G that is strongly connected is called a strongly connected component of graph G.*

Let $G \in K_n$ be a tournament graph. We use $G_1 ... G_k$ to denote its strongly connected components which satisfy that for all $u \in G_i$ and $v \in G_j$ such that $i < j$, edge $uv \in G$. The proof of Lemma 5 below can be found in [2].

Definition 11. *A directed graph G of order $n \geq 3$ is pancyclic if it contains a cycle of length l for each $l = 3, 4, ... n$, and is vertex-pancyclic if each vertex v of G lies on a cycle of length l for each $l = 3, 4, ... n$.*

Lemma 5. *Every strongly connected tournament graph is vertex-pancyclic.*

Corollary 2. *Let G be a tournament graph with strongly connected components $G_1 ... G_k$. If there is no cycle of length l in G, then $|G_i| < l$ for all $1 \leq i \leq k$.*

Our protocol $f_{n,m}^*$ described in Figure 6 is an algorithm working on tournament graphs. The algorithm checks whether $3 \,|\, n - m$.

- When $n - m \equiv 1 \pmod 3$, if there exists a cycle of 4 vertices, delete all the vertices in the cycle; otherwise, delete the lowest ranked vertex in G. As a result, we have $n' - m \equiv 0 \pmod 3$ where n' is the number of remaining candidates after deletion.
- When $n - m \equiv 2 \pmod 3$, if there exists a cycle of 5 vertices in G, delete all the vertices in the cycle; otherwise, delete the two lowest ranked vertices. Similarly, it can also be reduced to the case of $n' - m \equiv 0 \pmod 3$.
- When $n - m \equiv 0 \pmod 3$, if there exist cycles of 3 vertices, continuously delete them until either **1)** no such cycle exists, then choose the m highest ranked ones as winners; or **2)** there're m vertices left, then choose all of the remaining candidates as winners.

The proof of the following theorem can be found in the full version [3].

Theorem 2. *For all T_n, I and a-feasible graph G, $\boldsymbol{Ben}(f_{n,m}, T_n, I, G) \leq 0$.*

Ensure $n - m \geq 3$ and graph $G \in K_n$
let $G_1, G_2, \ldots G_k$ be the strongly connected components of graph $G = (N_n, E)$
if $n - m \equiv 1 \pmod 3$ **then**
  if there exists a cycle C of length 4 in G **then**
    delete all the 4 vertices in C from graph G
  else
    let t be the smallest vertex (integer) in G_k, and delete vertex t from G
  endif
else if $n - m \equiv 2 \pmod 3$ **then**
  if there exists a cycle C of length 5 in G **then**
    delete all the 5 vertices in C from graph G
  else if $|G_k| = 1$
    let $t_1 \in G_k$ and t_2 be the smallest vertex (integer) in G_{k-1}, delete t_1, t_2
  else
    let t_1 and t_2 be the two smallest vertices (integers) in G_k, delete t_1, t_2
  end if
end if
while the number of vertices in G is larger than m **do**
  if there exists a cycle C of length 3 in G **then**
    delete all the 3 vertices in C from graph G
  else
    vertices can be sorted as $k_1 \ldots k_{m'}$ such that $k_i k_j \in E$, $\forall\, 1 \leq i < j \leq m'$
    output set $\{ k_1, k_2, \ldots k_m \}$ and return
  end if
end while
output all the remaining vertices in G and return

Fig. 6. Details of Selection Protocol $f^*_{n,m}$

5 Conclusion Remarks

In this article, we discussed the possibility of an incentive compatible selection protocol to exist, by which the benefits of either individual players or a group of players are maximized by playing truthfully. Under the collective incentive compatible model, our result indicates that cheating strategies are available in at least 1/8 tournaments, if we assume the probability for each player to be in the bad group is 1/2. On the other hand, we showed that there does exist an incentive compatible selection protocol under the alliance incentive compatible model, by presenting a deterministic algorithm.

Many problems remain and require further analysis. Under the first model, could the general bound of 1/8 be improved? Could we find good selection protocols in the sense that the number of tournaments with cheating strategies is close to this bound? Though we have proved the inexistence of ideal protocol under this model, does there exist any probabilistic protocol, under which the probability of having cheating strategies is negligible?

Finally, we'd like to raise the issue of output truthful mechanism design. In our model, an output truthful mechanism would output a list of k players, each of which is among the top k players in the true ranking. It would be interesting to know whether there is such a mechanism or not. For a related problem we are going to describe next, this is possible. Consider a committee of 2n+1 to select one out of candidates. The expected output is the one favored by the majority of the committee. The following protocol will return the true outcome but not everyone will vote truthfully: After the voting, a fixed amount of bonus will be distributed to the voters who voted for the winner. Using this mechanism, every committee member will vote for the candidate favored by the majority though not everyone likes him or her.

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