Definition 4.1 (Inconspicuous Strategy Profile). A strategy profile is inconspicuous if each manipulator permutes their preference lists by moving only one man to a higher rank.

For convenience, we introduce a new notation $Pro(w)$ for each $w \in W$. A proposal list $Pro(w)$ of a woman is a list of all men who have proposed to her in the Gale-Shapley algorithm, and the orderings of its entries are the same as her stated preference list. A reduced proposal list contains the top two entries (first entry if only one entry exists) of $Pro(w)$, denoted by $Pro_r(w)$. Clearly, each woman w is matched to the first man of $Pro_r(w)$.

Theorem 4.2. *For any stable matching with respect to the true preference lists that can be obtained by permutation manipulations, there exists a preference profile for the manipulators, in which each manipulator only needs to move at most one man to some higher ranking, that yields the same matching.*

Theorem 4.2 suggests that for each woman w, let m_1 and m_2 be the two men in $Pro_r(w)$[4], then w can modify her true preference list by moving m_2 to the place right after m_1 to generate the same induced matching (see Algorithm 2 for details).

Algorithm 2: Find a Pareto-optimal and inconspicuous preference profile

Use Algorithm 1 to compute a strategy profile $P'(L)$ for L;
Compute $Pro_r(w)$ for each $w \in L$ with respect to $P'(L)$;
for *w in L* **do**
    Modify the true preference list $P(w)$ by moving the second man in $Pro_r(w)$ to the position right after the first man in $Pro_r(w)$;
return *the modified preference profile P*;

5 INCENTIVE PROPERTIES

Although we only have been focusing on constructing Pareto-optimal strategy profiles, a Pareto-optimal strategy profile, which is also inconspicuous, actually forms a Nash equilibrium.

Lemma 5.1. *Suppose there is only one manipulator w. Then the best matching μ' that w can obtain via permutation manipulation is stable with respect to the true preference P.*

Proof. Let P' be the preference profile corresponding to μ'. Assume on the contrary that μ' is not stable with respect to P. Then there must be a blocking pair. However, any pair (m, w') with $w' \neq w$ cannot block μ' under P, since they have the same preferences in both P and P'. It follows that the woman in the blocking pair must be w. Let (m, w) be the blocking pair. We move m to the top of $P'(w)$. If we run the Gale-Shapley algorithm with the new preference profile, m will still propose to w and will finally be matched to w since m is now the favorite man of w. But $m \succ_w \mu'(w)$, which contradicts to the fact that μ' is the best matching that w can obtain. □

Theorem 5.2. *A strategy profile, that is Pareto-optimal and inconspicuous, forms a Nash equilibrium.*

[4]If woman w only receives one proposal, she cannot implement any manipulation.

Proof. Denote by P and μ the true preference profile of agents and the corresponding matching. Let P_1 be the preference profile returned by Algorithm 2 given P, and μ_1 be the corresponding matching. It is clear that for each $w \in L$, Algorithm 2 only changes the order of the men ranked strictly lower than $\mu_1(w)$. For the sake of contradiction, assume there exists a manipulator $w' \in L$ such that w' can get a strictly better partner m ($m \succ^P_{w'} \mu_1(w')$) by misreporting a different preference list. Let P_2 and μ_2 be the preference profile after misreporting and the corresponding matching. Without loss of generality, we assume that m is the best partner (according to both P and P_1) that w' can obtain. Then we know from Lemma 5.1 that, μ_2 is stable with respect to P_1, and thus for each $w \in W$, we have that $\mu_2(w) \succeq^{P_1}_w \mu_1(w)$. It follows that $\mu_2(w) \succeq^P_w \mu_1(w)$, since Algorithm 2 does not change the order of the men who are ranked higher than $\mu_1(w)$. It follows that μ_2 is also stable with respect to P, and μ_2 Pareto-dominates μ_1. However, μ_2 is not found by Algorithm 2. A contradiction. □

Since all Pareto-optimal strategy profiles can be turned into an inconspicuous one by Algorithm 2, we have the following corollary.

Corollary 5.3. *All Pareto-optimal matchings can be induced by a Nash equilibrium.*

Therefore, Pareto-optimal matchings exactly address both the cooperation and the competition among the women in the coalition.

6 STRICTLY BETTER-OFF OUTCOMES

The above results show that the Gale-Shapley algorithm is vulnerable to coalition manipulation. However, if a manipulation is costly such that every manipulator needs to be strictly better off after the manipulation to preserve individual rationality, we show a hardness result:

Theorem 6.1. *It is NP-complete to find a strategy profile, the induced matching of which is strictly better off for all manipulators.*

Therefore, when the manipulation is costly, a manipulation coalition is unlikely to form and the Gale-Shapley algorithm is immune to coalition manipulations. According to Theorem 6.1, one immediate corollary is that the number of Pareto-optimal matchings cannot be polynomial in the number of men and women. For otherwise, we can enumerate all such matchings by Algorithm 1 to develop a polynomial time algorithm. Last but not least, we show that the problem to compute the number of Pareto-optimal matchings, which are strictly better off for all manipulators, is #P-complete.

Theorem 6.2. *It is #P-complete to compute the number of Pareto-optimal matchings, which are strictly better off for all manipulators.*

7 CONCLUSION

Motivated by a real life phenomenon risen in recent years in the college admissions in China, we consider manipulations by subsets of women in the Gale-Shapley algorithm. We show that a Nash equilibrium with Pareto-optimal matching can be efficiently computed in general. These results confirm that the leagues of universities can benefit from forming coalitions. On the contrary, we show that it is NP-complete to find a strictly better off matching for all the manipulators, implying that Gale-Shapley algorithm is immune from permutation manipulations when the manipulations are costly.

REFERENCES

[1] 2011. China's top universities start joint entrance exams. http://english.pku.edu.cn/News_Events/News/Outlook/7740.htm. (Feb. 21, 2011). Accessed: 2017-07-08.

[2] Atila Abdulkadiroğlu, Parag A Pathak, and Alvin E Roth. 2005. The new york city high school match. *American Economic Review* (2005), 364–367.

[3] Atila Abdulkadiroglu and Tayfun Sönmez. 2003. School choice: A mechanism design approach. *The American Economic Review* 93, 3 (2003), 729–747.

[4] David J Abraham, Avrim Blum, and Tuomas Sandholm. 2007. Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges. In *Proceedings of the 8th ACM conference on Electronic commerce*. ACM, 295–304.

[5] Itai Ashlagi, Yashodhan Kanoria, and Jacob Leshno. 2013. Unbalanced Random Matching Markets. In *Proceedings of the Fourteenth ACM Conference on Electronic Commerce (EC '13)*. ACM, New York, NY, USA, 27–28. https://doi.org/10.1145/2482540.2482590

[6] Haris Aziz, Hans Georg Seedig, and Jana Karina von Wedel. 2015. On the Susceptibility of the Deferred Acceptance Algorithm. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 939–947.

[7] Yan Chen and Onur Kesten. 2017. Chinese college admissions and school choice reforms: A theoretical analysis. *Journal of Political Economy* 125, 1 (2017), 99–139.

[8] Lester E Dubins and David A Freedman. 1981. Machiavelli and the Gale-Shapley algorithm. *American mathematical monthly* (1981), 485–494.

[9] D. Gale and L. S. Shapley. 1962. College Admissions and the Stability of Marriage. *The American Mathematical Monthly* 69, 1 (1962), 9–15.

[10] David Gale and Marilda Sotomayor. 1985. Ms. Machiavelli and the stable matching problem. *Amer. Math. Monthly* (1985), 261–268.

[11] Yannai A. Gonczarowski. 2014. Manipulation of Stable Matchings Using Minimal Blacklists. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation (EC '14)*. ACM, New York, NY, USA, 449–449. https://doi.org/10.1145/2600057.2602840

[12] Sushmita Gupta, Kazuo Iwama, and Shuichi Miyazaki. 2015. Stable Nash Equilibria in the Gale-Shapley Matching Game. *arXiv preprint arXiv:1509.04344* (2015).

[13] Dan Gusfield. 1987. Three fast algorithms for four problems in stable marriage. *SIAM J. Comput.* 16, 1 (1987), 111–128.

[14] Dan Gusfield and Robert W. Irving. 1989. *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, Cambridge, MA, USA.

[15] Chien-Chung Huang. 2006. Cheating by men in the Gale-Shapley stable matching algorithm. In *Algorithms–ESA 2006*. Springer, 418–431.

[16] Nicole Immorlica and Mohammad Mahdian. 2005. Marriage, honesty, and stability. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*. Society for Industrial and Applied Mathematics, 53–62.

[17] Robert W Irving. 1985. An efficient algorithm for the "stable roommates" problem. *Journal of Algorithms* 6, 4 (1985), 577–595.

[18] Robert W Irving and Paul Leather. 1986. The complexity of counting stable marriages. *SIAM J. Comput.* 15, 3 (1986), 655–667.

[19] Robert W Irving and David F Manlove. 2009. Finding large stable matchings. *Journal of Experimental Algorithmics (JEA)* 14 (2009), 2.

[20] Robert W Irving, David F Manlove, and Sandy Scott. 2000. The hospitals/residents problem with ties. In *Algorithm Theory-SWAT 2000*. Springer, 259–271.

[21] Donald E. Knuth. 1976. *Mariages Stables*. Les Presses de I'Universite de Montreal.

[22] Hirotatsu Kobayashi and Tomomi Matsui. 2009. Successful manipulation in stable marriage model with complete preference lists. *IEICE TRANSACTIONS on Information and Systems* 92, 2 (2009), 116–119.

[23] Hirotatsu Kobayashi and Tomomi Matsui. 2010. Cheating strategies for the Gale-Shapley algorithm with complete preference lists. *Algorithmica* 58, 1 (2010), 151–169.

[24] Yicheng Liu, Pingzhong Tang, and Wenyi Fang. 2014. Internally Stable Matchings and Exchanges. In *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence, July 27 -31, 2014, Québec City, Québec, Canada*. 1433–1439.

[25] Yicheng Liu, Pingzhong Tang, Tingting Xu, and Hang Zheng. 2016. Optimizing Trading Assignments in Water Right Markets. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence, February 12-17, 2016, Phoenix, Arizona, USA*. 551–557.

[26] David G McVitie and Leslie B Wilson. 1971. The stable marriage problem. *Commun. ACM* 14, 7 (1971), 486–490.

[27] Maria Silvia Pini, Francesca Rossi, K Brent Venable, and Toby Walsh. 2009. Manipulation and gender neutrality in stable marriage procedures. In *Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems-Volume 1*. International Foundation for Autonomous Agents and Multiagent Systems, 665–672.

[28] Alvin E Roth. 1982. The economics of matching: Stability and incentives. *Mathematics of operations research* 7, 4 (1982), 617–628.

[29] Alvin E Roth. 1996. The national residency matching program as a labor market. *Journal of the American Medical Association* 275, 13 (1996), 1054–1056.

[30] Alvin E Roth, Tayfun Sönmez, and M Utku Ünver. 2004. Kidney Exchange. *The Quarterly journal of economics* 119, 2 (2004), 457–488.

[31] Alvin E Roth, Tayfun Sönmez, and M Utku Ünver. 2005. A kidney exchange clearinghouse in New England. *American Economic Review* (2005), 376–380.

[32] Noriyoshi Sukegawa and Yoshitsugu Yamamoto. 2012. Preference profiles determining the proposals in the Gale–Shapley algorithm for stable matching problems. *Japan journal of industrial and applied mathematics* 29, 3 (2012), 547–560.

[33] Chung-Piaw Teo, Jay Sethuraman, and Wee-Peng Tan. 2001. Gale-shapley stable marriage problem revisited: Strategic issues and applications. *Management Science* 47, 9 (2001), 1252–1267.

[34] Rohit Vaish and Dinesh Garg. 2017. Manipulating Gale-Shapley Algorithm: Preserving Stability and Remaining Inconspicuous. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*. 437–443. https://doi.org/10.24963/ijcai.2017/62

[35] Wei Zhan, Zhiyuan Li, Pingzhong Tang, Yicheng Liu, and Tingting Xu. 2017. Stability of generalized two-sided markets with transaction thresholds. In *Proceedings of the AAMAS*.